# On the existence of radial graphs with constant scalar curvature

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Geometric aspects on capillary problems and related topics
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# Question

Under what conditions a closed $(n-1)$-dimensional embedded submanifold $\Lambda$ of $\mathbb{R}^{n+1}$ spans a compact hypersurface of constant (extrinsic) curvature?

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- Let $e_1, \ldots, e_n$ a l.o.f.f. on $\mathbb{S}^n$. The components of the metric and the second fundamental form of $\Sigma$ are given by

$$g_{ij} = \rho^2 \delta_{ij} + \nabla_i \rho \nabla_j \rho$$

and

$$h_{ij} = \frac{1}{(\rho^2 + |\nabla \rho|^2)^{1/2}} \left( \rho^2 \delta_{ij} + 2 \nabla_i \rho \nabla_j \rho - \rho \nabla_{ij} \rho \right).$$

# Some previous results

**Theorem (Serrim '69)** Assume $\bar{\Omega} \subset \mathbb{S}^n_+$ and $H \leqslant 0$ satisfies

$$H_{\partial\Omega}(x) \geqslant -\frac{n}{n-1} H\phi(x) \qquad x \in \partial\Omega.$$

Then there exists a unique solution of (P).

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**Theorem (Guan - Spruck '93)** Assume that $\Omega$ does not contain any hemisphere and there exists a strictly locally convex (s.l.c.) radial graph $\bar{\Sigma}$ with $\partial\bar{\Sigma} = \mathrm{graph}(\phi)$. If

$$0 < K < K(\bar{\Sigma}) = \inf_{p \in \bar{\Sigma}} K(p),$$

then there exists a s.l.c. radial graph $\Sigma$ of constant Gauss curvature $K$ with boundary $\partial\Sigma = \mathrm{graph}(\phi)$.

# Some previous results

**Theorem (CNS-V '88)** Let $\Omega \subset \mathbb{R}^n$ be a bounded strictly convex domain. Suppose that there exists an admissible (vertical) graph $\bar{\Sigma}$ with $\partial\bar{\Sigma} = \partial\Omega$. If

$$0 < R < R(\bar{\Sigma}) = \inf_{p \in \bar{\Sigma}} R(p),$$

then there exists an admissible graph $\Sigma$ of constant scalar curvature $R$ with boundary $\partial\Sigma = \partial\Omega$.

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**Theorem (— '15)** Assume that $\bar{\Omega} \subset \mathbb{S}^n_+$ is a mean convex domain. Then, for $0 < R < n(n-1)$, there exists a radial graph $\Sigma$ of constant scalar curvature $R$ and boundary $\partial\Sigma = \partial\Omega$.

# Curvature Functions

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Let $\Gamma$ be an open, convex, symmetric cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin and containing the positive cone

$$\Gamma^+ = \{\kappa \in \mathbb{R}^n \,:\, \text{each component } \kappa_i > 0\}.$$

# Curvature Functions

Let $\Gamma$ be an open, convex, symmetric cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin and containing the positive cone

$$\Gamma^+ = \{\kappa \in \mathbb{R}^n : \text{each component } \kappa_i > 0\}.$$

We say that $f : \Gamma \longrightarrow (0, +\infty)$ is a *curvature function* if it satisfies

Symmetry: $f(\lambda_1, \ldots, \lambda_n) = f(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)})$, for all $\sigma$;

Ellipticity: $f_i = \dfrac{\partial f}{\partial \lambda_i} > 0$;

Concavity: $f$ is a concave function;

Homogeneity: $f(t\lambda) = tf(\lambda)$, $t > 0$;

Compatibility: $\limsup\limits_{\lambda \to \partial\Gamma} f(\lambda) = 0$;

# Technical assumptions

- For every constant $C > 0$ and every compact set $E$ in $\Gamma$ there is a constant $R = R(C, E) > 0$ such that

$$f(\lambda_1, \cdots, \lambda_n + R) \geqslant C, \quad \text{for all } \lambda \in E, \tag{1}$$

- For all $\lambda \in \Gamma_{\mu,\nu} = \{\lambda \in \Gamma : \mu < f(\lambda) < \nu\}$ it holds

$$\sum f_i(\lambda)\lambda_i^2 \leqslant C_0(\lambda_j \chi_{\{\lambda_j > 0\}} + \sum_{k \neq j} f_k(\lambda)\lambda_k^2). \tag{2}$$

# Main Theorem

## Theorem

*Let $\Omega$ be a smooth bounded domain such that $\bar{\Omega} \subset \mathbb{S}^n_+$ and $H_{\partial\Omega} \geqslant 0$, and let $\psi$ be a smooth positive function defined on $\bar{\Omega}$. Assume $f$ satisfy (1)-(2) and there exists a smooth admissible [1] radial graph $\bar{\Sigma}$: $\bar{X}(x) = \bar{\rho}(x)x$ over $\bar{\Omega}$ that satisfies*

$$f(\kappa_{\bar{\Sigma}}[\bar{X}]) > \psi(x) \quad in\ \Omega,$$
$$\bar{\rho} = \phi \quad on\ \partial\Omega$$

*and is locally strictly convex (up to the boundary) in a neighbourhood of $\partial\Omega$. Then there exists a smooth radial graph $\Sigma : X(x) = \rho(x)x$ satisfying*

$$f(\kappa_{\Sigma}[X]) = \psi(x) \quad in\ \Omega,$$
$$\rho = \phi \quad on\ \partial\Omega.$$

[1] $\Sigma$ is admissible if $\kappa_{\Sigma}([X]) \in \Gamma$ for all $X \in \Sigma$.

# Existence

Consider the auxiliary functions

$$\Psi^t(\rho x) = \left(\frac{\overline{\rho}(x)}{\rho}\right)^3 \big(t\psi(x) + (1-t)\underline{\psi}(x)\big), \quad t \in [0,1]$$

and

$$\Phi^s(\rho x) = s\psi(x) + (1-s)\left(\frac{\overline{\rho}(x)}{\rho}\right)^3 \psi(x), \quad s \in [0,1]$$

defined in the solid cylinder $\Delta = \{X \in \mathbb{R}^{n+1} \,:\, \frac{X}{\|X\|} \in \overline{\Omega}\}$.

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**Remark.**

- $\Phi^1(\rho x) = \psi(x)$
- $\Psi^1 = \Phi^0$
- $\frac{\partial}{\partial \rho}(\rho \Psi^t(\rho x)) \leqslant 0$
- $\frac{\partial}{\partial \rho}(\rho \Phi^s(\rho x)) \leqslant 0$ if $\rho \leqslant \overline{\rho}$.

# Existence

Consider the auxiliary equations

$$\begin{aligned} H(\nabla^2 v, \nabla v, v) &= \Psi^t(X) \quad \text{in } \Omega \\ v &= -\ln \phi \quad \text{on } \partial\Omega \end{aligned} \qquad (EqA)_t \tag{3}$$

and

$$\begin{aligned} H(\nabla^2 v, \nabla v, v) &= \Phi^s(X) \quad \text{in } \Omega \\ v &= -\ln \phi \quad \text{on } \partial\Omega \end{aligned} \qquad (EqB)_s \tag{4}$$

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**Remark.**

- $\underline{v} = -\ln \overline{\rho}$ is a solution of $(EqA)_0$
- $\underline{v} = -\ln \overline{\rho}$ is a strictly subsolution of $(EqA)_t$ for each $t > 0$
- $\frac{\partial}{\partial v}(H - \Psi^t) \leqslant 0$
- $(EqA)_1 = (EqB)_0$.

# A priori estimates

## Theorem

*Let $v \geqslant \underline{v}$ be an admissible solution of*

$$f(\kappa_\Sigma[X]) = \Upsilon(X) \quad in\ \Omega$$
$$v = \varphi \quad on\ \partial\Omega.$$

*Suppose* $\frac{\partial}{\partial\rho}(\rho\Upsilon) \leqslant 0$. *Then we have the estimate*

$$\|v\|_{C^2(\bar{\Omega})} \leqslant C$$

*where $C$ depends on* $\inf_\Omega \underline{v}$, $\|\underline{v}\|_{C^2(\bar{\Omega})}$, $\|\Upsilon\|_{C^2(\Delta_L)}$, *the convexity of $\bar{\Sigma}$ in a neighbourhood of $\partial\Omega$ and other known data.*

# Existence

It follows from the a priori estimates that

- There exists a unique solution $v^0$ of $(EqB)_0$.
- If $v^s$ is a solution of $(EqB)_s$ and $v^s \geqslant \underline{v}$ then

$$\|v^s\|_{C^{4,\alpha}(\bar{\Omega})} < C_1.$$

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Consider the open set

$$\begin{aligned}\mathcal{O} = \{ & w \in C_0^{4,\alpha}(\bar{\Omega}) : w > 0 \text{ in } \Omega,\ \nabla_n w > 0 \text{ on } \partial\Omega, \\ & w + \underline{v} \text{ is admissible and } \|w\|_{C^{4,\alpha}(\bar{\Omega})} \leqslant C_1 + \|\underline{v}\|_{C^{4,\alpha}(\bar{\Omega})}\}.\end{aligned}$$

and the map

$$M_s[w] = H(\nabla^2(w+\underline{v}), \nabla(w+\underline{v}), w+\underline{v}) - \Phi^s(w+\underline{v}), \quad w \in \mathcal{O}.$$

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We have

- $w^0 = v^0 - \underline{v}$ is the unique solution of $M_0[w] = 0$ in $\mathcal{O}$
- There is no solution of $M_s[w] = 0$ on $\partial\mathcal{O}$
- The Fréchet derivative of $M_0$ at $w^0$ is invertible.

# Existence

Then, by the degree theory developed by Yan Yan Li[2], we can see that the degree of $M_s$ on $\mathcal{O}$ at 0 $\deg(M_s, \mathcal{O}, 0)$ is well defined and independet of $s$. Moreover,

$$\deg(M_0, \mathcal{O}, 0) = \pm 1 \neq 0$$

and we conclude that

$$\deg(M_s, \mathcal{O}, 0) \neq 0 \text{ for all } s \in [0, 1].$$

Let $w^1$ be a solution of $M_1[w] = 0$. Then the function $v^1 = w^1 + \underline{v}$ is the desired solution.

[2] Li, Y.Y., Degree theory for second order nonlinear elliptic operators and its applications. *Comm. in PDE's.* 14(11)(1989).

# Height and Gradient Bounds

Let $\rho$ be the solution of the minimal surface equation in $\Omega$ that satisfies $\rho = \phi$ on $\partial\Omega$.
Then

$$\underline{\rho} \leqslant \rho \leqslant \bar{\rho} \quad \text{in } \Omega$$

and

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Under the condition

$$\frac{\partial}{\partial\rho}\big(\rho\Upsilon(\rho x)\big) \leqslant 0,$$

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**Remark.** Once established the second derivative boundary estimates, the interior ones follow as above.

# The Barrier Method

Set

$$u = 1/\rho \text{ and } \varphi = 1/\phi.$$

The mixed second derivative boundary estimates are obtained applying the barrier method to the function $\nabla_\alpha u$. We will make use of the following version of the Maximum Principle.

## Theorem (Maximum Principle)

*Let $v, w \in C^2(\bar{\Omega})$ and $u$ an admissible function on $\Omega$. Assume that*

$$G^{ij}\nabla_{ij} w \leqslant C_0(1 + |\nabla w|) \quad in\ U$$
$$G^{ij}\nabla_{ij} v \geqslant C_0(1 + |\nabla v|) \quad in\ U,$$

*where $C_0$ is a constant and $U \subset \Omega$. If $v \leqslant w$ on $\partial U$, then $v \leqslant w$ in $\bar{U}$.*

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Rmk:

$u = 1/\rho \quad \Rightarrow \quad G(\nabla^2 u, \nabla u, u) = \Upsilon$

$$G^{ij} = \frac{\partial G}{\partial \nabla_{ij} u}$$

## Lemma (Fundamental Inequality)

*For some positive constants $K$ and $M$ sufficiently large depending on $\|u\|_{C^1(\bar{\Omega})}, \|\Upsilon\|_{C^1(\Delta_L)}$ and other known data, the function*

$$\Phi = \nabla_k(u-\varphi) - \frac{K}{2}\sum_{l<n}\big(\nabla_l(u-\varphi)\big)^2$$

*satisfies*

$$G^{ij}\nabla_{ij}\Phi \leqslant M(1+|\nabla\Phi| + G^{ij}\delta_{ij} + G^{ij}\nabla_i\Phi\nabla_j\Phi) \quad in \quad \Omega_\delta.$$

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We choose

$$w = 1 - e^{-a_0\Phi} + b_0(u - \underline{u})$$

to get

$$G^{ij}\nabla_{ij}w \leqslant C_0(1+|\nabla w|).$$

# Barrier Function

## Lemma

*There exist some uniform positive constants* $t, \delta, \varepsilon$ *sufficiently small and* $N$ *sufficiently large depending on* $\inf_{\bar{\Omega}} \underline{u}$, $\|\underline{u}\|_{C^2(\bar{\Omega})}$, $\sup_{\Delta_L} \Upsilon$, *the convexity of* $\underline{u}$ *in a neighbourhood of* $\partial\Omega$ *and other known data, such that the function*

$$\Theta = u - \underline{u} + td - Nd^2, \quad d = dist(\cdot, \partial\Omega)$$

*satisfies*

$$G^{ij}\nabla_{ij}\Theta \leqslant -(1 + |\nabla\Theta| + G^{ij}\delta_{ij}) \quad in\ \Omega_\delta$$

*and*

$$\Theta \geqslant 0 \quad on\ \partial\Omega_\delta.$$

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*and*

$$\Theta \geqslant 0 \quad on\ \partial\Omega_\delta.$$

We choose

$$v = -c_0 \mathrm{dist}(\cdot, x_0)^2 - d_0\Theta$$

to get

$$G^{ij}\nabla_{ij}v \geqslant C_0(1 + |\nabla v|)$$

and $v \leqslant w$ on $\partial\Omega_\delta$.

# Double Normal Estimate

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Let $\kappa'$ the roots of $\det(h_{\alpha\beta} - tg_{\alpha\beta}) = 0$. It follows from the previous estimates that the principal curvatures $\kappa_i$ behave like

$$\kappa_\alpha = \kappa'_\alpha + o(1), \quad 1 \leqslant \alpha \leqslant n-1,$$
$$\kappa_n = \frac{h_{nn}}{g_{nn}}\left(1 + O\left(\frac{1}{h_{nn}}\right)\right),$$

as $|h_{nn}| \to \infty$.

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as $|h_{nn}| \to \infty$. Let $\Gamma'$ be the projection of $\Gamma$ on $\mathbb{R}^{n-1}$. Thus, there exists a uniform positive constant $N_0 > 0$ satisfying

$$\kappa' \in \Gamma' \quad \text{if} \quad \nabla_{nn} u \geqslant N_0. \qquad (*)$$

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$$\kappa' \in \Gamma' \quad \text{if} \quad \nabla_{nn} u \geqslant N_0. \qquad (*)$$

## Lemma

*Let $N_0 > 0$ be the constant defined in (*) and suppose that $\nabla_{nn} u \geqslant N_0$. Then there exists a uniform constant $c_0 > 0$ such that*

$$d(x) = dist(\cdot, \partial\Gamma') \geqslant c_0 \qquad on\ \partial\Omega.$$

Therefore an upper bound for $\kappa_n$ follows from the previous established estimates and the assumption that $f$ is of unbounded type.

# References


- Caffarelli, L., Nirenberg L. and Spruck, J., The Dirichlet Problem for Nonlinear Second-Order Elliptic Equations IV: Starshaped compact Weingarten hypersurfaces, *Current Topics in P.D.E.*, Tokyo, 1986.
- Caffarelli, L., Nirenberg, L. and Spruck, J., Nonlinear Second-Order Elliptic Equations V. The Dirichlet Problem for Weingarten Hypersurfaces. *Comm. Pure Applied Math.*, 41(1988), 47–70.
- Guan, B. and Spruck, J., Boundary Value Problem on $S^n$ for Surfaces of Constant Gauss Curvature. *Ann. of Math.*, 138(1993), 601–624.
- Lopez, R., *Constant Mean Curvature Surfaces with Boundary*, Springer-Verlag, New York-Heidelberg-Berlin, 2010.
- Serrin, J., The Problem of Dirichlet for Quasilinear Elliptic Differential Equations with Many Variables. *Philos. Trans. Roy. Soc. London Ser. A.*, 264(1969), 413–496.
- Su, C., Starshaped Locally Convex Hypersurfaces with Prescribed Curvature and Boundary. *arXiv:* 1310.4730 (2013).
- Trudinger, N., On the Dirichlet Problem for Hessian Equations. *Acta Math.*, 175(1995), 151–164.

**Thank you!**