# Interfaces determined by capillarity and gravity in a two-dimensional porous medium


by

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# Introduction

**Motivation:**

- Haines jumps (Furuberg-Maloy-Feder 1992& 1996)
- Modelling of two-phase Porous Media Flow (Jäger, Schweiser, ... )

**Model:**

- Squared container partially filled with water
- Pores are formed by space between uniformly distributed grains
- The interface is the union of single interfaces that meet two grains
- Energy = Surface energy + potential energy
- Existence of a large number of equilibria

**Aims:**

- Description of minimizers
- Evolution: Transitions between minimizers

# Setting

**The medium:**

- **Square container of size** $L \times L$
- $N$ **grains of radius** $R > 0$
- **Average distance** $d > 0$
- **Uniformly distributed grains:**

$$N = \nu \frac{L^2}{d^2}, \quad \nu = \text{'density of grains'}$$

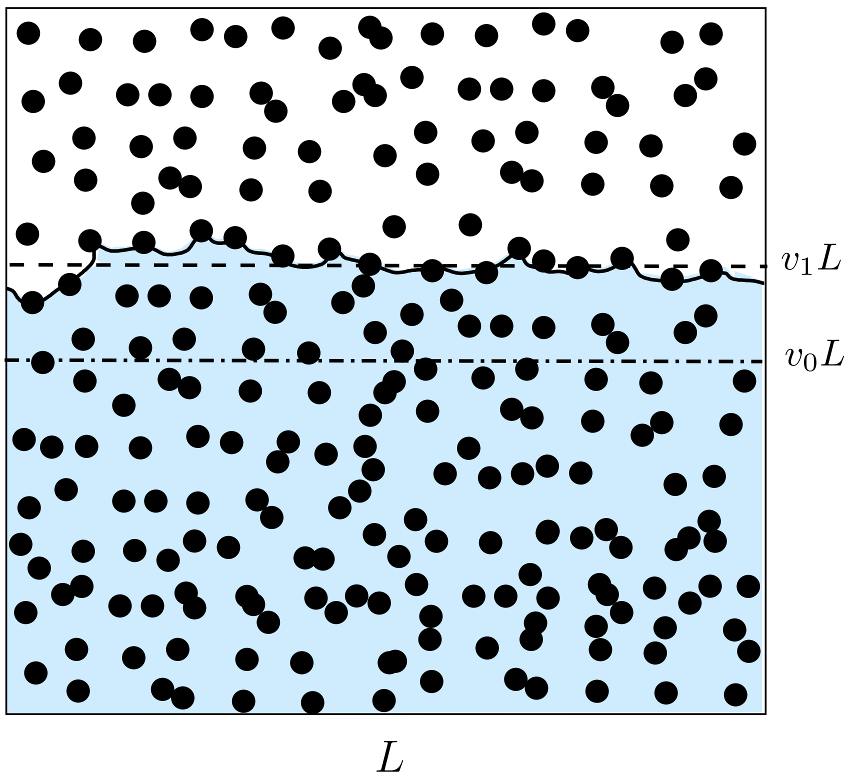
**The liquid phase:**

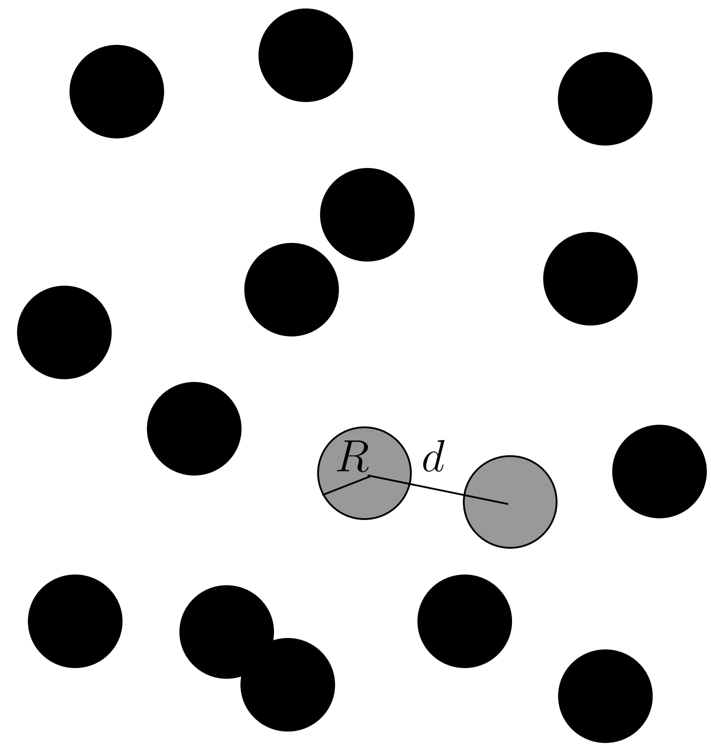
- $v_0 =$ **volume fraction of water**
- $v_1 =$ **volume fraction of water** $+$ **grains:**

$$v_1 L^2 \approx v_0 L^2 + n\pi R^2, \qquad n = \text{number of grains under the interface}$$

# Setting

**Sketch of the model:**



# The interface

$\Gamma$ **union of** *elemental components*

$$\Gamma = \cup_{i \in J} \{\gamma_i\}, \qquad \gamma_i \text{ curve joining two grains}$$

*Indexing follows orientation*

**An elemental component satisfies**

$$\sigma H = \rho\, g x_2 - p$$

$+$ **contact angle condition (Young's law)**

$\sigma$ **surface tension**

$H$ **signed curvature**

$p = \rho g v_1 L$**, hydrostatic pressure**

$g$ **gravity**

# Parameter regimes

**Dimensionless setting:** **We take $d = 1$ with $R$ and $L$ dimensionless parameters**

$$H = B(x_2 - \lambda) \qquad \lambda = v_1 L\,, \qquad B = \frac{\rho g d^2}{\sigma}$$ **(Bond number)**

**Asymptotic Regimes:**

- **Cases with $B \gg 1$:**

  $\gamma_i$**'s horizontal** $+ O(1/\sqrt{B})$ **Boundary Layer**

- **Cases with $B \ll 1$:**

  $\gamma_i$**'s can connect in many directions**

**Subregimes of case $B \gg 1$:** $L \to \infty \;\Rightarrow\; N \to \infty$

**For simplicity:** $R \ll 1$**,** $\max\{1/\sqrt{B}, R\} = 1/\sqrt{B}$

**Parameter relating (dimensionless) $L$ and $B$:**

$$\theta := \frac{L}{\sqrt{B}}$$

# Gluing of solutions

**Global capillary solutions satisfy**

$$B\frac{(x_2(s)-\lambda)^2}{2}+\cos\beta(s)=C$$

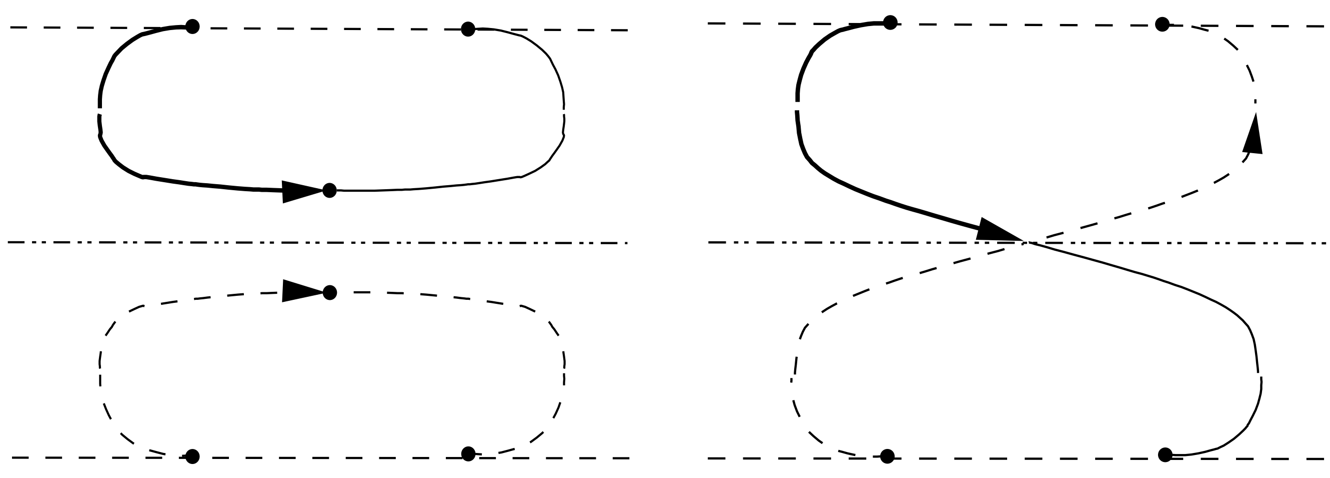
$$x_1'(s)=\cos\beta(s)\,,\quad x_2'(s)=\sin\beta(s)\,,\quad s=\textbf{arc-length}$$

**Can be solved by elliptic integrals**

*(Myshkis 1987)*

**Complete curves are obtained by gluing of graphs**

*(details in Calle, C and Velázquez 2015, arXiv:1505.03676)*



(a) $C>1$ (b) $0<C<1$

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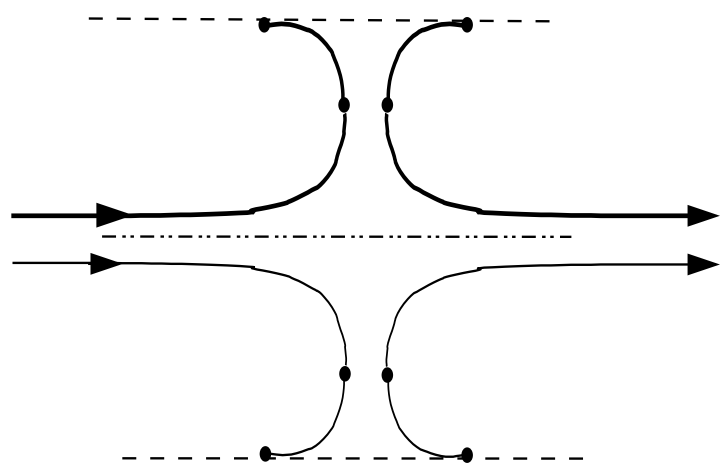

(c) $C=1$

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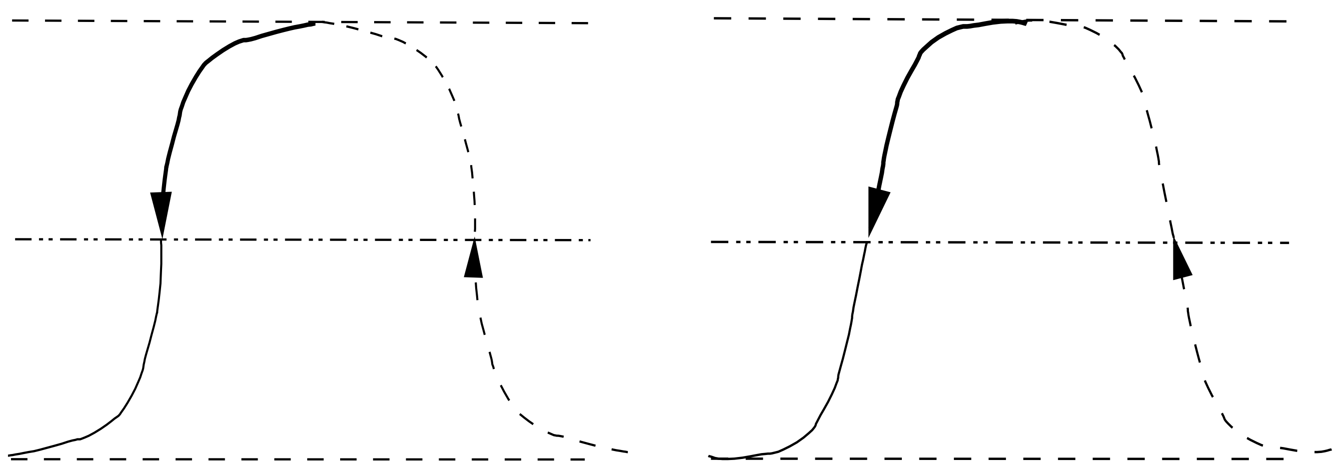


(d) $C=0$ (e) $-1<C<0$

# Probabilistic Setting

**Centers notation:**

$$\xi_k = (\xi_k^{(1)}, \xi_k^{(2)}) \in \mathbb{R}^2\,, \quad k \in \{1, \dots, \nu L^2\}$$

**Centers uniformly and independently distributed:**

$$\Omega_\nu(L) = \text{all configurations of } \nu L^2 \text{ centers in } [0, L]^2$$

**Probability of finding $m$ centers in $V \subset [0, L]^2$:**

$$P(V) = \left( \frac{1}{L^2} \int_V d\xi \right)^m$$

**Probability measure to $\Omega_\nu(L)$:**

$$\mu_\nu(d\xi) = \frac{1}{L^{2N}} \prod_{k=1}^{\nu L^2} d\xi_k$$

**Interface solution depends on configuration:**

$$\Gamma(\omega, \lambda, \theta)\,,\ \omega \in \Omega_\nu(L)$$

# Basic Tools

- **Estimates on $\Gamma$ outside a strip:**

  **Proposition** For every $L$, $\omega \in \Omega_\nu(L)$ and $v_0$, there exists $D_0 > 0$, such that if

$$\Gamma(\omega, \lambda, \theta) \cap \left\{ |\, x_2 - v_1 L| \, \geq \frac{D}{\sqrt{B}} \right\} \neq \varnothing \text{ for } D \geq D_0,$$

  then there exists a constant $K > 0$ such that every pair of centers of grains joined by $\Gamma$, $\xi_i$ and $\xi_l$, and contained in $\left\{ |\, x_2 - v_1 L| \, \geq \frac{D}{\sqrt{B}} \right\}$, satisfy

$$\|\xi_i - \xi_l\| \leq \frac{K}{D\sqrt{B}}$$

- **Stirling estimates:**

$$\binom{n}{m} \leq C(n) e^{W(n,m)}$$

# The regime $B \gg 1$ and $1 \lesssim L \ll \sqrt{B}$ or $\theta \ll 1$

**Theorem** For any $0 < \varepsilon < 1$, there exists $\theta_\varepsilon$ such that for all $\theta \leq \theta_\varepsilon$ there exists $\Omega_\varepsilon \subset \Omega_\nu(L)$ with $\mu_\nu(\Omega_\varepsilon) \geq 1 - \varepsilon$ and such that for any $\omega \in \Omega_\varepsilon$, the solutions $\Gamma(\omega, \lambda, \theta)$ satisfy

$$\Gamma(\omega, \lambda, \theta) \equiv \{x_2 - v_1 L = h(\omega)\}$$

with

$$|h(\omega)| \leq \frac{\tilde{K}}{\sqrt{B}}$$

**Proof:**

- Let $\Omega_0 = \{\omega \in \Omega_\nu(L) : \{\xi_k\} \cap \{|x_2 - v_1 L| < \frac{\tilde{K}}{\sqrt{B}}\} = \emptyset\}$ (configurations with no grains in the strip). Then

$$\mu_\nu(\Omega_0) = \left(1 - \frac{\tilde{K}}{L\sqrt{B}}\right)^{\nu L^2} \to 1 \quad \text{as} \quad \theta \to 0$$

- If $\omega \in \Omega_0$ such that $\Gamma(\omega, \lambda, \theta)$ is in $\{|x_2 - v_1 L| > \frac{\tilde{K}}{\sqrt{B}}\}$, then $\#\Gamma \geq \sqrt{B} L \ll L^2$, contradiction.

# The regime $B \gg 1$ and $L = O(\sqrt{B})$ or $\theta = O(1)$

**Consider**

$$U_k = \{\omega \in \Omega_\nu(L) : \ \exists \Gamma(\omega, \lambda, \theta) \ \& \ k = \#\{\xi_j \in \omega : \ \Gamma(\omega, \lambda, \theta) \cap B_R(\xi_j) \neq \emptyset\}\}$$

**(Configurations with interfaces that connect exactly $k$ grains).**

**Theorem** **There exists $N_0 \leq 2D_0\theta\nu < \nu L^2$ and $L$ sufficiently large such that, if $k \leq N_0$,**

$$\mu_\nu(U_k) \lesssim e^{-2D_0\theta\nu} \sum_{i=k}^{N_0} \frac{1}{i!} (2D_0\theta\nu)^i \, .$$

**Moreover, for all $\varepsilon > 0$, there exists a $L_\varepsilon > 0$ such that for all $L \geq L_\varepsilon$ there exists $\Omega_\varepsilon \subset \Omega_\nu(L)$ such that $\mu_\nu(\Omega_\varepsilon) \geq 1 - \varepsilon$, and that for all $\omega \in \Omega_\varepsilon$ the solutions $\Gamma(\omega, \lambda, \theta)$ satisfy $|C(\Gamma) - 1| \leq \varepsilon$.**

# The regime $B \gg 1$ and $L = O(\sqrt{B})$ or $\theta = O(1)$

**Idea of the Proof:** **Write**

$$U_k = (U_k \cap \Omega_s) \cup (U_k \cap \Omega_s^c \cap U_{P \geq N_0}) \cup (U_k \cap \Omega_s^c \cap U_{P < N_0}) \,,$$

**where**

- $\Omega_s \subset \Omega_\nu(L)$ **such that** $\exists \Gamma(\omega, \lambda, \theta)$ **for** $\omega \in \Omega_s$ **such that**

$$\Gamma(\omega, \lambda, \theta) \cap \{|x_2 - v_1 L| > D_0/\sqrt{B}\} \neq \emptyset$$

- $U_{P \geq N_0}$ **set of configurations such that** $\#\{\xi_j \in \{|x_2 - v_1 L| < D_0/\sqrt{B}\}\} \geq N_0$ **with** $N_0 = 2\nu D_0 \theta = O(1)$ **(maximum number of expected centers)**
- **We also need** $\omega_k$ **the configurations with exactly** $k$ **grains in** $\{|x_2 - v_1 L| < D_0/\sqrt{B}\}$**.**

# The regime $B \gg 1$ and $L = O(\sqrt{B})$ or $\theta = O(1)$

**Idea of the Proof:** **Write**

$$U_k = (U_k \cap \Omega_s) \cup (U_k \cap \Omega_s^c \cap U_{P \geq N_0}) \cup (U_k \cap \Omega_s^c \cap U_{P < N_0}),$$

**then**

- **Counting argument:** $\mu_\nu(\Omega_s) \sim 1/L^2 \to 0$ **as** $L \to \infty$**.**
- $\mu(\omega_k) = \binom{\nu L^2}{k} \left(\frac{2D_0\theta}{L^2}\right)^k \left(1 - \frac{2D_0\theta}{L^2}\right)^{\nu L^2 - k}$**.**
- **That gives:**

$$\mu_\nu(U_{P \geq N_0}) = \sum_{j=N_0}^{\nu L^2} \binom{\nu L^2}{j} \left(\frac{2D_0\theta}{L^2}\right)^j \left(1 - \frac{2D_0\theta}{L^2}\right)^{\nu L^2 - j} \to 0 \quad \text{as} \quad L \to \infty$$

**Hence:**

$$\mu_\nu(U_k) \lesssim \sum_{j=k}^{N_0} \binom{\nu L^2}{j} \left(\frac{2D_0\theta}{L^2}\right)^j \left(1 - \frac{2D_0\theta}{L^2}\right)^{\nu L^2 - j}$$

# The regime $B \gg 1$ and $L \gg \sqrt{B}$

**Theorem** **Let $0 < h = O(1)$ and $\Omega_h \subset \Omega_\nu(L)$ denote the set of all configurations $\omega$ such that there exists a $\Gamma(\omega, \lambda, \theta)$ satisfying $\max |\xi_j^{(2)} - v_1 L| \geq h$. Then,**

$$\mu_\nu(\Omega_h) \to 0 \quad \textbf{as} \quad B \to \infty$$

**if $\sqrt{B} \ll L \ll \sqrt{B} \log B$.**

**Idea of the Proof:**

- **Characterise $\Omega_h$: In particular $\Omega_h \subset U_h(K, D_0, \theta, L)$ where $U_h(K, D_0, \theta, L)$ have configurations of grains with the right geometric properties to allow interfaces as in $\Omega_h$**
- **Distinguish $U^g \in U_h(K, D_0, \theta, L)$ with all such grains outside $\{|x_2 - v_1 L| < D_0/\sqrt{B}\}$ and those with grains in it**

# The regime $B \gg 1$ and $L \gg \sqrt{B}$ (contd.)

**Idea of the Proof (contd.)**

1. **First case: counting argument shows** $\sum \mu_\nu(U^g) \to 0$ **as** $B \to \infty$ **if** $L^2 \gg B$**.**
2. **Second case: geometric argument shows** $\sum \mu_\nu(U^g) \to 0$ **as** $B \to \infty$ **if** $L \ll \sqrt{B} \log B$**.**

**Second case main ingredients:**

- **Identification of grains that could form the interface intersecting the strip** $\{|x_2 - v_1 L| < D_0/\sqrt{B}\}$**.**
- **Estimation on measure of the longest possible interface.**
- **Technical lemma on iteration of measure properties.**

# The regime $L = O(1/B)$ as $B \to 0$

**Observation:** **In the limit $B \to 0$ only $LB = O(1)$ makes sense:**

- **If $L \ll 1/B$ then $H \sim 0$ (only horizontal solutions).**
- **If $L \gg 1/B$ then $H \sim \infty$ (no connections possible).**

**Setting:** **Given $v_0$, restrict domain to:**

$$(0, L) \times ((v_1 - \varepsilon_0)L, (v_1 + \varepsilon_0)L)\ , \quad 0 < \varepsilon_0 < \min\{v_1, 1 - v_1, LB\}$$

**Theorem** **Assume $BL = O(1)$. Then, given a *compatible* curve $\Lambda \in C^1([0,1])$, there exist $\nu_0 > 0$ and $L_0 > 0$ such that for all $\nu \geq \nu_0$ and $L \geq L_0$ there exist $\mathcal{U} \in \Omega_\nu(L)$ and $B \ll \varepsilon \ll 1$, $\varepsilon \to 0$ as $L \to \infty$ such that**

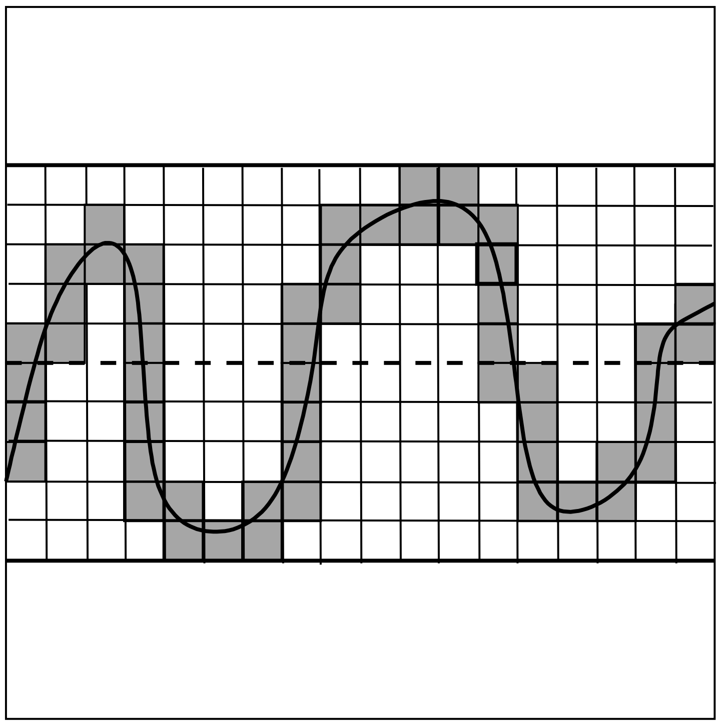
$$\mu_\nu(\mathcal{U}) \geq \delta_\varepsilon \quad \text{with} \quad 0 < \delta_\varepsilon \longrightarrow 1^- \quad \text{as} \quad L \to \infty\ ,$$

**and that for any $\omega \in \mathcal{U}$, there exists $\Gamma(\omega, \lambda, \theta)$ such that**

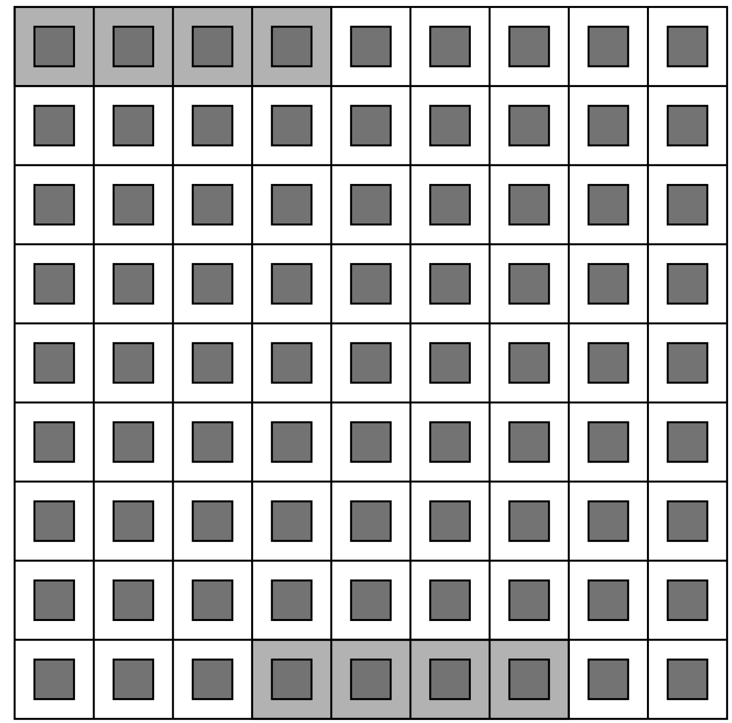
$$\tilde{\Gamma}(\omega, \lambda, \theta) \subseteq T_{\sqrt{2}\varepsilon}(\Lambda)\ ,\ \Lambda \subseteq T_{\sqrt{2}\varepsilon}(\tilde{\Gamma}(\omega, \lambda, \theta))\ .$$

# The regime $L = O(1/B)$ as $B \to 0$ (contd.)

**Sketch of the proof:**



(f) Domain divided into squares $Q$ of size $(\varepsilon L)^2$



(g) A square $Q$ divided into squares $S_\kappa$ of $O(1)$ size

# The regime $L = O(1/B)$ as $B \to 0$ (contd.)

**Sketch of the proof (contd.):**

**Solve a percolation problem in each $Q$:**

- **A *site* $S_\kappa$ is "open" if contains at least one grain, and "close" otherwise:**
- $$P(S_\kappa \text{ closed}) = (1 - |S_\kappa|/L^2)^{\nu L^2} \to e^{-\nu} \quad \text{as} \quad L \to \infty$$
- **Site Percolation: $\exists \nu_0$ such that**
$$P(Q \text{ closed}) \to 0 \quad \text{as} \quad L \to \infty \quad \text{if} \quad \nu \geq \nu_0.$$
- $P(\cap Q\text{'s connected}) = 1 - P(\cup Q \text{ closed}) \geq 1 - \sum P(Q \text{ closed})$.

# THANK YOU!