Interfaces determined by capillarity and gravity in a two-dimensional porous medium

by

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**Introduction**

**Motivation:**
- Modelling of two-phase Porous Media Flow *(Jäger, Schweiser, ... )*

**Model:**
- Squared container partially filled with water
- Pores are formed by space between uniformly distributed grains
- The interface is the union of single interfaces that meet two grains
- Energy = Surface energy + potential energy
- Existence of a large number of equilibria

**Aims:**
- Description of minimizers
- Evolution: Transitions between minimizers
Setting

The medium:

- Square container of size $L \times L$
- $N$ grains of radius $R > 0$
- Average distance $d > 0$
- Uniformly distributed grains:

$$N = \nu \frac{L^2}{d^2}, \quad \nu = \text{'density of grains'}$$

The liquid phase:

- $v_0 = \text{volume fraction of water}$
- $v_1 = \text{volume fraction of water + grains:}$

$$v_1 L^2 \approx v_0 L^2 + n\pi R^2, \quad n = \text{number of grains under the interface}$$
Setting

Sketch of the model:

\[ L \]

\[ v_1 L \]

\[ v_0 L \]
The interface

\[ \Gamma \text{ union of } \text{elemental components} \]

\[ \Gamma = \bigcup_{i \in J} \{ \gamma_i \}, \quad \gamma_i \text{ curve joining two grains} \]

Indexing follows orientation

An elemental component satisfies

\[ \sigma H = \rho g x_2 - p \]

+ contact angle condition (Young’s law)

\( \sigma \) surface tension
\( H \) signed curvature
\( p = \rho g v_1 L \), hydrostatic pressure
\( g \) gravity
Parameter regimes

Dimensionless setting: We take $d = 1$ with $R$ and $L$ dimensionless parameters

$$H = B(x_2 - \lambda) \quad \lambda = v_1 L, \quad B = \frac{\rho gd^2}{\sigma} \quad \text{(Bond number)}$$

Asymptotic Regimes:

- Cases with $B \gg 1$:
  $\gamma_i$’s horizontal $+ O(1/\sqrt{B})$ Boundary Layer

- Cases with $B \ll 1$:
  $\gamma_i$’s can connect in many directions

Subregimes of case $B \gg 1$: $L \to \infty \Rightarrow N \to \infty$

For simplicity: $R \ll 1, \max\{1/\sqrt{B}, R\} = 1/\sqrt{B}$

Parameter relating (dimensionless) $L$ and $B$:

$$\theta := \frac{L}{\sqrt{B}}$$
Global capillary solutions satisfy

\[ B \frac{(x_2(s) - \lambda)^2}{2} + \cos \beta(s) = C \]

\[ x_1'(s) = \cos \beta(s), \quad x_2'(s) = \sin \beta(s), \quad s = \text{arc-length} \]

Can be solved by elliptic integrals

(Myshkis 1987)

Complete curves are obtained by gluing of graphs

Gluing of solutions

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Centers notation:

\[ \xi_k = (\xi_k^{(1)}, \xi_k^{(2)}) \in \mathbb{R}^2, \quad k \in \{1, \ldots, \nu L^2\} \]

Centers uniformly and independently distributed:

\[ \Omega_\nu(L) = \text{all configurations of } \nu L^2 \text{ centers in } [0, L]^2 \]

Probability of finding \( m \) centers in \( V \subset [0, L]^2 \):

\[ P(V) = \left( \frac{1}{L^2} \int_V d\xi \right)^m \]

Probability measure to \( \Omega_\nu(L) \):

\[ \mu_\nu(d\xi) = \frac{1}{L^{2N}} \nu L^2 \prod_{k=1}^{\nu L^2} d\xi_k \]

Interface solution depends on configuration:

\[ \Gamma(\omega, \lambda, \theta), \quad \omega \in \Omega_\nu(L) \]
Basic Tools

 Estimates on $\Gamma$ outside a strip:

**Proposition** For every $L, \omega \in \Omega_\nu(L)$ and $v_0$, there exists $D_0 > 0$, such that if

$$\Gamma(\omega, \lambda, \theta) \cap \left\{\left| x_2 - v_1 L \right| \geq \frac{D}{\sqrt{B}} \right\} \neq \emptyset \text{ for } D \geq D_0,$$

then there exists a constant $K > 0$ such that every pair of centers of grains joined by $\Gamma$, $\xi_i$ and $\xi_l$, and contained in $\left\{\left| x_2 - v_1 L \right| \geq \frac{D}{\sqrt{B}} \right\}$, satisfy

$$\|\xi_i - \xi_l\| \leq \frac{K}{D \sqrt{B}}.$$ 

Stirling estimates:

$$\binom{n}{m} \leq C(n)e^{W(n,m)}$$
The regime $B \gg 1$ and $1 \lesssim L \ll \sqrt{B}$ or $\theta \ll 1$

**Theorem** For any $0 < \varepsilon < 1$, there exists $\theta_\varepsilon$ such that for all $\theta \leq \theta_\varepsilon$ there exists $\Omega_\varepsilon \subset \Omega_\nu(L)$ with $\mu_\nu(\Omega_\varepsilon) \geq 1 - \varepsilon$ and such that for any $\omega \in \Omega_\varepsilon$, the solutions $\Gamma(\omega, \lambda, \theta)$ satisfy

$$\Gamma(\omega, \lambda, \theta) \equiv \{x_2 - v_1 L = h(\omega)\}$$

with

$$|h(\omega)| \leq \frac{\tilde{K}}{\sqrt{B}}$$

**Proof:**

- Let $\Omega_0 = \{\omega \in \Omega_\nu(L) : \{\xi_k\} \cap \{|x_2 - v_1 L| < \frac{\tilde{K}}{\sqrt{B}}\} = \emptyset\}$ (configurations with no grains in the strip). Then

$$\mu_\nu(\Omega_0) = \left(1 - \frac{\tilde{K}}{L\sqrt{B}}\right)^{\nu L^2} \rightarrow 1 \quad \text{as} \quad \theta \rightarrow 0$$

- If $\omega \in \Omega_0$ such that $\Gamma(\omega, \lambda, \theta)$ is in $\{|x_2 - v_1 L| > \frac{\tilde{K}}{\sqrt{B}}\}$, then $\#\Gamma \geq \sqrt{B}L \ll L^2$, contradiction.
The regime \( B \gg 1 \) and \( L = O(\sqrt{B}) \) or \( \theta = O(1) \)

Consider

\[
U_k = \{ \omega \in \Omega_\nu(L) : \exists \Gamma(\omega, \lambda, \theta) \& k = \# \{ \xi_j \in \omega : \Gamma(\omega, \lambda, \theta) \cap B_R(\xi_j) \neq \emptyset \} \}
\]

( Configurations with interfaces that connect exactly \( k \) grains).

**Theorem** There exists \( N_0 \leq 2D_0 \theta \nu < \nu L^2 \) and \( L \) sufficiently large such that, if \( k \leq N_0 \),

\[
\mu_\nu(U_k) \lesssim e^{-2D_0 \theta \nu} \sum_{i=k}^{N_0} \frac{1}{i!} (2D_0 \theta \nu)^i .
\]

Moreover, for all \( \varepsilon > 0 \), there exists a \( L_\varepsilon > 0 \) such that for all \( L \geq L_\varepsilon \) there exists \( \Omega_\varepsilon \subset \Omega_\nu(L) \) such that \( \mu_\nu(\Omega_\varepsilon) \geq 1 - \varepsilon \), and that for all \( \omega \in \Omega_\varepsilon \) the solutions \( \Gamma(\omega, \lambda, \theta) \) satisfy \( |C(\Gamma) - 1| \leq \varepsilon \).
The regime $B \gg 1$ and $L = O(\sqrt{B})$ or $\theta = O(1)$

Idea of the Proof: Write

$$U_k = (U_k \cap \Omega_s) \cup (U_k \cap \Omega_s^c \cap U_{P \geq N_0}) \cup (U_k \cap \Omega_s^c \cap U_{P < N_0}),$$

where

- $\Omega_s \subset \Omega_\nu(L)$ such that $\exists \Gamma(\omega, \lambda, \theta)$ for $\omega \in \Omega_s$ such that

  $$\Gamma(\omega, \lambda, \theta) \cap \{|x_2 - v_1 L| > D_0/\sqrt{B}\} \neq \emptyset$$

- $U_{P \geq N_0}$ set of configurations such that $\# \{\xi_j \in \{|x_2 - v_1 L| < D_0/\sqrt{B}\} \geq N_0$ with $N_0 = 2\nu D_0 \theta = O(1)$ (maximum number of expected centers)

- We also need $\omega_k$ the configurations with exactly $k$ grains in $\{|x_2 - v_1 L| < D_0/\sqrt{B}\}$. 

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The regime $B \gg 1$ and $L = O(\sqrt{B})$ or $\theta = O(1)$

Idea of the Proof: Write

$$U_k = (U_k \cap \Omega_s) \cup (U_k \cap \Omega^c_s \cap U_{P \geq N_0}) \cup (U_k \cap \Omega^c_s \cap U_{P < N_0}),$$

then

- **Counting argument:** $\mu_{\nu}(\Omega_s) \sim 1/L^2 \to 0$ as $L \to \infty$.
- $\mu(\omega_k) = \left(\frac{\nu L^2}{k}\right) \left(\frac{2D_0 \theta}{L^2}\right)^k \left(1 - \frac{2D_0 \theta}{L^2}\right)^{\nu L^2 - k}$.
- That gives:

$$\mu_{\nu}(U_{P \geq N_0}) = \sum_{j=N_0}^{\nu L^2} \binom{\nu L^2}{j} \left(\frac{2D_0 \theta}{L^2}\right)^j \left(1 - \frac{2D_0 \theta}{L^2}\right)^{\nu L^2 - j} \to 0 \quad \text{as} \quad L \to \infty$$

Hence:

$$\mu_{\nu}(U_k) \lesssim \sum_{j=k}^{N_0} \binom{\nu L^2}{j} \left(\frac{2D_0 \theta}{L^2}\right)^j \left(1 - \frac{2D_0 \theta}{L^2}\right)^{\nu L^2 - j}$$
The regime $B \gg 1$ and $L \gg \sqrt{B}$

**Theorem** Let $0 < h = O(1)$ and $\Omega_h \subset \Omega_\nu(L)$ denote the set of all configurations $\omega$ such that there exists a $\Gamma(\omega, \lambda, \theta)$ satisfying $\max |\xi_j^{(2)} - v_1 L| \geq h$. Then,

$$\mu_\nu(\Omega_h) \to 0 \quad \text{as} \quad B \to \infty$$

if $\sqrt{B} \ll L \ll \sqrt{B} \log B$.

**Idea of the Proof:**

- **Characterise $\Omega_h$:** In particular $\Omega_h \subset U_h(K, D_0, \theta, L)$ where $U_h(K, D_0, \theta, L)$ have configurations of grains with the right geometric properties to allow interfaces as in $\Omega_h$.

- **Distinguish** $U^g \in U_h(K, D_0, \theta, L)$ with all such grains outside $\{|x_2 - v_1 L| < D_0/\sqrt{B}\}$ and those with grains in it.
The regime $B \gg 1$ and $L \gg \sqrt{B}$ (contd.)

Idea of the Proof (contd.)

1. First case: counting argument shows $\sum \mu_\nu(U^g) \to 0$ as $B \to \infty$ if $L^2 \gg B$.

2. Second case: geometric argument shows $\sum \mu_\nu(U^g) \to 0$ as $B \to \infty$ if $L \ll \sqrt{B} \log B$.

Second case main ingredients:

- Identification of grains that could form the interface intersecting the strip $\{|x_2 - v_1 L| < D_0/\sqrt{B}\}$.
- Estimation on measure of the longest possible interface.
- Technical lemma on iteration of measure properties.
The regime $L = O(1/B)$ as $B \to 0$

**Observation:** In the limit $B \to 0$ only $LB = O(1)$ makes sense:

- If $L \ll 1/B$ then $H \sim 0$ (only horizontal solutions).
- If $L \gg 1/B$ then $H \sim \infty$ (no connections possible).

**Setting:** Given $v_0$, restrict domain to:

$$(0, L) \times ((v_1 - \varepsilon_0)L, (v_1 + \varepsilon_0)L), \quad 0 < \varepsilon_0 < \min\{v_1, 1 - v_1, LB\}$$

**Theorem** Assume $BL = O(1)$. Then, given a compatible curve $\Lambda \in C^1([0, 1])$, there exist $\nu_0 > 0$ and $L_0 > 0$ such that for all $\nu \geq \nu_0$ and $L \geq L_0$ there exist $U \in \Omega_{\nu}(L)$ and $B \ll \varepsilon \ll 1, \varepsilon \to 0$ as $L \to \infty$ such that

$$\mu_{\nu}(U) \geq \delta_{\varepsilon} \quad \text{with} \quad 0 < \delta_{\varepsilon} \to 1^{-} \quad \text{as} \quad L \to \infty,$$

and that for any $\omega \in U$, there exists $\Gamma(\omega, \lambda, \theta)$ such that

$$\tilde{\Gamma}(\omega, \lambda, \theta) \subseteq T_{\sqrt{2\varepsilon}}(\Lambda), \quad \Lambda \subseteq T_{\sqrt{2\varepsilon}}(\tilde{\Gamma}(\omega, \lambda, \theta)).$$
The regime $L = O(1/B)$ as $B \to 0$ (contd.)

Sketch of the proof:

(f) Domain divided into squares $Q$ of size $(\varepsilon L)^2$

(g) A square $Q$ divided into squares $S_\kappa$ of $O(1)$ size
The regime $L = O(1/B)$ as $B \to 0$ (contd.)

Sketch of the proof (contd.):

Solve a percolation problem in each $Q$:

- A site $S_κ$ is “open” if contains at least one grain, and “close” otherwise:
  \[
  P(S_κ \text{ closed}) = (1 - |S_κ|/L^2)^{νL^2} \to e^{-ν} \quad \text{as} \quad L \to ∞
  \]

- Site Percolation: $∃ν_0$ such that
  \[
  P(Q \text{ closed}) \to 0 \quad \text{as} \quad L \to ∞ \quad \text{if} \quad ν ≥ ν_0.
  \]

- $P(∩Q’s \text{ connected}) = 1 - P(∪Q \text{ closed}) ≥ 1 - ∑ P(Q \text{ closed})$. 
THANK YOU!