Stability of Unduloids under the Volume Preserving Mean Curvature Flow

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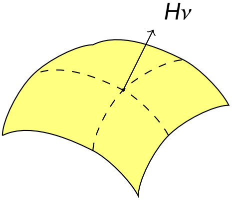
Volume Preserving Mean Curvature Flow

$$\tilde{\mathbf{X}} : M^n \to \tilde{\mathbf{X}}(M^n) =: \tilde{\Omega} \subset \mathbb{R}^{n+1}$$

$$\frac{\partial \mathbf{X}}{\partial t} = (h - H)\,\nu, \quad \mathbf{X}(\cdot, 0) = \tilde{\mathbf{X}}, \quad h = \frac{\int_{M^n} H\, d\mu}{\int_{M^n} d\mu} \tag{1}$$



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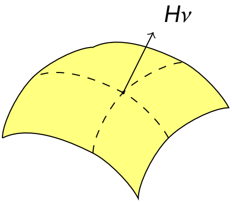

- Together with boundary condition (if applicable)

Graphs Over Hypersurfaces - $\frac{\partial \mathbf{X}}{\partial t} = (h - H)\,\nu,\ \mathbf{X}(\cdot, 0) = \tilde{\mathbf{X}}$

Considering the case where the initial hypersurface is a normal graph over an embedded hypersurface

$$\tilde{\mathbf{X}}(\mathbf{p}) = \mathbf{X}_{\rho_0}(\mathbf{p}) = \mathbf{X}_0(\mathbf{p}) + \rho_0(\mathbf{p})\,\nu_0(\mathbf{p}), \quad \mathbf{p} \in M^n, \qquad \rho_0 : M^n \to \mathbb{R}$$

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The flow is equivalent to an equation for the height function

$$\frac{\partial \rho}{\partial t} = L(\rho)\big(h(\rho) - H(\rho)\big), \ \ \rho(\cdot, 0) = \rho_0 \tag{2}$$

$$L(\rho) = \sqrt{1 + \tilde{g}_\rho^{ij} \nabla_i \rho \nabla_j \rho}, \quad h(\rho) = \frac{\int_{M^n} H(\rho)\, d\mu_\rho}{\int_{M^n} d\mu_\rho}$$

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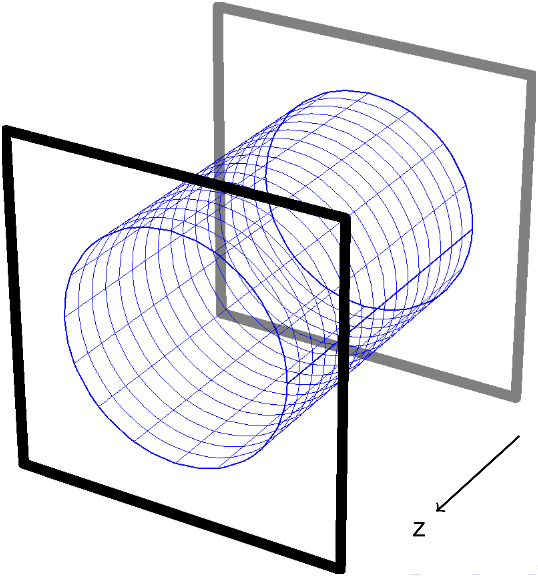
- $\tilde{g}^{ij}_{\rho}$ is the inverse of $\left(\delta^k_i + \rho h^k_i\right) g_{kl} \left(\delta^l_j + \rho h^l_j\right)$.

Cylindrical Graphs - $\frac{\partial\rho}{\partial t} = L(\rho)\big(h(\rho) - H(\rho)\big),\ \rho(\cdot,0) = \rho_0$



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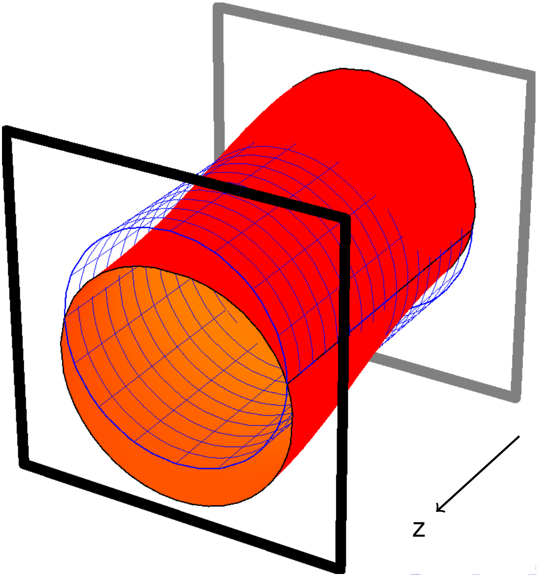

Results for Cylinders - $\frac{\partial\rho}{\partial t} = L(\rho)\left(h(\rho) - H(\rho)\right),\ \rho(\cdot, 0) = \rho_0$

Theorem (H. 2014)

For all $\rho_0 \in U := \left\{ f \in h^{1,\beta}_{\frac{\partial}{\partial z}}\left(\mathscr{C}^n_{R,d}\right) : f > -R \right\}$ *(2), together with the boundary condition* $\frac{\partial\rho}{\partial z}\Big|_{z=0,d} = 0$*, has a unique maximal solution*

$$\rho \in C\left([0,\delta), U\right) \cap C^\infty\left((0,\delta), C^\infty\left(\mathscr{C}^n_{R,d}\right)\right).$$

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Furthermore if $R > \frac{d\sqrt{n-1}}{\pi}$*, then for each* $k \in \mathbb{N}$ *there exists a neighbourhood of zero* $V_k \subset U$ *such that for all* $\rho_0 \in V_k$ *we have* $\delta = \infty$ *and* $\rho(t)$ *converges exponentially fast, with respect to the* C^k *topology, to a function whose graph is a cylinder.*

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- There exist non-axially symmetric hypersurfaces converging to cylinders

Axial Symmetry- $\frac{\partial \rho}{\partial t} = L(\rho)\left(h(\rho) - H(\rho)\right),\ \rho(\cdot, 0) = \rho_0$

- In the axially symmetric case we have:

$$H(\rho) := \frac{-\frac{d^2\rho}{dz^2}}{\left(1 + \left(\frac{d\rho}{dz}\right)^2\right)^{3/2}} + \frac{n-1}{\rho\sqrt{1 + \left(\frac{d\rho}{dz}\right)^2}}$$

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- To remove the boundary conditions we consider an equivalent equation on the circle of radius $\frac{d}{\pi}$:

$$\frac{\partial u}{\partial t} = G(u) := L(u)\big(h(u) - H(u)\big),\ u(\cdot, 0) = u_0, \tag{3}$$

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with solutions in the space $h_e^{2,\alpha}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right)$, where:

$$h_e^{2,\alpha}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right) = \left\{u \in h^{2,\alpha}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right) : u(-z) = u(z)\right\}$$

Splitting the Space - $\frac{\partial u}{\partial t} = L(u)\big(h(u) - H(u)\big),\ u(\cdot, 0) = u_0$

- We split the function space into $h^{2,\alpha}_{e,0}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right) := P_0\left[h^{2,\alpha}_e\left(\mathscr{S}^1_{\frac{d}{\pi}}\right)\right]$, $P_0[u] := u - \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} u\,dz$, and the positive real line

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There exist $U_1 \subset h^{2,\alpha}_{e,0}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right) \times \mathbb{R}^+$ and $U_2 \subset h^{2,\alpha}_{e}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right)$, neighbourhoods of $\left(0, \frac{\pi\sqrt{n-1}}{d}\right)$ and $\frac{d\sqrt{n-1}}{\pi}$, respectively, and a diffeomorphism $\psi : U_1 \to U_2$ such that:

$$Vol\big(\psi(\bar{u},\eta)\big) = \frac{2d\omega_n(n-1)^n}{\eta^n}, \text{ and } \psi(\bar{u},\eta) = \bar{u} + ⨍_{\mathscr{S}^1_{\frac{d}{\pi}}} \psi(\bar{u},\eta)\,dz.$$

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- The converse is also true: if $u \in U_2$ define $\bar{u} = P_0[u]$ and $\eta = (n-1)\left(\frac{2d\omega_n}{Vol(u)}\right)^{1/n}$ then $u = \psi(\bar{u},\eta)$

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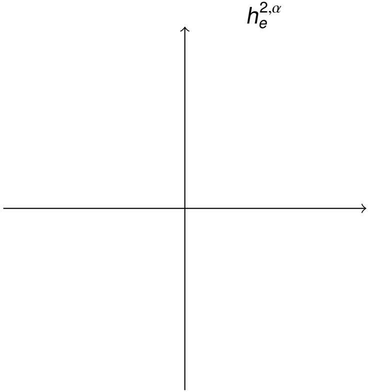

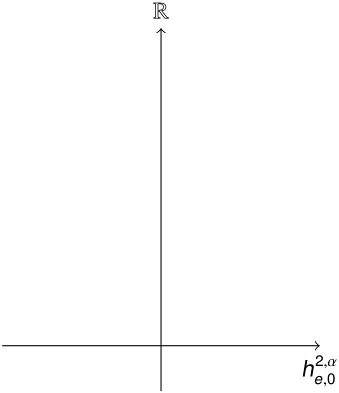

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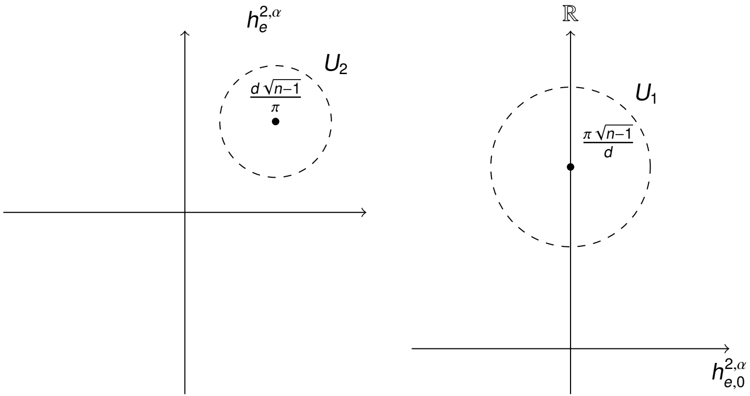

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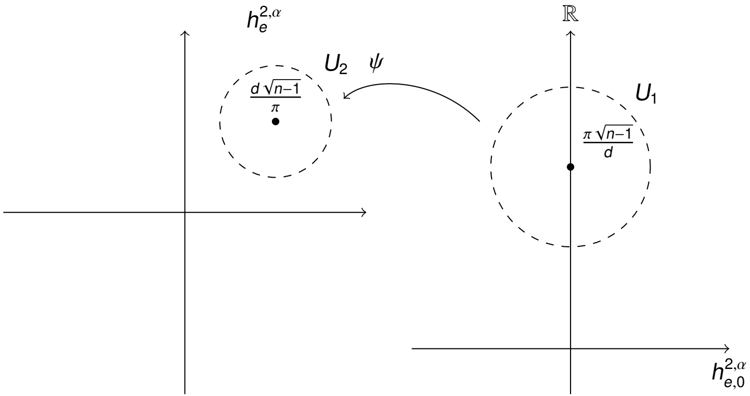

Equivalence - $\frac{\partial u}{\partial t} = L(u)\left(h(u) - H(u)\right),\ u(\cdot, 0) = u_0$

$$\frac{\partial \bar{u}}{\partial t} = \bar{G}(\bar{u}, \eta) := P_0\left[G\left(\psi\left(\bar{u}(t), \eta\right)\right)\right], \\ \bar{u}(\cdot, 0) = \bar{u}_0 \tag{4}$$

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- Solving this system is equivalent to solving (3) with $u_0 = \psi(\bar{u}_0,\eta)$, the solution is $u = \psi(\bar{u},\eta)$
- A solution to (3) also gives a solution to (4), with η determined by the volume of u_0, i.e. $\bar{u} = P_0\,[u]$

Stationary Solutions - $\frac{\partial \bar{u}}{\partial t} = \bar{G}(\bar{u}, \eta),\ \bar{u}(\cdot, 0) = \bar{u_0}$

There exists $\delta > 0$ and a continuously differentiable curve of stationary solutions $(\bar{r}_s, \eta_s)$, $|s| < \delta$, such that $(\bar{r}_0, \eta_0) = \left(0, \frac{\pi\sqrt{n-1}}{d}\right)$ and $\bar{r}_s \neq 0$ for $s > 0$. Further, there exists a neighbourhood of $\left(0, \frac{\pi\sqrt{n-1}}{d}\right)$ such that any non-trivial stationary solution in this neighbourhood is on the curve.

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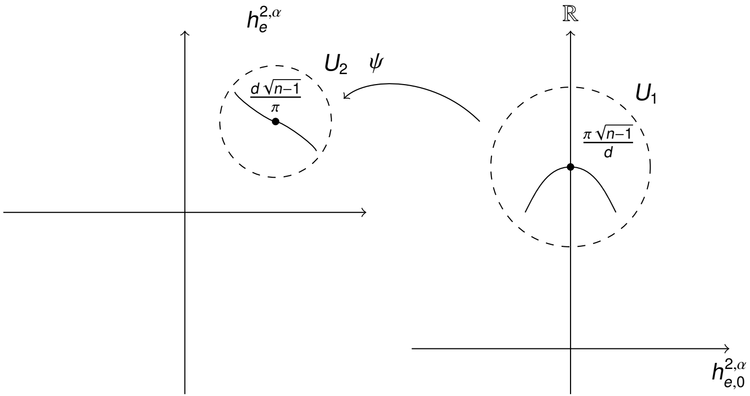

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- If $n \geq 11$ there exists $\epsilon > 0$ such that $Re\left(\sigma\left(D_1\bar{G}(\bar{r}_s, \eta_s)\right)\right) \subset \mathbb{R}^-$ for $0 < |s| < \epsilon$.

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For all $0 < |s| < \epsilon$ there exists $V_s \subset h^{2,\alpha}_{e,0}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right)$, a neighbourhood of $\bar{r}_s$, and $\omega > 0$ such that if $\bar{u}_0 \in V_s$ then the solution to (4) with $\eta = \eta_s$ exists for all time and satisfies

$$\left\|\bar{u}(\cdot, t) - \bar{r}_s\right\|_{h^{2,\alpha}} \leq Ce^{-\omega t}\left\|\bar{u}_0 - \bar{r}_s\right\|_{h^{2,\alpha}}.$$

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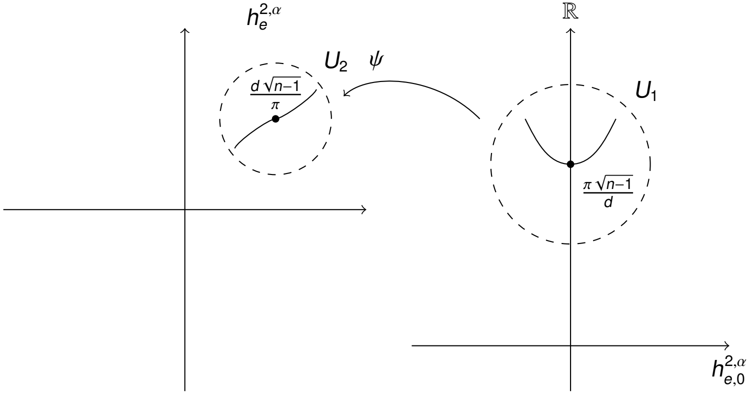

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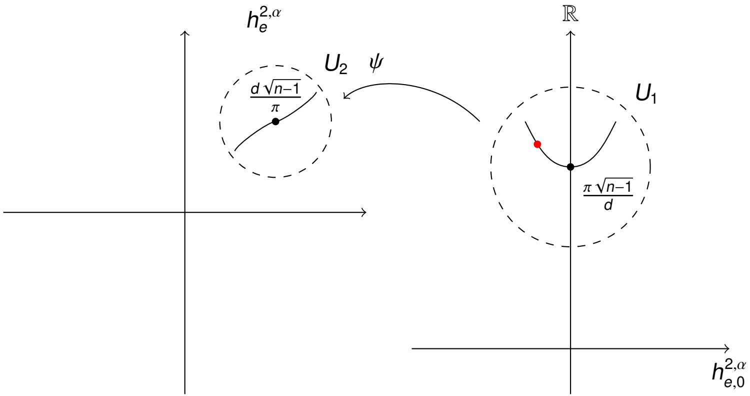

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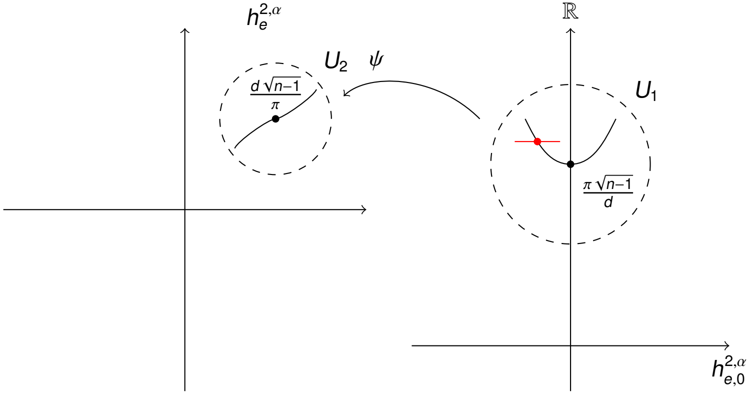

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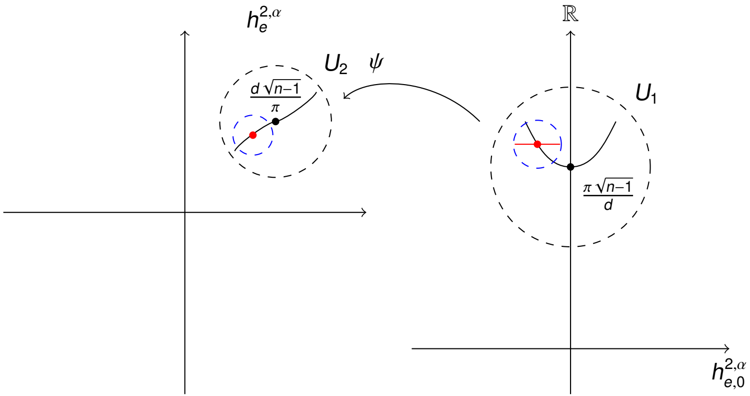

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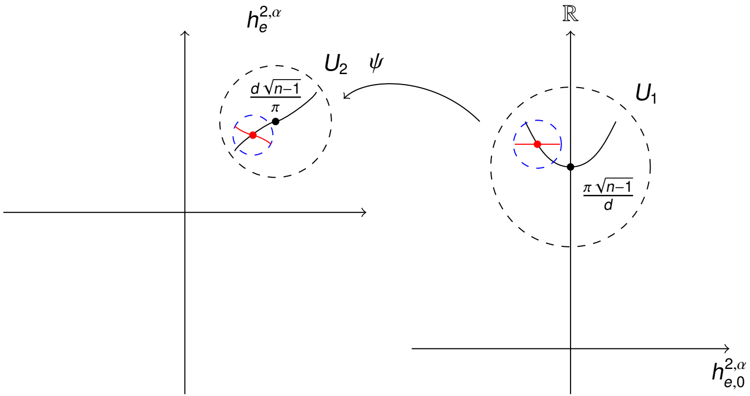

Stable Unduloids - $\frac{\partial\rho}{\partial t} = L(\rho)\left(h(\rho) - H(\rho)\right), \; \rho(0) = \rho_0$

- $r_s := \psi(\bar{r}_s, \eta_s)|_{[0,d]}$, $|s| < \delta$, is a continuously differentiable family of profile curves that define CMC unduloids, with r_0 defining a cylinder of radius $\frac{d\sqrt{n-1}}{\pi}$.

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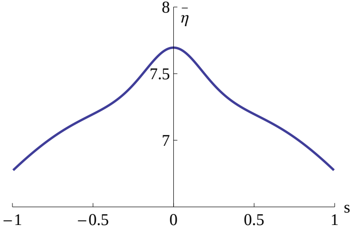
Theorem (H. 2015)

When $n \geq 11$, for all $0 < |s| < \epsilon$ there exists $U_s \subset h^{2,\alpha}_{\frac{d}{dz}}\left([0,d]\right)$ a neighbourhood of r_s and $\omega > 0$ such that if $\rho_0 \in U_s$ and $Vol\left(\rho_0\right) = Vol\left(r_s\right)$ then the solution to (2) exists for all time and satisfies

$$\left\|\rho(\cdot, t) - r_s\right\|_{h^{2,\alpha}} \leq Ce^{-\omega t}\left\|P_0[u_{\rho_0} - u_{r_s}]\right\|_{h^{2,\alpha}}.$$

Bifurcation Parameter Curves



(a) Normalised η_s when $n = 7$

(b) Normalised η_s when $n = 9$

(c) Normalised η_s when $n = 10$

(d) Normalised η_s when $n = 11$

Thank You