Stability of Axisymmetric Liquid Bridges

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CONTENT

- *J. Coll. Interf. Sci.*, **417**, # 3, pp. 37-50 (2014),
- *Z. Angew. Math. Phys.*, **66**, # 6, pp. 3447-3471 (2015),
- *J. Geom. Symmetry Phys.*, **39**, pp. 77-98 (2015)

Isoperimetric problem with free endpoints

Let a planar curve C with parameterization $\{x(t), y(t)\}$, $t_2 \le t \le t_1$, be given with its endpoints $\{x(t_j), y(t_j)\}$, $j = 1, 2$ allowed to move along two given curves S_j parameterized as $\{X_j(\tau_j), Y_j(\tau_j)\}$, $0 \le \tau_j \le \tau_j^*$ (variable τ_j runs along S_j).

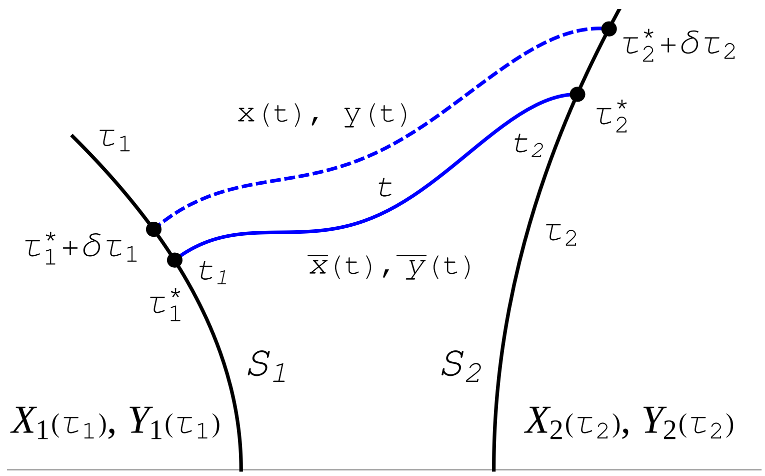


Figure 1:

Consider functionals $E[x, y]$ and $V[x, y]$ with constraint $V[x, y] = 1$,

$$E[x, y] = \int_{t_2}^{t_1} \mathsf{E}(x, y, x_t, y_t)dt + \sum_{j=1}^{2} \int_0^{\tau_j^*} \mathsf{A}_j(X_j, Y_j, X_{j,\tau_j}, Y_{j,\tau_j})d\tau_j,$$

$$V[x, y] = \int_{t_2}^{t_1} \mathsf{V}(x, y, x_t, y_t)dt + \sum_{j=1}^{2}(-1)^j \int_0^{\tau_j^*} \mathsf{B}_j(X_j, Y_j, X_{j,\tau_j}, Y_{j,\tau_j})d\tau_j,$$

where E and V be positive homogeneous functions of degree one in x_t and y_t.

$$x(t) = \bar{x}(t) + u(t), \qquad y(t) = \bar{y}(t) + v(t)$$

The first isoperimetric problem (IP)

Find an extremal curve $\bar{C} = \{\bar{x}(t), \bar{y}(t)\}$ *with free endpoints* $\bar{x}(t_j), \bar{y}(t_j)$ *which belong to two given curves* S_j *such that the functional* $E[x, y]$ *reaches its minimum while* V *is constrained.*

- A functional $W[x,y] = E[x,y] - \lambda V[x,y]$ with multiplier

$$W[x,y] = \int_{t_2}^{t_1} F(x,y,x_t,y_t)dt - \sum_{j=1}^{2}(-1)^j \int_0^{\tau_j^*} G_j(X_j, Y_j, X_{j,\tau_j}, Y_{j,\tau_j})d\tau_j,$$

where

$$F = \mathsf{E} - \lambda\mathsf{V}, \quad G_1 = \lambda\mathsf{B}_1 + \mathsf{A}_1, \quad G_2 = \lambda\mathsf{B}_2 - \mathsf{A}_2.$$

$$F = \frac{\partial F}{\partial x_t}x_t + \frac{\partial F}{\partial y_t}y_t, \quad G_j = \frac{\partial G_j}{\partial X_{j,\tau_j}}X_{j,\tau_j} + \frac{\partial G_j}{\partial Y_{j,\tau_j}}Y_{j,\tau_j}.$$

- The total variation of $W[x,y]$

$$\mathbb{D}W = \mathbb{D}_0 W + \mathbb{D}_1 W - \mathbb{D}_2 W,$$

$$\mathbb{D}_0 W = \int_{t_2}^{t_1} [\Delta_1 F + \Delta_2 F]\, dt, \qquad \mathbb{D}_j W = \int_0^{\tau_j^* + \delta\tau_j} G_j d\tau_j - \int_0^{\tau_j^*} G_j d\tau_j,$$

$$\begin{aligned}
\Delta_1 F &= \frac{\partial F}{\partial x}u + \frac{\partial F}{\partial x_t}u' + \frac{\partial F}{\partial y}v + \frac{\partial F}{\partial y_t}v', \\
\Delta_2 F &= \frac{u^2}{2}\frac{\partial^2 F}{\partial x^2} + uu'\frac{\partial^2 F}{\partial x \partial x_t} + \frac{u'^2}{2}\frac{\partial^2 F}{\partial x_t^2} + \frac{v^2}{2}\frac{\partial^2 F}{\partial y^2} + vv'\frac{\partial^2 F}{\partial y \partial y_t} \\
&+ \frac{v'^2}{2}\frac{\partial^2 F}{\partial y_t^2} + uv\frac{\partial^2 F}{\partial x \partial y} + uv'\frac{\partial^2 F}{\partial x \partial y_t} + u'v\frac{\partial^2 F}{\partial x_t \partial y} + u'v'\frac{\partial^2 F}{\partial x_t \partial y_t},
\end{aligned}$$

• First variation δW and Euler-Lagrange equation

$$\delta W = \int_{t_2}^{t_1} \Delta_1 F dt + G_1^* \delta\tau_1 - G_2^* \delta\tau_2,$$

where $G_j^* = G_j$ and $\partial G_j^*/\partial X_j = \partial G_j/\partial X_j$ computed at $\tau_j = \tau_j^*$.

• Boundary conditions (BC) for $u(t), v(t)$:

The endpoints of the perturbed curve C are located on curves S_j

$$\bar{x}(t_j) = X(\tau_j^*), \qquad \bar{x}(t_j) + u(t_j) = X(\tau_j^* + \delta\tau_j),$$
$$\bar{y}(t_j) = Y(\tau_j^*), \qquad \bar{y}(t_j) + v(t_j) = Y(\tau_j^* + \delta\tau_j),$$

The perturbations $u(t_j), v(t_j)$

$$u(t_j) = \sum_{k=1}^{\infty} u_k(t_j), \qquad v(t_j) = \sum_{k=1}^{\infty} v_k(t_j)$$

where

$$u_k(t_j) = \frac{1}{k!}\frac{d^k X_j}{d\tau_j^k}\delta^k\tau_j, \quad v_k(t_j) = \frac{1}{k!}\frac{d^k Y_j}{d\tau_j^k}\delta^k\tau_j.$$

$$\delta W = \int_{t_2}^{t_1} \left(u\frac{\delta F}{\delta x} + v\frac{\delta F}{\delta y} \right) dt - \sum_{j=1}^{2} (-1)^j \left[\frac{\partial F_j}{\partial x'}\frac{dX_j}{d\tau_j} + \frac{\partial F_j}{\partial y'}\frac{dY_j}{d\tau_j} + G_j^* \right] \delta\tau_j,$$

• Euler-Lagrange equations (ELE)

$$\frac{\delta F}{\delta x} = \frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial x'} = 0, \qquad \frac{\delta F}{\delta y} = \frac{\partial F}{\partial y} - \frac{d}{dt}\frac{\partial F}{\partial y'} = 0$$

• Transversality conditions

$$\frac{\partial F_2}{\partial x'}\frac{dX_2}{d\tau_2} + \frac{\partial F_2}{\partial y'}\frac{dY_2}{d\tau_2} + G_2^* = 0, \quad \frac{\partial F_1}{\partial x'}\frac{dX_1}{d\tau_1} + \frac{\partial F_1}{\partial y'}\frac{dY_1}{d\tau_1} + G_1^* = 0$$

where $F_j = F,\ \partial F_j/\partial x = \partial F/\partial x,$ *etc.* computed at $t = t_j$.

• Weierstrass representation of second variation $\delta^2 W$

$$\delta^2 W = \int_{t_2}^{t_1} \Delta_2 F dt - \frac{1}{2}\sum_{j=1}^{2}(-1)^j \left(\frac{\partial F_j}{\partial x'}\frac{d^2 X_j}{d\tau_j^2} + \frac{\partial F_j}{\partial y'}\frac{d^2 Y_j}{d\tau_j^2} + \frac{dG_j^*}{d\tau_j}\right)\delta^2\tau_j.$$

• Weierstrass perturbation function $w(t)$

$$w(t) = u(t)\bar{y}_t - v(t)\bar{x}_t$$

$$w(t_j) = \eta(t_j, \tau_j^*)\delta\tau_j, \quad \eta(t_j, \tau_j^*) = \bar{y}_t \frac{dX_j}{d\tau_j} - \bar{x}_t \frac{dY_j}{d\tau_j}$$

Weierstrass part of the second variation $\delta_B^2 W = \int_{t_2}^{t_1} \Delta_2 F dt$

$$\delta_B^2 W[x,y] = \frac{1}{2}\Xi_0[w] + \frac{1}{2}\left(Lu_1^2 + 2Mu_1v_1 + Nv_1^2\right)\Big|_{t_2}^{t_1},$$

$$\begin{aligned}
\Xi_0[w] &= \int_{t_2}^{t_1}\left(H_1 w'^2 + H_2 w^2\right) dt \\
H_1 &= \frac{F_{x'x'}}{\bar{y}_t^2} = \frac{F_{y'y'}}{\bar{x}_t^2} = -\frac{F_{x'y'}}{\bar{x}_t\bar{y}_t}, \\
H_2 &= \frac{F_{xx} - \bar{y}_{tt}^2 H_1 - L_t}{\bar{y}_t^2} = \frac{F_{yy} - \bar{x}_{tt}^2 H_1 - N_t}{\bar{x}_t^2}, \\
L &= F_{xx'} - \bar{y}_t\bar{y}_{tt}H_1, \quad N = F_{yy'} - \bar{x}_t\bar{x}_{tt}H_1, \\
M &= F_{xy'} + \bar{x}_t\bar{y}_{tt}H_1 = F_{yx'} + \bar{y}_t\bar{x}_{tt}H_1.
\end{aligned}$$

- The total second variation $\delta^2 W$

$$\delta^2 W = \frac{1}{2}\Xi_0[w] + K_1(\delta\tau_1)^2 - K_2(\delta\tau_2)^2,$$

$$2K_j = 2\xi_j + L(t_j)\left(\frac{dX_j}{d\tau_j}\right)^2 + 2M(t_j)\frac{dX_j}{d\tau_j}\frac{dY_j}{d\tau_j} + N(t_j)\left(\frac{dY_j}{d\tau_j}\right)^2$$

$$\begin{aligned} 2\xi_j \ = \ & \frac{\partial F_j}{\partial x'}\frac{d^2X_j}{d\tau_j^2} + \frac{\partial F_j}{\partial y'}\frac{d^2Y_j}{d\tau_j^2} + \frac{\partial G_j^*}{\partial X_j}\frac{dX_j}{d\tau_j} + \\ & \frac{\partial G_j^*}{\partial Y_j}\frac{dY_j}{d\tau_j} + \frac{\partial G_j^*}{\partial X_j'}\frac{d^2X_j}{d\tau_j^2} + \frac{\partial G_j^*}{\partial Y_j'}\frac{d^2Y_j}{d\tau_j^2} \end{aligned}$$

Homogeneous boundary conditions: fixed endpoints

$$u(t_j) = v(t_j) = w(t_j) = 0, \quad j = 1, 2.$$

- Bolza representation of the constraint $\delta V[x, y] = 0$

$$\Xi_1[w] = \int_{t_2}^{t_1} H_3 w(t) dt = 0,$$

$$H_3 = \mathsf{V}_{xy'} - \mathsf{V}_{x'y} + H_4(\bar{x}_t \bar{y}_{tt} - \bar{y}_t \bar{x}_{tt}),$$

$$H_4 = \frac{\mathsf{V}_{x'x'}}{\bar{y}_t^2} = \frac{\mathsf{V}_{y'y'}}{\bar{x}_t^2} = -\frac{\mathsf{V}_{x'y'}}{\bar{x}_t \bar{y}_t}$$

In the case of liquid drop between two convex bodies

$$\mathsf{V} = x^2 y', \quad \mathsf{B}_j = X_j^2 Y_j', \quad H_3 = \bar{x}, \quad H_4 = 0$$

- The second isoperimetric problem

Find $w(t)$ providing minimal value of the second variation $\Xi_0[w]$ with constraint $\Xi_1[w] = 0$

- A functional $\Xi_2[w] = \Xi_0[w] + 2\mu\Xi_1[w]$ with multiplier μ

$$\Xi_2[w] = \int_{t_2}^{t_1} \mathcal{H}(t, w, w') dt,$$

$$\mathcal{H}(t, w, w') = H_1 w'^2 + H_2 w^2 + 2\mu H_3 w$$

Necessary conditions for a strong minimum

1. Euler-Lagrange

   Inhomogeneous Jacobi equation

$$(H_1 w')' - H_2 w = \mu H_3, \quad w(t_1) = w(t_2) = 0$$

2. Legendre

$$H_1(t) > 0$$

3. Jacobi criterion

   Interval $[t_1, t_2]$ does not contain points t' conjugated to the endpoints

4. Weierstrass

   Weierstrass function $\mathcal{E}(t, w, w', f) > 0, \quad f(t) \neq w'(t)$

$$\mathcal{E} = \mathcal{H}(t, w, f) - \mathcal{H}(t, w, w') + (w' - f)\mathcal{H}_{w'}(t, w, w')$$

$$\mathcal{E} = H_1(t)[f(t) - w'(t)]^2 > 0$$

The Jacobi criterion is the most difficult part of conditions

• Weierstrass-Howe criterion for conjugated points

Let $\bar{w}_1(t)$ and $\bar{w}_2(t)$ be the fundamental solutions of homogeneous Jacobi equation, and the particular solution $\mu\bar{w}_3(t)$ of inhomogeneous Jacobi equation may be found by standard procedure.

The Howe matrix $D(t_2, t_1)$

$$D(t_2,t_1) = \begin{pmatrix} \bar{w}_1(t_2) & \bar{w}_2(t_2) & \bar{w}_3(t_2) \\ \bar{w}_1(t_1) & \bar{w}_2(t_1) & \bar{w}_3(t_1) \\ J_1(t_1) - J_1(t_2) & J_2(t_1) - J_2(t_2) & J_3(t_1) - J_3(t_2) \end{pmatrix}$$

$$J_k(t) = \int^t H_3(t')\bar{w}_k(t')dt'$$

The determinant equation

$$\boxed{\Delta(t_2,t_1) = \det D(t_2,t_1) = 0}$$

describes a continuous curve $\mathcal{D}(t_2, t_1)$ of conjugated points.

• Existence of extremals

Require that the extremal $\{\bar{x}(t), \bar{y}(t)\}$ does not intersect with the curves S_j. Intersection points are located in the plane $\{t_1, t_2\}$ at the lines $t_j = t_j^\bullet$

$$\eta(t_j^\bullet, \tau_j^*) = 0$$

• Stability domain for IP with fixed endpoints

Consider a point $M_1 = (a, b)$ in the lower halfplane $\{t_2 < t_1\}$ and two more points: $M_2 = (a, a)$ and $M_3 = (b, b)$. Call a point M_1 **the Jacobi point** if the line M_1M_2 does not intersect both $\mathcal{D}(t_2, t_1)$ and $t_2 = t_2^\bullet$, and M_1M_3 does not intersect both $\mathcal{D}(t_2, t_1)$ and $t_1 = t_1^\bullet$. Define a set $\mathbb{J}(t_2, t_1)$ as a union of points M_1

Two sets in the halfplane $\{t_2 < t_1\}$

$$\mathbb{J}(t_2, t_1) =$$

$$\left\{ (a, b) \;\middle|\; \begin{array}{l} \Delta(t, a) \neq 0,\ \Delta(b, t) \neq 0, \quad t_2 < b \leq t \leq a < t_1, \\ \eta(t_2, \tau_2^*) \neq 0, \quad t_2^\bullet < b \leq t_2 \leq a < t_1, \\ \eta(t_1, \tau_1^*) \neq 0, \quad t_2 < b \leq t_1 \leq a < t_1^\bullet. \end{array} \right\}$$

$$\mathbb{L}(t_2, t_1) \;=\; \{(t_2, t_1) | H_1(t) > 0,\ t \in [t_2, t_1]\}\,,$$

Stability domain $\mathsf{Stab}_1(t_2, t_1)$

$$\mathsf{Stab}_1(t_2, t_1) = \mathbb{J}(t_2, t_1) \cap \mathbb{L}(t_2, t_1)$$

Inhomogeneous boundary conditions: free endpoints

The endpoints of perturbed curve $\{\bar{x}+u, \bar{y}+v\}$ always belong to two given curves S_j

$$\boxed{w(t_j) = \eta(t_j, \tau_j^*)\delta\tau_j,} \quad \eta(t_j, \tau_j^*) = \bar{y}_t \frac{dX_j}{d\tau_j} - \bar{x}_t \frac{dY_j}{d\tau_j}$$

and satisfy constraint equation

$$\int_{t_2}^{t_1} H_3(t) w(t) dt = 0$$

General solution $\bar{w}$ of the Jacobi equation

$$\bar{w}(t) = C_1 \bar{w}_1(t) + C_2 \bar{w}_2(t) + \mu \bar{w}_3(t)$$

Linear equations of BC

$$\bar{w}_1(t_j) C_1 + \bar{w}_2(t_j) C_2 + \bar{w}_3(t_j)\mu = \bar{w}(t_j)$$

$$I_1 C_1 + I_2 C_2 + I_3 \mu = 0, \qquad I_k = J_k(t_1) - J_k(t_2)$$

General solution $\bar{w}(t)$

$$\bar{w}(t) = A_1(t)\delta\tau_1 + A_2(t)\delta\tau_2,$$

$$A_j(t) = \frac{\eta_j B_j(t)}{\Delta(t_2, t_1)}, \qquad B_j(t) = B_j(t, t_2, t_1)$$

$$B_1(t) = -\begin{vmatrix} \bar{w}_1(t) & \bar{w}_2(t) & \bar{w}_3(t) \\ \bar{w}_1(t_2) & \bar{w}_2(t_2) & \bar{w}_3(t_2) \\ I_1 & I_2 & I_3 \end{vmatrix}$$

$$B_2(t) = \begin{vmatrix} \bar{w}_1(t) & \bar{w}_2(t) & \bar{w}_3(t) \\ \bar{w}_1(t_1) & \bar{w}_2(t_1) & \bar{w}_3(t_1) \\ I_1 & I_2 & I_3 \end{vmatrix}$$

The second variation as a quadratic form

$$\boxed{\delta^2 W = Q_{11}\left(\delta\tau_1\right)^2 + 2Q_{12}\delta\tau_1\delta\tau_2 + Q_{22}\left(\delta\tau_2\right)^2}$$

$$Q_{11}(t_2, t_1) = \frac{\eta_1^2 P_{11}}{2\Delta} + K_1, \qquad P_{11} = H_1(t_1)B_1'(t_1),$$

$$Q_{22}(t_2, t_1) = \frac{\eta_2^2 P_{22}}{2\Delta} - K_2, \qquad P_{22} = -H_1(t_2)B_2'(t_2),$$

$$Q_{12}(t_2, t_1) = \frac{\eta_1\eta_2 P_{12}}{2\Delta}, \qquad P_{12} = P_{21} = H_1(t_1)B_2'(t_1),$$

$$\eta_j = \eta(t_j, \tau_j^*), \qquad K_j = K_j(t_j, \tau_j^*)$$

• Stability theorem

Let $Q_{ij}(t_2, t_1)$ be given, then $\delta^2 W$ is positive definite if the following inequlities hold,

$$Q_{11}(t_2, t_1) \geq 0, \qquad Q_{22}(t_2, t_1) \geq 0,$$
$$Q_{33}(t_2, t_1) = Q_{11}Q_{22} - Q_{12}^2 \geq 0.$$

Three sets $\mathbb{Q}_j(t_2, t_1)$ and their intersection

$$\mathbb{Q}_j(t_2, t_1) := \{(a, b) \mid (a, b) \in \{t_2 < t_1\},\ Q_{jj}(t_2, t_1) \geq 0\}$$

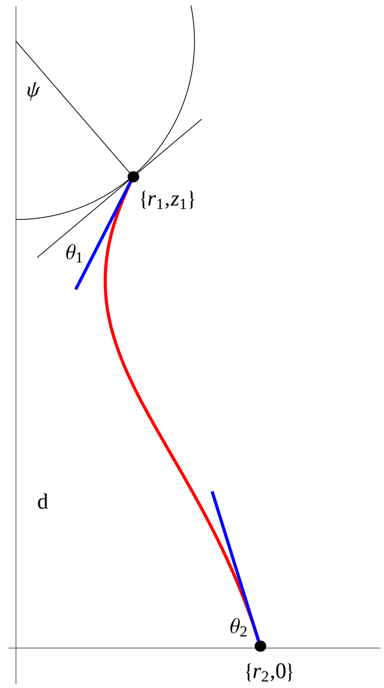
$$\mathbb{Q}(t_2, t_1) := \mathbb{Q}_1(t_2, t_1) \cap \mathbb{Q}_2(t_2, t_1) \cap \mathbb{Q}_3(t_2, t_1)$$

• Stability domain for IP with free endpoints

$$\mathsf{Stab}_2(t_2, t_1) = \mathsf{Stab}_1(t_2, t_1) \cap \mathbb{Q}(t_2, t_1)$$

$$\mathsf{Stab}_2(t_2, t_1) \subseteq \mathsf{Stab}_1(t_2, t_1)$$

Application to axisymmetric liquid bridges



The functional $W = E - \lambda V$ and its integrands

$$W = \int_{\phi_2}^{\phi_1} F(r, r', z, z')d\phi - \sum_{j=1}^{2}(-1)^j \int_0^{\psi_j^*} G_j d\psi_j,$$

$$F = \left[\gamma_{lv}\sqrt{r'^2 + z'^2} - \frac{\lambda r z'}{2}\right] r,$$

$$G_j = \left[\frac{\lambda R_j Z_j'}{2} - (-1)^j(\gamma_{ls_j} - \gamma_{vs_j})\sqrt{R_j'^2 + Z_j'^2}\right] R_j,$$

- Young–Laplace equation (ELE)

$$2H = \frac{z'}{r(r'^2 + z'^2)^{1/2}} + \frac{z''r' - z'r''}{(r'^2 + z'^2)^{3/2}}, \quad H = \frac{\lambda}{2\gamma_{lv}}$$

• Young–Dupré relation (Transversality)

$$\gamma_{lv}\cos\theta_j + \gamma_{ls_j} = \gamma_{vs_j}$$

$$\theta_j = (-1)^{j-1}\left(\arctan\frac{\bar{z}'(\phi_j)}{\bar{r}'(\phi_j)} - \arctan\frac{Z'(\psi_j^*)}{R'(\psi_j^*)}\right)$$

• Axisymmetric perturbations

$$\eta_j = \bar{z}'(\phi_j)R'(\psi_j^*) - \bar{r}'(\phi_j)Z'(\psi_j^*), \qquad K_j = U_j\eta_j,$$

$$U_j = -\frac{R_j}{2\sqrt{\bar{r}_j'^2 + \bar{z}_j'^2}}\left(\frac{\bar{z}_j''R_j' - \bar{r}_j''Z_j'}{\bar{r}_j'^2 + \bar{z}_j'^2} - \frac{Z_j''R_j' - R_j''Z_j'}{R_j'^2 + Z_j'^2}\right)$$

• Stability matrix $Q_{ij}(\phi_1, \phi_2)$

$$Q_{11} = \eta_1\left(\frac{\eta_1 P_{11}}{2\Delta} + U_1\right), \qquad Q_{12} = \frac{\eta_1\eta_2 P_{12}}{2\Delta},$$

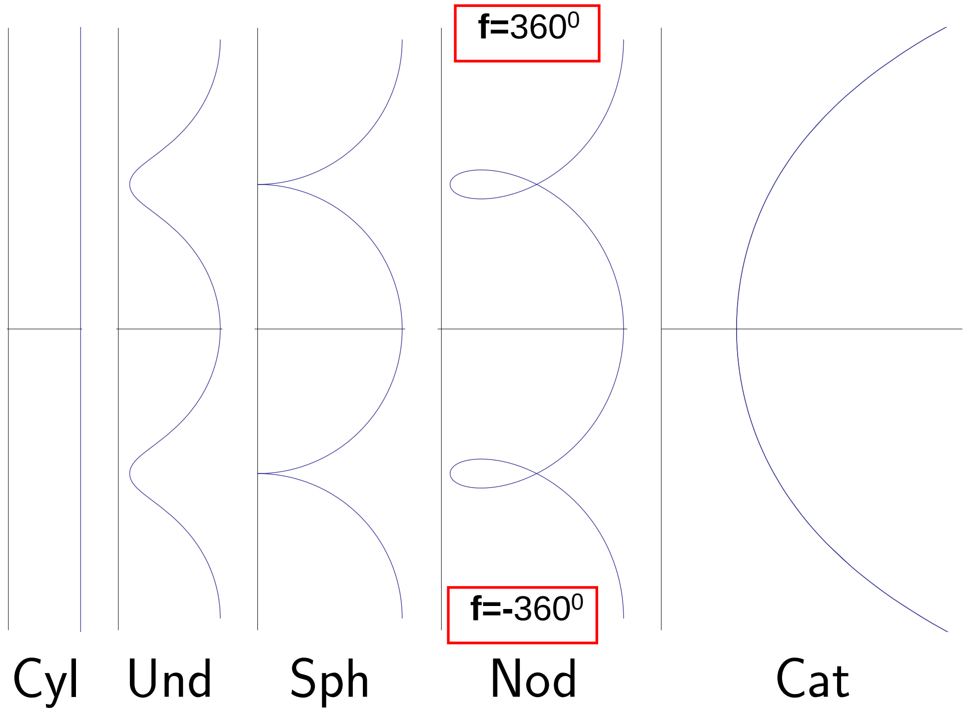
$$Q_{22} = \eta_2\left(\frac{\eta_2 P_{22}}{2\Delta} - U_2\right)$$

• Jacobi equation and fundamental solutions ($H_3 = \bar{r}$)

$$\boxed{(H_1 w')'\bar{r}' - (H_1\bar{r}'')'\,w = \mu\bar{r}'\bar{r},} \quad H_1 = \frac{\bar{r}}{\left(\bar{r}'^2 + \bar{z}'^2\right)^{3/2}},$$

$$\bar{w}_1 = \bar{r}'(\phi), \quad \bar{w}_2 = \bar{w}_1\int^{\phi}\frac{dt}{H_1\bar{r}'^2}, \quad H_2 = \frac{(H_1\bar{r}'')'}{\bar{r}'},$$

Delaunay surfaces and their evolution



(1) $B=0$, (2) $B<1$, (3) $B=1$, (4) $B>1$, (5) ...

- Kenmotsu-Myshkis parameterization

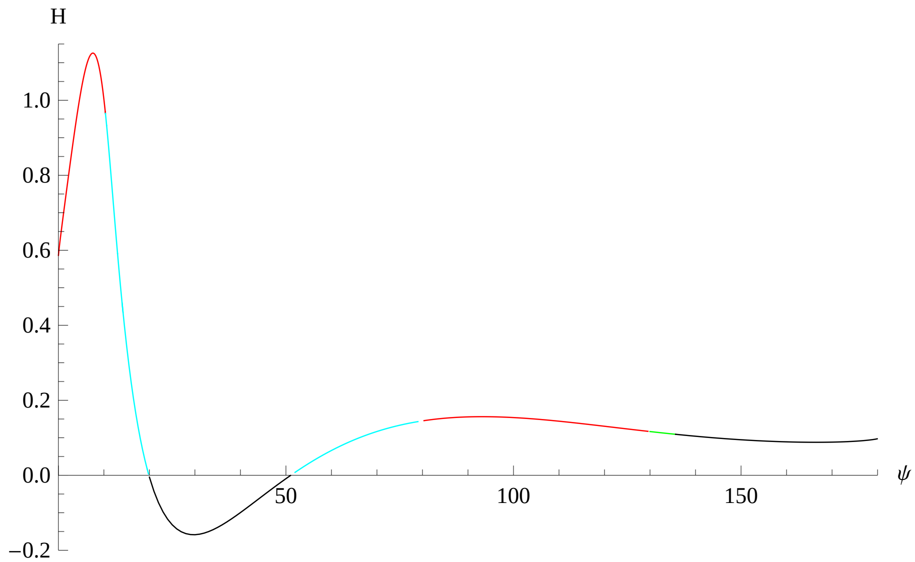
$$\bar{r}(\phi)=\sqrt{1+B^2+2B\cos(\phi)}, \qquad \bar{z}(\phi)=M(\phi,B),$$

$$M(\phi,B)=(1+B)E\left(\phi/2,m\right)+(1-B)F\left(\phi/2,m\right),$$

where $F(x,m)$ and $E(x,m)$ stand for elliptic integrals of the 1st and 2nd kind with modulus $m=2\sqrt{B}/(1+B)$

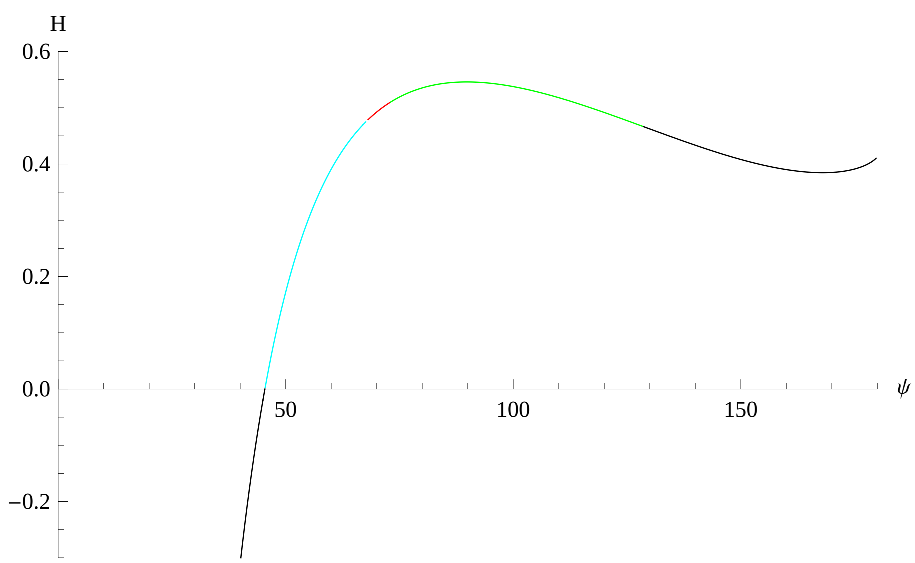
• Evolution of menisci shapes

Solid bodies do not contact



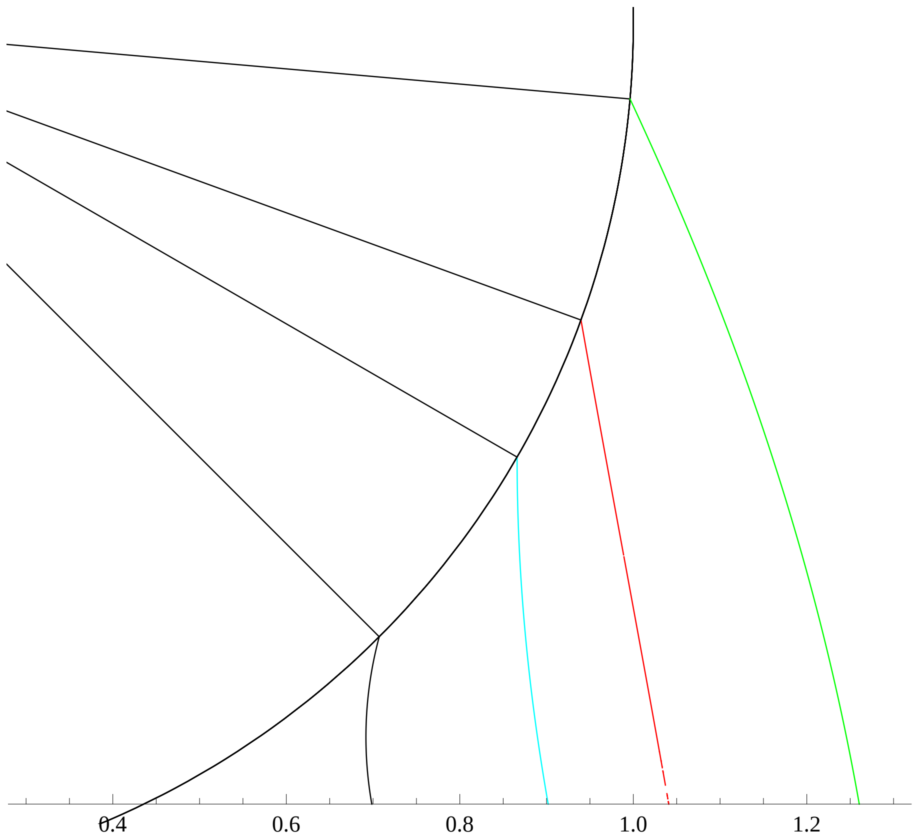
$$\mathrm{Und}_1^- \longrightarrow \mathrm{Und}_0^- \longrightarrow \mathrm{Cat} \longrightarrow \mathrm{Nod}^- \longrightarrow \mathrm{Cat} \longrightarrow \mathrm{Und}_0^- \longrightarrow \mathrm{Und}_1^- \longrightarrow \mathrm{Und}_0^+ \longrightarrow \mathrm{Sph} \longrightarrow \mathrm{Nod}^+$$

Solid bodies do contact

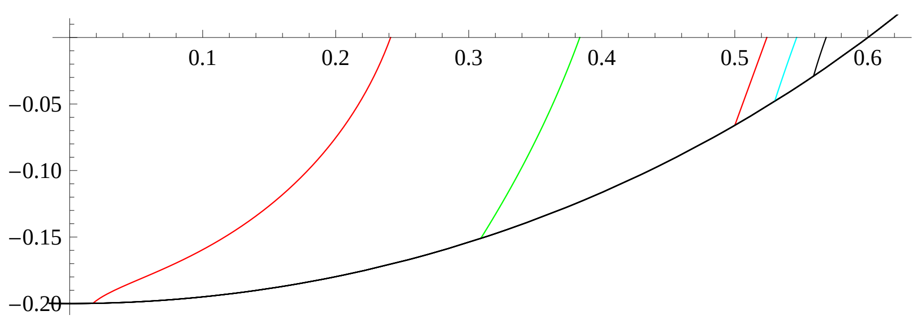


$$\mathrm{Nod}^- \longrightarrow \mathrm{Cat} \longrightarrow \mathrm{Und}_0^- \longrightarrow \mathrm{Und}_1^- \longrightarrow \mathrm{Und}_0^+ \longrightarrow \mathrm{Sph} \longrightarrow \mathrm{Nod}^+$$

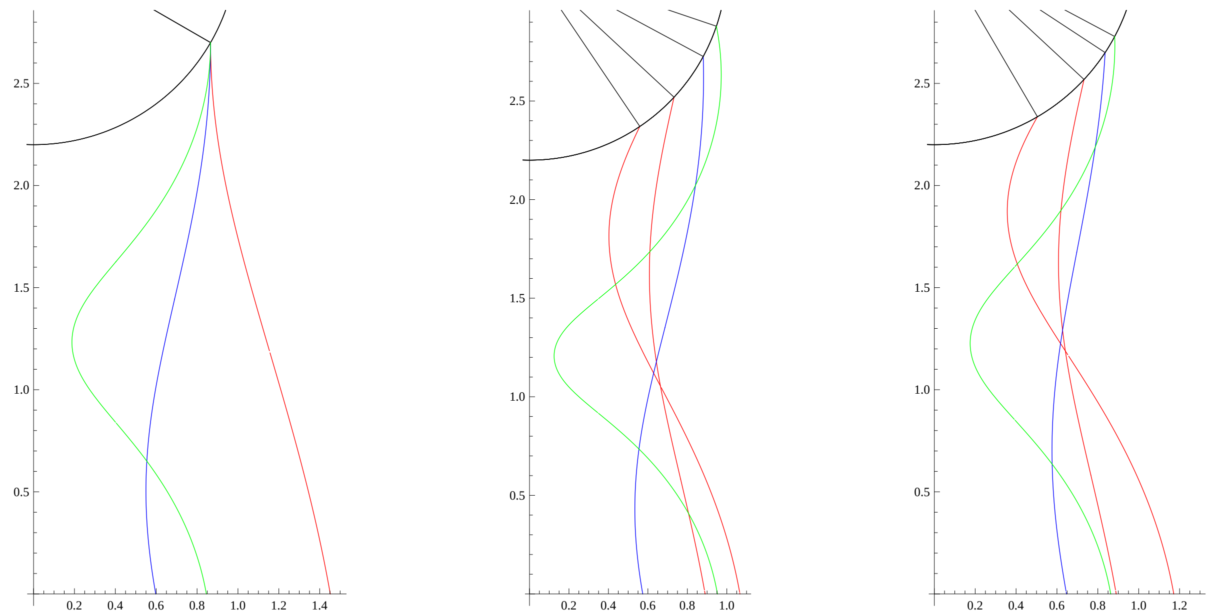
Menisci between convex solid ball interected with plate



Menisci between concave solid ball interected with plate



- Solutions of the Young-Laplace equation are not unique for given volume and contact angles



$\psi = const$ $V = const$ $S = const$

The menisci Und_1^- (*red*), Und_1^+ (*blue*), and Und_2^+ (*green*) for $\theta_1 = 30^o$, $\theta_2 = 80^o$ and $d = 2.2$ computed in three different setups:

1. for $\psi = 60^o$ there are three menisci with different volumes V and surface areas S;

2. for fixed $V = 3.6$ there are four menisci with $\psi_1 = 34^o$ and $\psi_2 = 47.2^o$ – Und_1^-, $\psi_3 = 61.7^o$ – Und_1^+, $\psi_4 = 71.3^o$ – Und_2^+ and different surface areas S;

3. for fixed $S = 11.3$ there are four profiles with $\psi_1 = 30.3^o$ and $\psi_2 = 47.1^o$ – Und_1^-, $\psi_3 = 56.7^o$ – Und_1^+, $\psi_4 = 61.95^o$ – Und_2^+ and different volumes V.

Liquid bridges with zero curvature

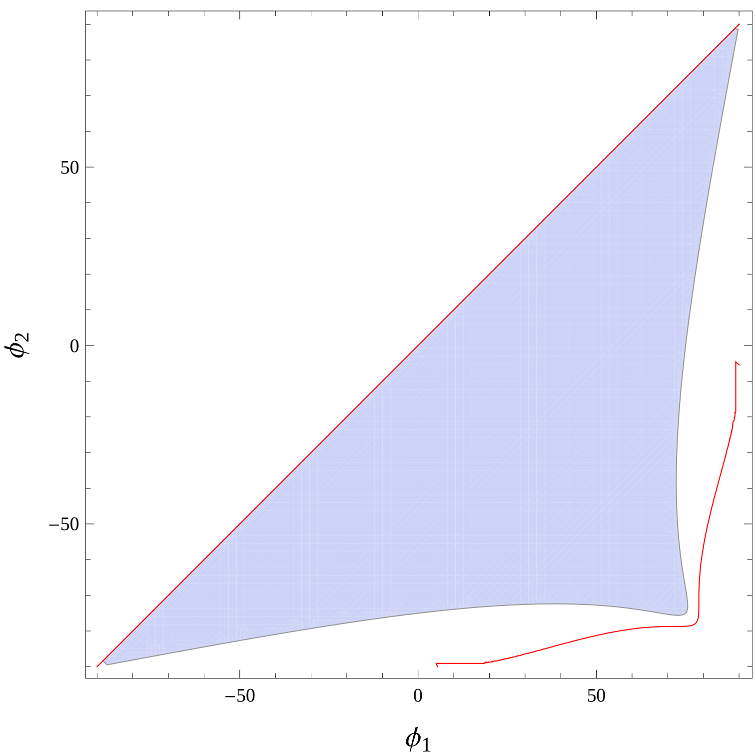
$$\bar{r} = \sec\phi, \quad \bar{z} = \ln\frac{\cos\phi}{1-\sin\phi} + C, \quad \bar{r}'^2 + \bar{z}'^2 = \bar{r}^4,$$

$$H_1 = \frac{1}{\bar{r}^5}, \quad H_2 = -4\frac{\bar{r}^2 - 1}{\bar{r}^5},$$

$$w'' - 5w'\tan\phi + 4w\tan^2\phi = \mu\sec^6\phi,$$

$$\bar{w}_1 = \tan\phi\ \sec\phi, \quad \bar{w}_2 = \sec^2\phi - \bar{w}_1\ln(\tan\phi + \sec\phi)$$

- Catenoid menisci between two solid plates



Symmetric Cat meniscus between two plates is stable if $\theta \geq 14.97^o$.

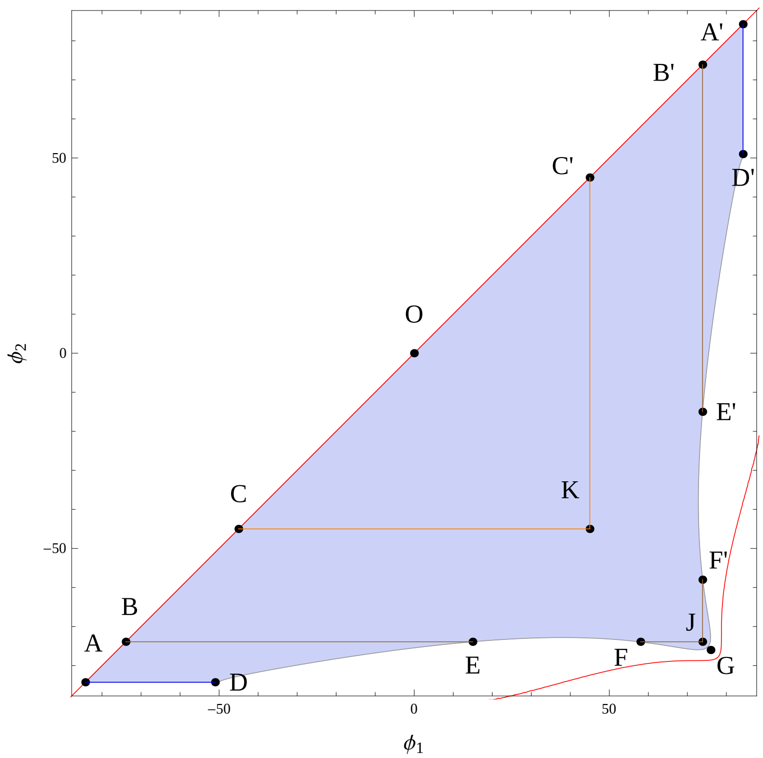
Zhou, L., On stability of a catenoidal liquid bridge, *Pacific J. Math.* **178**, 185-198 (1997)

• Catenoid menisci between two solid convex spheres

$$R_j = A\sin\psi_j, \quad Z_j = g_j + (-1)^j A\cos\psi_j,$$

$$\theta_j = \frac{\pi}{2} + (-1)^j\phi_j - \arctan\frac{1}{\sqrt{A^2\cos^2\phi_j - 1}},$$

$$\phi_1^{\bullet}(1, A) = \arccos\frac{1}{\sqrt{A}}$$



$\mathsf{Stab}_2(t_2, t_1)$ are represented by interiors of curvilinear polygons:
{OCBADEFGF'E'D'A'B'C'O}, $\phi^{\bullet}(1, 100) = 84.3^o$;
{OCBEFJF'E'A'B'C'O}, $\phi^{\bullet}(1, 13) = 73.9^o$;
{OCKC'O} $\phi^{\bullet}(1, 2) = 45^o$.

The *red curves* show the location of conjugate points, the *blue,brown, and orange lines* show the location of points where $\eta(\phi_j^{\bullet}, \psi_j^*) = 0$

Liquid bridges with nonzero curvature

$$H_1 = H_3 = \bar{r}, \quad H_2 = -\left(\bar{r} + 2\bar{r}''\right), \quad \bar{r}'^2 + \bar{z}'^2 = 1,$$

$$w'' - \frac{B\sin\phi}{\bar{r}^2} w' + \left(1 - \frac{2B\cos\phi}{\bar{r}^2} - \frac{2B^2\sin^2\phi}{\bar{r}^4}\right) w = \mu$$

$$\bar{w}_1 = \frac{\sin\phi}{\bar{r}}, \quad \bar{w}_2 = \cos\phi + (1+B)M_1(\phi, m)\bar{w}_1$$

$$M_1(\phi, m) = E\left(\frac{\phi}{2}, m\right) - \left(1 - \frac{m^2}{2}\right) F\left(\frac{\phi}{2}, m\right)$$

• Stability of Cylinder Menisci

$$\bar{r} = 1, \quad \bar{z} = \phi, \quad \bar{w}_1 = \sin\phi, \quad \bar{w}_2 = \cos\phi, \quad \bar{w}_3 = 1,$$

$$\Delta_{Cyl}(\phi_1, \phi_2) = \Delta\phi\ \Gamma_1\left(\frac{\Delta\phi}{2}\right)\sin\Delta\phi, \quad \Delta\phi = \phi_1 - \phi_2,$$

$$P_{11} = P_{22} = \Delta\phi\ \Gamma_1\left(\Delta\phi\right)\cos\Delta\phi, \quad P_{12} = -\Delta\phi\Gamma_2(\Delta\phi),$$

$$\Gamma_1(x) = 1 - \frac{\tan x}{x}, \quad \Gamma_2(x) = 1 - \frac{\sin x}{x}$$

• Cylinder Menisci between two solid plates

$$Q_{11} = Q_{22} = \frac{\Gamma_1\left(\Delta\phi\right)}{\Gamma_1\left(\Delta\phi/2\right)}\cot\Delta\phi, \quad Q_{33} = -\frac{1}{\Gamma_1\left(\Delta\phi/2\right)}$$

$$Q_{12} = -\frac{\Gamma_2\left(\Delta\phi\right)}{\Gamma_1\left(\Delta\phi/2\right)}\csc\Delta\phi$$

(a) fixed endpoints : $\Delta_{Cyl} < 0 \ \Rightarrow \ 0 < \Delta\phi < 2\pi,$

(b) free endpoints : $Q_{33} > 0 \ \Rightarrow \ 0 < \Delta\phi < \pi,$

(c) one endpoint is free and another is fixed :

$$Q_{11} > 0 \ \Rightarrow 0 < \Delta\phi < \varkappa\pi,$$
$$\varkappa = \min\{x_* \mid \tan x_* = x_*, \ x_* > 0\} \simeq 1.4303$$

• Cylinder Menisci between two solid ellipsoids

$$\frac{Q_{jj}}{A^2} = \frac{\epsilon_j \sin\psi_j^* \cos\psi_j^*}{\epsilon_j^2 \sin^2\psi_j^* + \cos^2\psi_j^*} + \frac{P_{jj}\cos^2\psi_j^*}{\Delta_{Cyl}},$$
$$\frac{Q_{12}}{A^2} = \frac{P_{12}\cos\psi_1^* \cos\psi_2^*}{\Delta_{Cyl}}$$

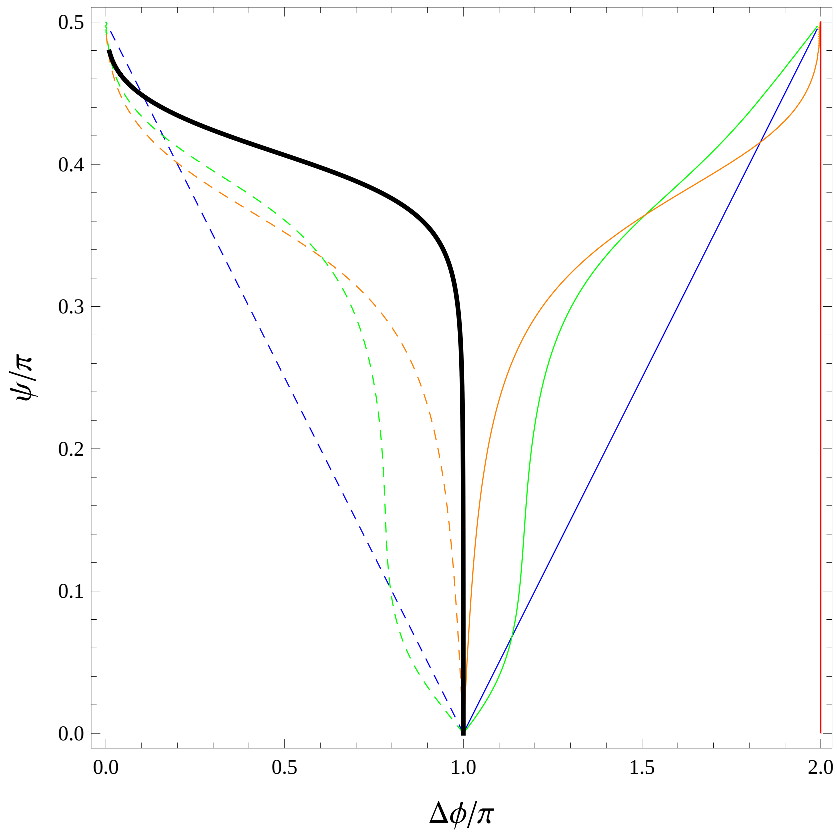
Ellipsoids shapes: exterior (convex, $\epsilon > 0$) and
interior (concave or hollow, $\epsilon < 0$)

Stability criteria

$$\cot\frac{\Delta\phi}{2} + \frac{\epsilon\tan\psi^*}{\epsilon^2\sin^2\psi^* + \cos^2\psi^*} = 0$$

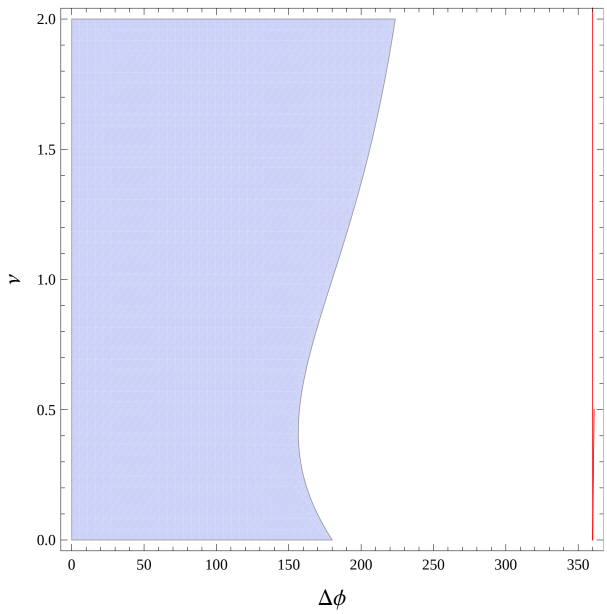
Two equal spheres $\epsilon = \pm 1$

$$\epsilon = 1, \qquad 2\psi^* = -\pi + \Delta\phi, \quad \pi \le \Delta\phi \le 2\pi,$$
$$\epsilon = -1, \quad 2\psi^* = \pi - \Delta\phi, \qquad 0 \le \Delta\phi \le \pi$$

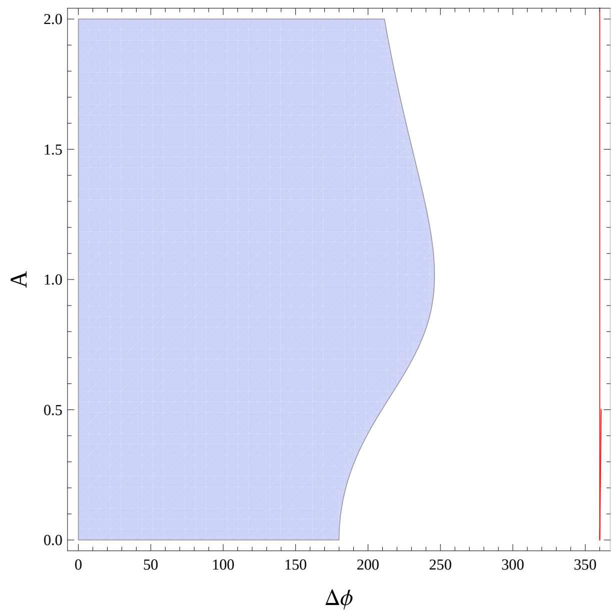


The right boundaries of stability domains $\mathsf{Stab}_2(\psi^*, \Delta\phi)$ for Cyl menisci between two convex (*plain*) and concave (*dashed*) ellipsoids:
blue color $\epsilon_1 = \epsilon_2 = 1,\ \epsilon_1 = \epsilon_2 = (-1)$, *two spheres*;
green color $\epsilon_1 = 3(-3),\ \epsilon_2 = 0.1(-0.1)$,
orange color: $\epsilon_1 = 0.05(-0.05),\ \epsilon_2 = 0.15(-0.15)$.
black color corresponds to Cyl meniscus between convex and concave ellipsoids $(\epsilon_1 = 0.05,\ \epsilon_2 = -0.05)$.
The left boundaries of stability domains $\mathsf{Stab}_2(\psi^*, \Delta\phi)$ coinside with the ψ-axis.
The *red line* $\Delta\phi = 2\pi$ is a right boundary of stability domain $\mathsf{Stab}_1(\psi^*, \Delta\phi)$ for Cyl menisci with fixed endpoints.

• Cyl Menisci between two paraboloids or two catenoids



(1)



(2)

1. **Cyl** between two equal paraboloids, $a, A, C, \nu > 0$

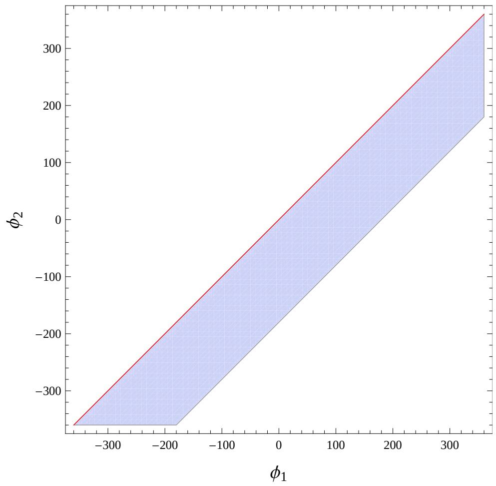
$$\frac{R_j}{aA} = \frac{\psi_j}{a}, \quad \frac{Z_j}{aA} = \frac{g}{aA} + (-1)^{j+1} C \left(\frac{\psi_j}{a}\right)^{\nu}$$

Conic surface $\nu = 1$.

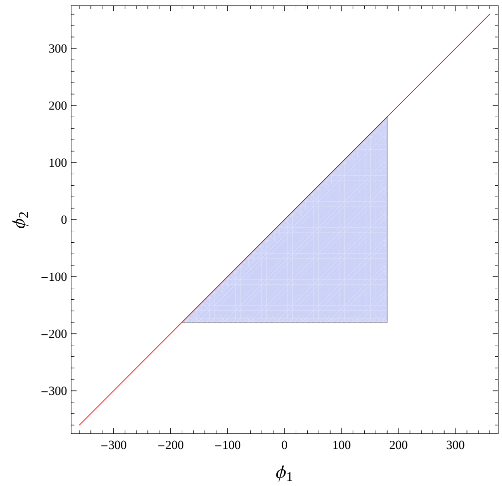
2. **Cyl** between two equal catenoids $A, C, b > 0$

$$\frac{R_j}{A} = \psi_j, \qquad \frac{Z_j}{A} = \frac{g}{A} + (-1)^{j+1} C \cosh(b\psi_j)$$

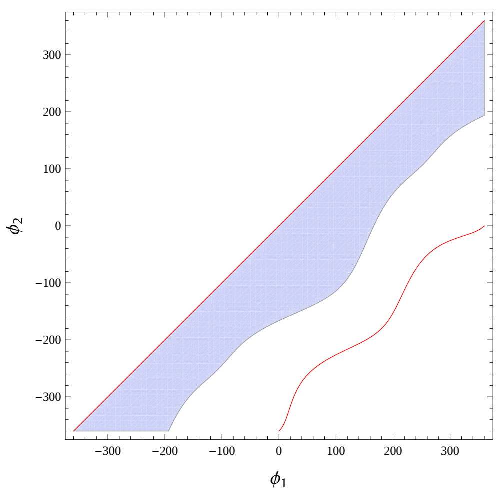
• Unduloid menisci between solid plates



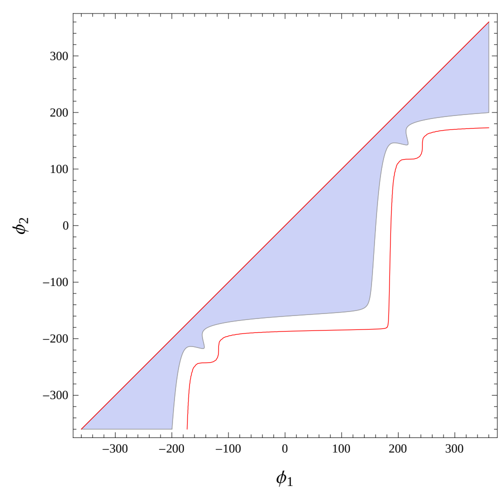
(Cyl, $B = 0$)



(Sph, $B = 1$)



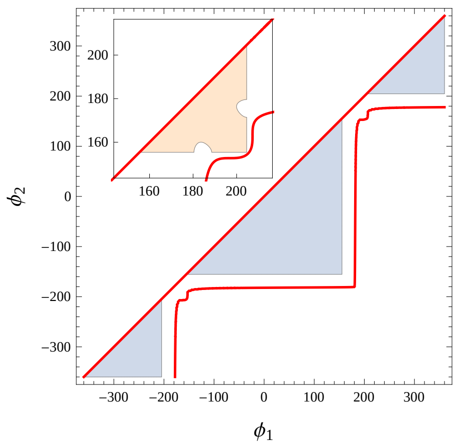
(Und, $B = 0.3$)



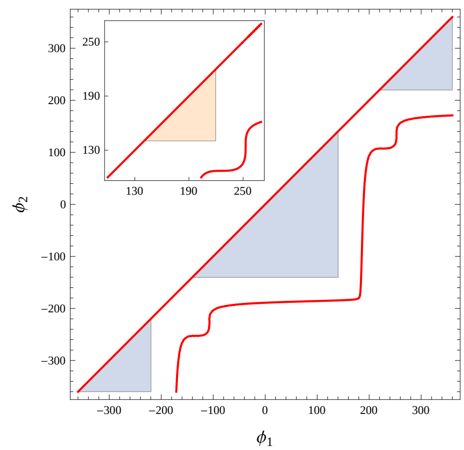
(Und, $B = 0.8$)

The stability domain $\mathsf{Stab}_2(\phi_1, \phi_2)$ for menisci between two plates.
The red curves show the location of conjugate points.

• Nodoid menisci between solid plates

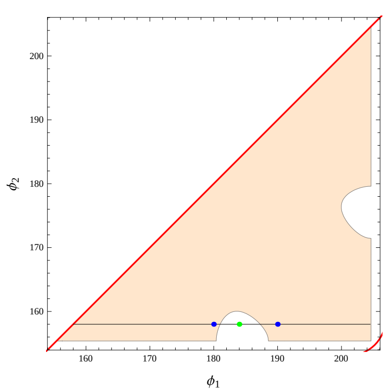


$(B = 1.1)$

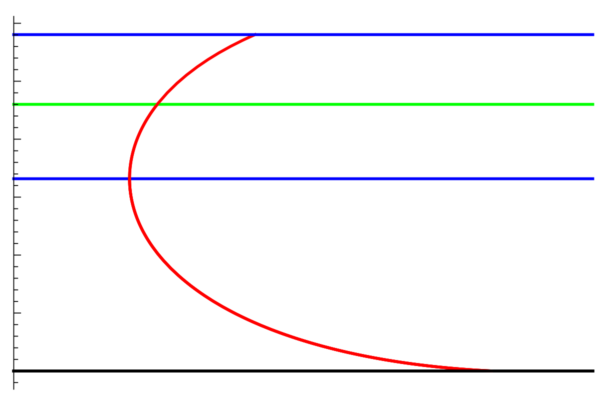


$(B = 1.3)$

The stability domain $\mathsf{Stab}_2(\phi_1, \phi_2)$ for **Nod** menisci. Different types of **Nod** menisci curvature are shown in *violet-blue* $(H > 0)$ and *orange* $(H < 0)$ colors. The orange domain is accompanied with its twin shifted by 360^o in ϕ_1, ϕ_2.



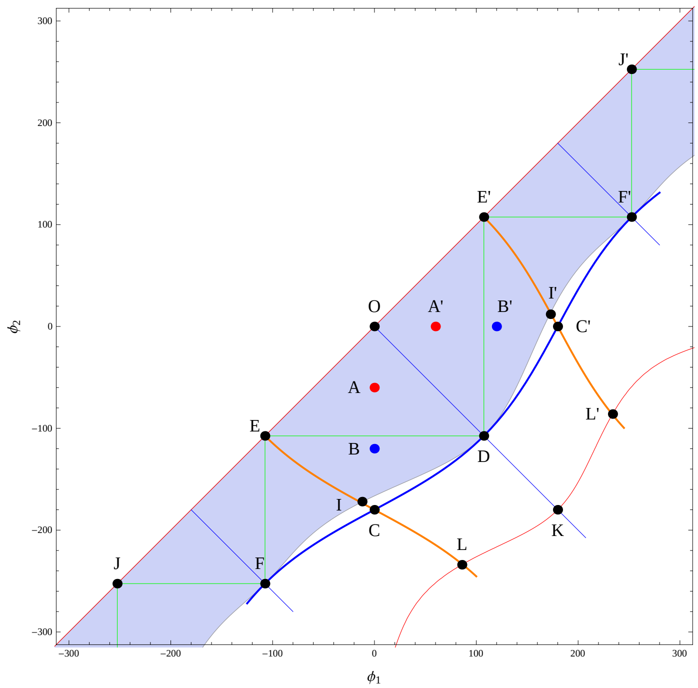
Stability domain



Three Menisci

Appearance-disappearance of stability along **Nod** menisci. At $\phi_1 \in [181.15^o, 187.2^o], \phi_2 = 158^o$ **Nod** is unstable.

Unduloid menisci with inflection point between two plates



The stability domain $\mathrm{Stab}_2(\phi_1, \phi_2)$ for Und menisci with $B = 0.3$. The *green lines* show *IP* separation from a plate.

If $\theta_1 = \theta_2 = \pi/2$ *the* Und *menisci are unstable, (points* C, C'*),*

If $\theta_1 = \theta_2$ *there are no stable menisci with one or more IPs, blue curves,*

$$\tan(\phi_1/2)\tan(\phi_2/2) = -(1+B)(1-B)$$

If $\theta_1, \theta_2 \neq \pi/2$, $\theta_1 + \theta_2 = \pi$ *there are stable menisci of large volume that have IPs, brown curves,*

$$\tan(\phi_1/2)\tan(\phi_2/2) = (1+B)(1-B)$$

M. Athanassenas, J. für Math., v. 377, 97-107 (1987)

T. Vogel, SIAM J. Appl. Math. v. 49, 1009-1028 (1989)

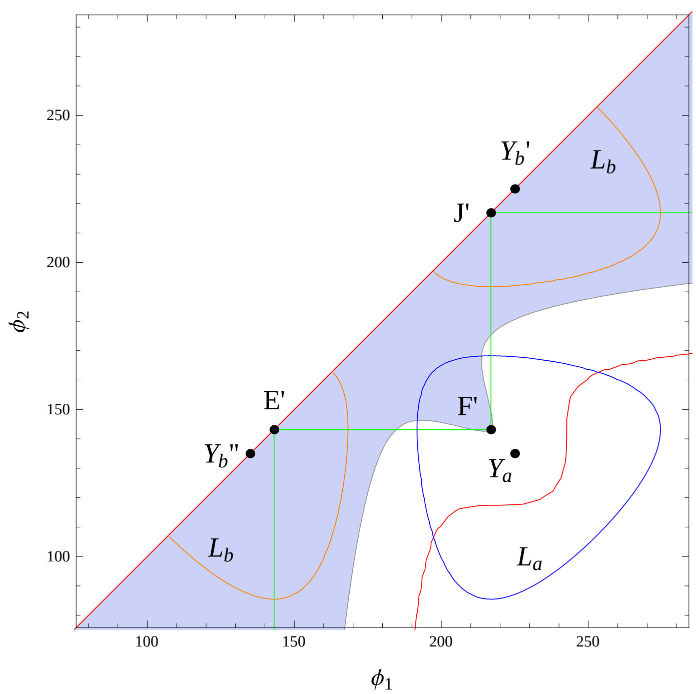
R. Finn, T. Vogel, Z. Anal. Anwend. 11, 3-23 (1992)

If $\theta_1 + \theta_2 = \pi/2$ *there are no stable menisci with one or more IPs. There are stable and unstable menisci without IPs, blue curve* L_a.

$$(1 + B\cos\phi_1)(1 + B\cos\phi_2) = -B^2 \sin\phi_1 \sin\phi_2$$

If $\theta_1 - \theta_2 = \pi/2$ *there are only menisci with one IP or without it. All menisci are stable, orange curves* L_b.

$$(1 + B\cos\phi_1)(1 + B\cos\phi_2) = B^2 \sin\phi_1 \sin\phi_2$$

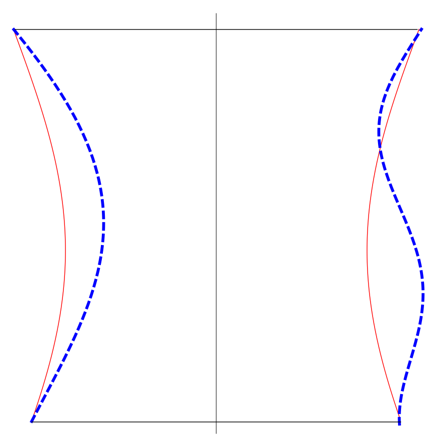


The stability domain $\mathrm{Stab}_2(\phi_1, \phi_2)$ for Und menisci with $B = 0.8$.
The *green lines* show *IP* separation from a plate.
The *red curve* show the location of conjugated points and the boundary of stability domain $\mathrm{Stab}_1(\phi_1, \phi_2)$ for Und menisci with fixed endpoints.

Stability under asymmetric perturbations for menisci between two plates and Schwarz symmetrization

Let an axisymmetric liquid bridge is trapped between two plates. Its surface energy reads

$$E(\Omega) = \mu A(\Sigma) + \mu_1 A(\sigma_1) + \mu_2 A(\sigma_2),$$
$$\partial(\Omega) = \Sigma \cup \sigma_1 \cup \sigma_2, \quad \sigma_j = \Omega \cap S_j, \quad \gamma_j = \partial(\sigma_j)$$



Lemma 2.1

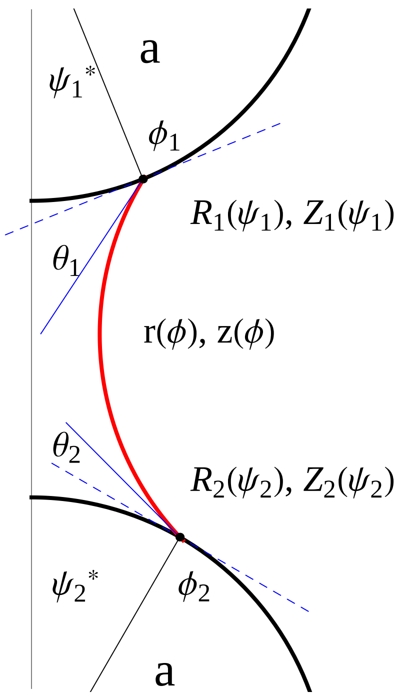
Suppose Ω is a set of axisymmetric menisci and suppose $E(\Omega_0) < E(\Omega')$ for all axisymmetric volume-conserving perturbations Ω' of $\Omega_0 \in \Omega$ with $\partial(\Omega')$ uniformly close to $\partial(\Omega_0)$.

Then $E(\Omega_0) \leq E(\Omega'')$ if Ω'' is an asymmetric volume conserving perturbation of Ω_0 with $\partial(\Omega'')$ uniformly close to $\partial(\Omega_0)$.

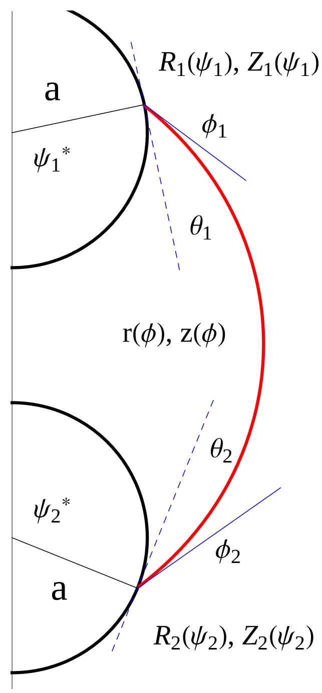
T. Vogel, SIAM J. Appl. Math. v. 47, 516-525 (1987)

Unduloid and nodoid menisci between solid balls

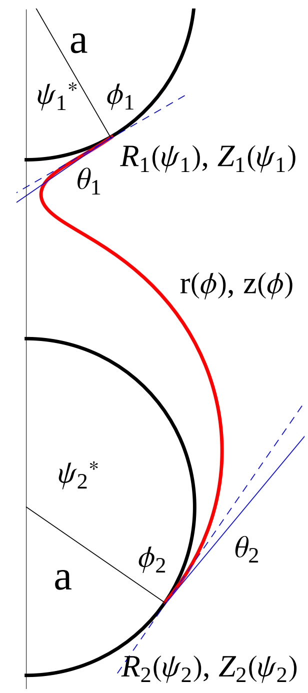
Three setups: face-to-face (F-F), back-to-back (B-B), face-to-back (F-B)



(F-F)



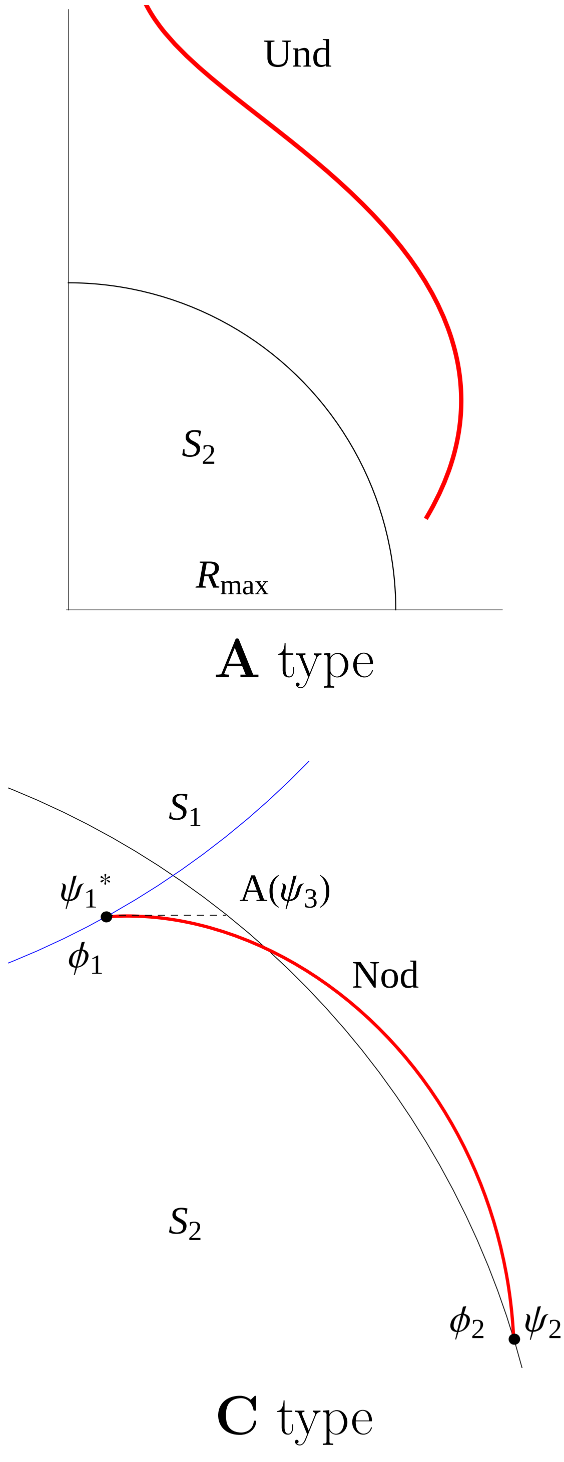
(B-B)



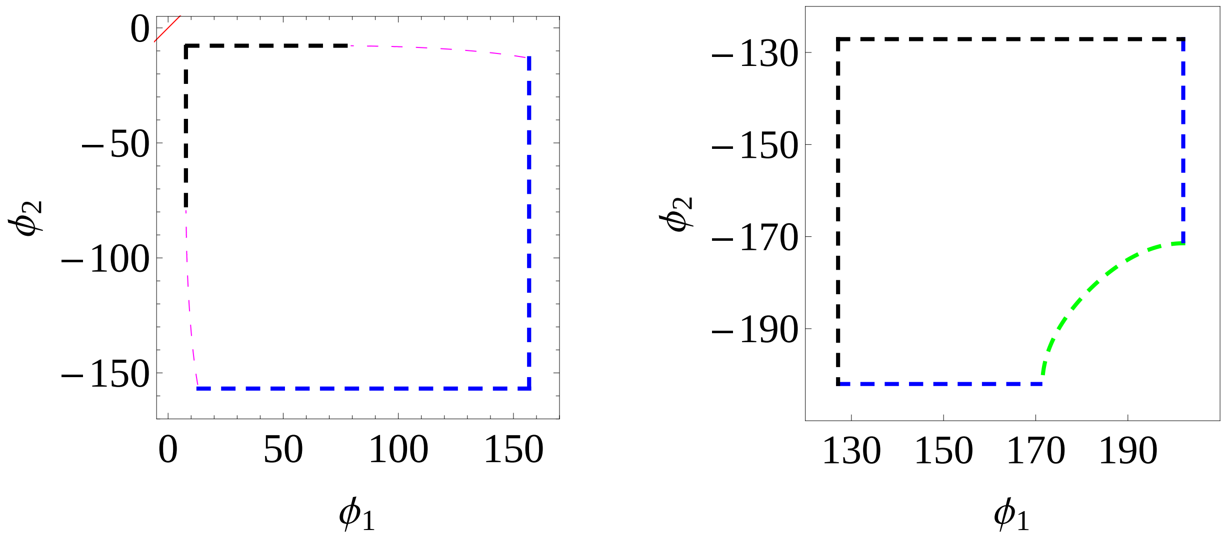
(F-B)

• Existence conditions

1. Type **A**: meniscus does not reach solid surface,
2. Type **B**: meniscus reaches solid surface with negative contact angle,
3. Type **C**: meniscus reaches S_1 at the endpoint which is immersed into the other S_2,
4. Type **D**: the center of S_2 is above the center of S_1.


Und
S_2
$R_{\max}$
A type
ϕ_1
S_1
Und
B type
S_1
$\psi_1{}^*$
$A(\psi_3)$
ϕ_1
Nod
S_2
ϕ_2
ψ_2
C type
ϕ_1
S_1
O_2
Nod
O_1
S_2
ϕ_2
D type

Coexistence of the different types constraints

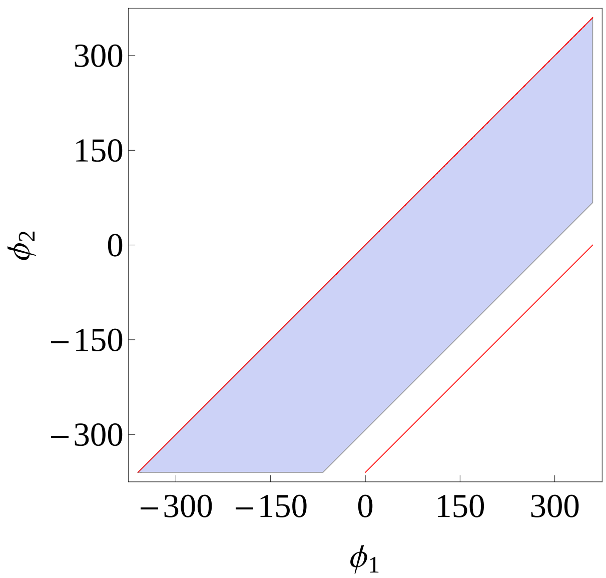


$\mathbf{A}$ (*black*), $\mathbf{B}$ (*blue*), $\mathbf{C}$ (*magenta*), $\mathbf{D}$ (*green*) types constraints for Nod between two equal spheres:

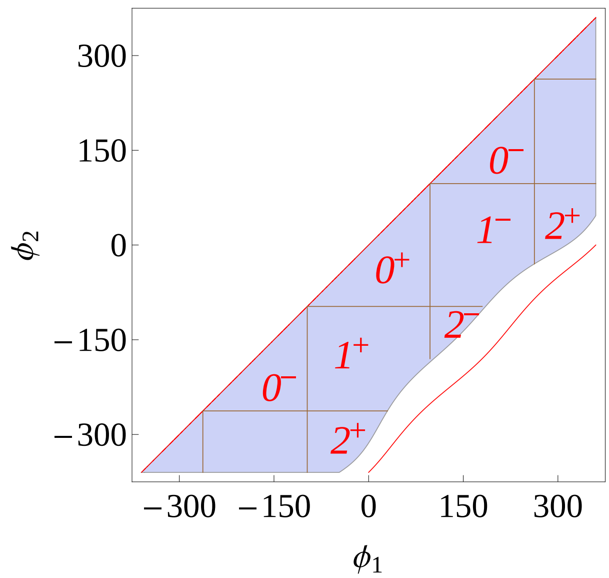
(a) F-F setup, $B = 1.205, a = 2.2$;

(b) B-B setup, $B = 1.5, a = 1.2$.

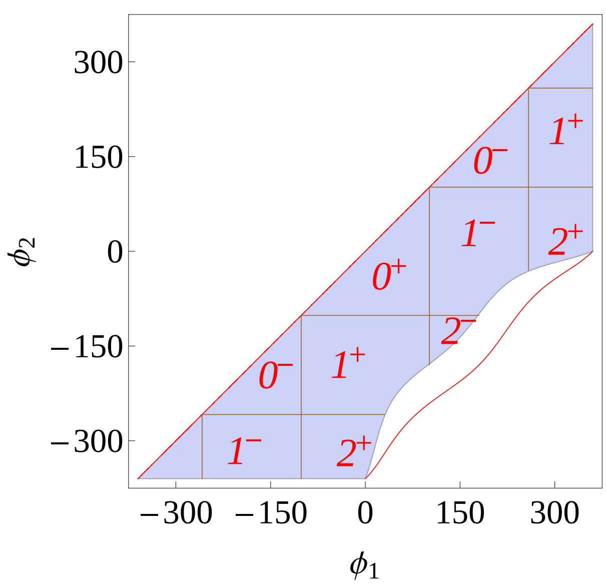
• Stability diagrams for F-F setup



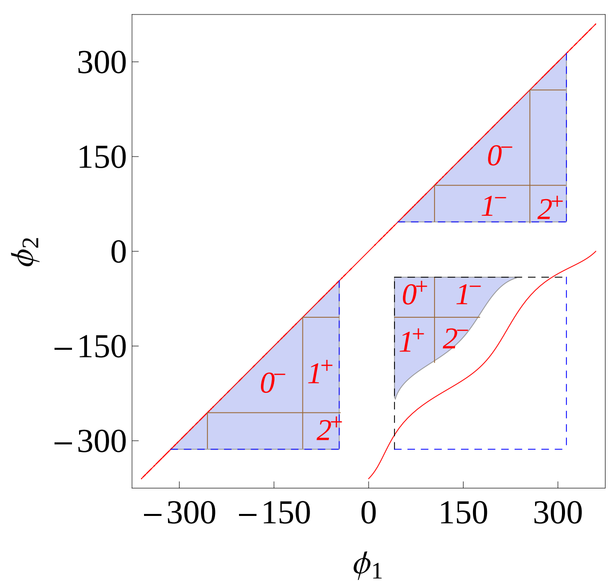
$(\text{Cyl},\ B = 0)$



$(\text{Und},\ B = 0.15)$

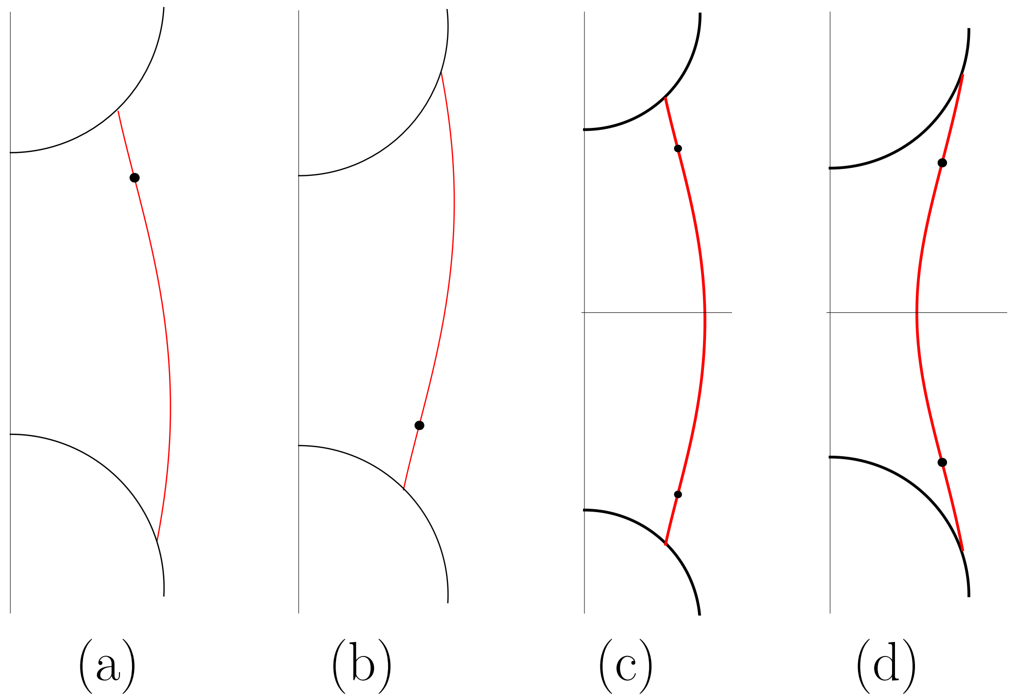


$(\text{Und},\ B = 0.2)$



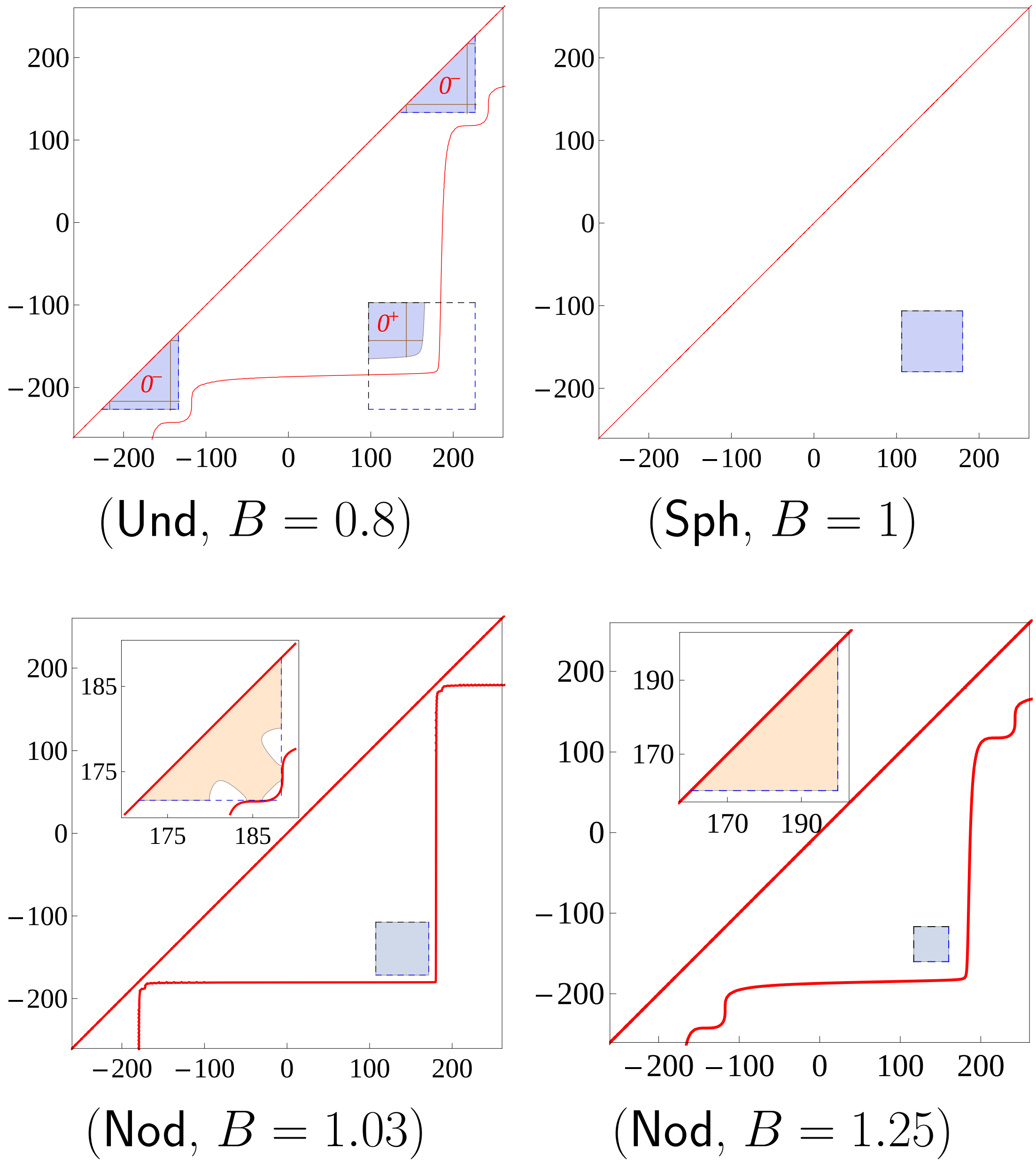
$(\text{Und},\ B = 0.25)$

Stable menisci with two inflection points do exist

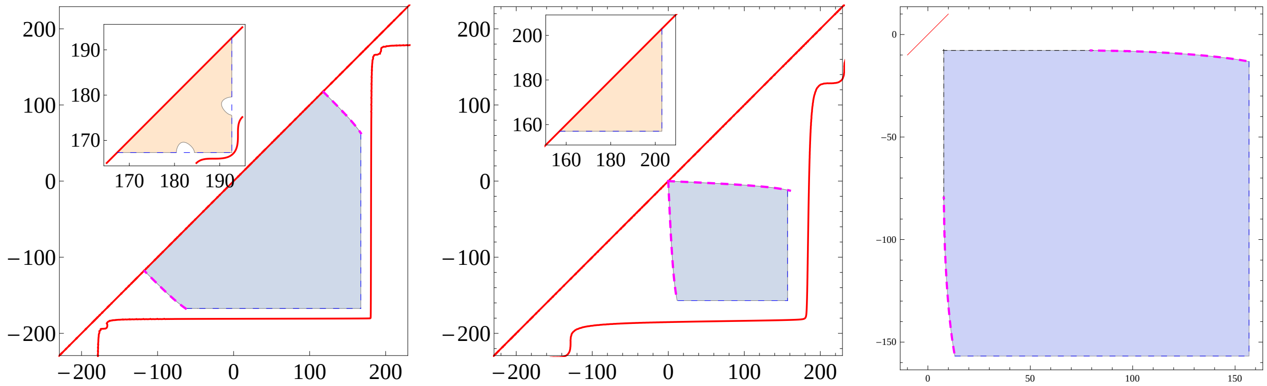


Stable Und menisci $B = 0.25$ with one and two IPs (*black points*) for F-F setup between two solid spheres, $a = 1.2$, and endpoints:

$$
\begin{array}{lll}
(a) & \mathsf{Und}_1^- & \phi_2 = -60^o, \; \phi_1 = +135^o, \\
(b) & \mathsf{Und}_1^+ & \phi_2 = +60^o, \; \phi_1 = -135^o, \\
(c) & \mathsf{Und}_2^- & \phi_2 = -135^o, \; \phi_1 = +135^o, \\
(d) & \mathsf{Und}_2^+ & \phi_2 = +60^o, \; \phi_1 = +300^o.
\end{array}
$$



Stability diagrams for F-F setup of menisci between two solid spheres of radius $a = 1.2$.



$(B = 1.05)$ $(B = 1.2)$ $(B = 1.205)$

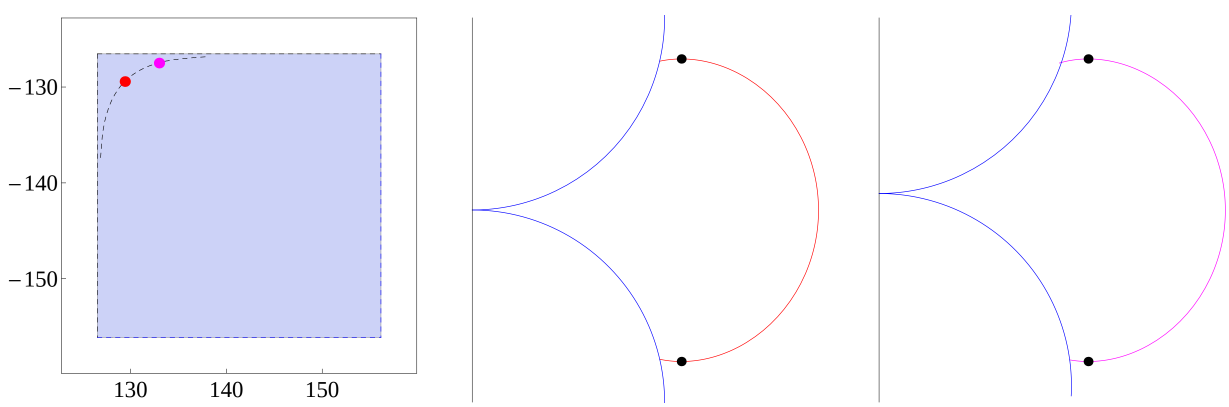
Stability diagrams for F-F setup of **Nod** menisci between two solid spheres of radius $a = 2.2$.

• Menisci between two equal contacting spheres

T. Vogel, J. Math. Fluid Mech., v. 16, 737-744 (2014)

Note. 3.5. *Do not exist stable (convex)* Nod *menisci between contacting balls with: a) rotation symmetry, b) $\Theta \geq \pi/2$, c) symmetry across the perpendicular bisector of the line segment between the balls centers.*

Open question: whether the last condition may be dropped.



(a) (b) (c)

A stable domain for Nod menisci ($B = 2.15$) with F-F setup between two touching spheres of radius $a = 1.75$ with (b) and without (c) reflection symmetry between the centers of the balls: (b) $\phi_1 = -\phi_2 \simeq 129.15^o$, $\psi \simeq 76^o$ and $\theta \simeq 115^o$; (c) $\phi_1 \simeq 133^o$, $\phi_2 \simeq -127.5^o$.

Red (a) and magenta (b) points stand for stable menisci.

THANK YOU FOR ATTENTION

GRACIAS POR SU ATENCIÓN