STABLE CAPILLARY HYPERSURFACES IN A WEDGE

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(Joint with Miyuki Koiso)

- Among all domains of fixed volume in $\mathbb{R}^{n+1}$, which one has

  least boundary area?

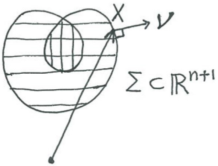
  isoperimetric problem $\Rightarrow$ the ball

- Which one has **critical** boundary area?

  More general domains enclosed by immersed

  hypersurfaces Σ:

  Oriented volume: $V(\Sigma) = \frac{1}{n+1} \int_\Sigma \langle X, \nu \rangle dS$.



- $|\Sigma|$ is critical among all hypersurfaces enclosing fixed

  oriented volume. $\Leftrightarrow$ Σ has constant mean curvature.

- Hopf conjecture: CMC immersion $\Rightarrow$ round sphere

  counterexample: Wente torus

- CMC + extra condition $\Rightarrow$ sphere ?

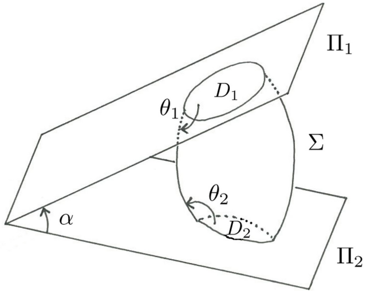
  Alexandrov: embedded + CMC ($\Sigma^n \subset \mathbb{R}^{n+1}$)

  Hopf: immersed CMC sphere ($\Sigma^2 \subset \mathbb{R}^3$)

  Barbosa-do Carmo: stable CMC ($\Sigma^n \subset \mathbb{R}^{n+1}$)

- Σ is stable if the second variation of $|\Sigma|$ is nonnegative for all volume preserving perturbations.

- Modified situation: $\Sigma \subset \mathbb{R}^{n+1} \to \Sigma \subset$ wedge W, $\partial\Sigma \subset \Pi_1 \cup \Pi_2$



- $E(\Sigma) := |\Sigma| - |D_1|\cos\theta_1 - |D_2|\cos\theta_2$: **total energy**

  $\widehat{V}(\Sigma) := \frac{1}{n+1}\int_{\Sigma\cup D_1\cup D_2}\langle X,\nu\rangle dS$

- Finn: A critical point of $E(\Sigma)$ among all hypersurfaces $\Sigma \subset W$ with $\widehat{V}(\Sigma) = const$ is a capillary surface with constant contact angles θ_1, θ_2.

- McCuan: $\nexists$ embedded annular capillary surface in W if $\theta_1 + \theta_2 \leq \pi + \alpha$.

  Park: embedded annular capillary surface in $W \Rightarrow$ round.

  McCuan: inversion

  Park: Bonnet transform $X + \frac{1}{H}\nu$: CMC surface $\rightarrow$ CMC surface
- Generalize Alexandrov, Hopf, Barbosa-do Carmo for capillary hypersurfaces in a wedge:

  Is there an embedded capillary surface of genus ≥ 1 in W?

  McCuan: No, if $\theta_i \leq \pi/2$.

  Is there an immersed capillary nonspherical surface in W?

  Yes, if $g = 0$. Wente, Bobenko, Heil
- What if Σ is stable?

Theorem

$\Sigma^n \subset \mathbb{R}^{n+1}$: immersed stable capillary hypersurface in a wedge W, $\theta_i \geq \pi/2$, disjoint from the edge of W.

$\partial\Sigma$: embedded for $n = 2$ or convex for $n \geq 3$.

Then Σ is part of a round sphere.

- McCuan and Park's theorems $\Rightarrow$ Σ with $\theta_i < \pi/2$ is less likely to exist.
- Σ with least total energy can intersect the edge of W and can be nonspherical.

- (CK) $\Sigma^n \subset \mathbb{R}^{n+1}$: immersed stable capillary hypersurface in a half-space, $\theta \geq \pi/2$.
  $\partial\Sigma$: embedded for $n = 2$ or convex for $n \geq 3$.
  $\Rightarrow \Sigma$ is a spherical cap.
- Wente (1980): An embedded capillary hypersurface in a half-space $\subset \mathbb{R}^{n+1}$ is a spherical cap.
- Nitsche: An immersed disk type capillary surface in a half-space $\subset \mathbb{R}^3$ is a spherical cap.
- Marinov (2012): A stable capillary surface in a half-space $\subset \mathbb{R}^3$ with embedded boundary is a spherical cap.
- Ainouz-Souam (2015): An immersed stable capillary hypersurface Σ in a half-space $\subset \mathbb{R}^{n+1}$ with $\theta \leq \pi/2$ and with embedded boundary is a spherical cap.

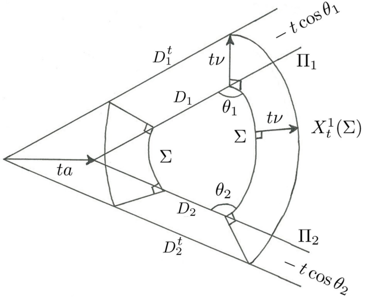
- **Barbosa-do Carmo** used variation field $(1 + H\langle X, \nu\rangle)\nu$.

  Wente: variations by parallel surfaces and homotheties.

  $\Rightarrow$ explicit computations of volume, area as polynomials.

- Proof. Σ_t^1 parallel hypersurface of Σ with distance t.

  $\Sigma : X, \quad \Sigma_t^1 : X + t\nu$



- Σ: constant contact angle $\Rightarrow \partial\Sigma_t^1 \subset$ hyperplanes // Π_i.

- $|\Sigma_t^1|$: Weyl's tube formula

  $S_r \subset \mathbb{R}^3$: sphere of radius $r \Rightarrow |S_r| = 4\pi r^2$,

  $|S_{r+t}| = 4\pi(r+t)^2 = 4\pi r^2 + 8\pi rt + 4\pi t^2$.

- $$|\Sigma_t^1| = |\Sigma| + \left(\int_\Sigma nHdS\right) t + \left(\int_\Sigma \sum_{i<j} k_i k_j dS\right) t^2 + \cdots + \left(\int_\Sigma k_1 k_2 \cdots k_n dS\right) t^n.$$

- $\exists\, a$ such that $\Sigma_t^2 := \Sigma_t^1 + ta$ has boundary $\partial\Sigma_t^2 \subset \Pi_1 \cup \Pi_2$.

  $\widehat{V}(\Sigma_t^2) = \widehat{V}(\Sigma_t^1)$

- $$\frac{d}{dt}\widehat{V}(\Sigma_t^2) = |\Sigma_t^2| - \cos\theta_1 |D_1^t| - \cos\theta_2 |D_2^t| = E(\Sigma_t^2)$$

- Σ: constant contact angle $\Rightarrow$ ∂D_i^t are parallel surfaces of ∂D_i.

  $\therefore\ |D_i^t| = |D_i| + \int_0^t |\partial D_i^t| \sin\theta_i dt$

  $= |D_i| + |\partial D_i| t \sin\theta_i + \frac{1}{2}\int_{\partial D_i}(n-1)\bar{H} dS \cdot t^2 \sin\theta_i + \cdots$

- $\frac{d}{dt}\widehat{V}(\Sigma_t^2) = |\Sigma_t^2| - \cos\theta_1 |D_1^t| - \cos\theta_2 |D_2^t|$

  $= \{|\Sigma| - \sum_i \cos\theta_i |D_i|\} + \{nH|\Sigma| - \sum_i \cos\theta_i \sin\theta_i |\partial D_i|\}\, t$

  $+ \left\{\int_\Sigma \sum_{i<j} k_i k_j \, dS - \frac{1}{2}\sum_i \cos\theta_i \sin^2\theta_i \int_{\partial D_i}(n-1)\bar{H} dS\right\} t^2 +$

  $\cdots$

  $= E(\Sigma_t^2) := e_0 + e_1 t + e_2 t^2 + \cdots + e_n t^n.$

- $\widehat{V}(\Sigma_t^2) := v_0 + v_1 t + \cdots + v_{n+1} t^{n+1}.$

  $\therefore\ \frac{d}{dt}\widehat{V}(\Sigma_t^2) = E(\Sigma_t^2) \Rightarrow\ v_1 = e_0,\ \ 2v_2 = e_1.$

- $\widehat{V}(\Sigma_t^2) = \widehat{V}(\Sigma_t^1) > \widehat{V}(\Sigma)$.

  Introduce $\Sigma_t^3 := s(t)\Sigma_t^2$, contraction centered at O such that $\widehat{V}(\Sigma_t^3) = \widehat{V}(\Sigma) = v_0$.

  $\therefore \partial\Sigma_t^3 \subset \Pi_1 \cup \Pi_2$.

- $$\begin{cases} \widehat{V}(\Sigma_t^3) = s(t)^{n+1}(v_0 + v_1 t + \cdots + v_{n+1}t^{n+1}) = v_0. \\ E(\Sigma_t^3) = s(t)^n(e_0 + e_1 t + e_2 t^2 + \cdots + e_n t^n). \end{cases}$$

- $s(t)^n = 1 - \frac{n}{n+1}\frac{v_1}{v_0}t + \left\{\frac{n(2n+1)}{2(n+1)^2}\left(\frac{v_1}{v_0}\right)^2 - \frac{n}{n+1}\left(\frac{v_2}{v_0}\right)\right\}t^2 + \cdots$

- $\therefore E(\Sigma_t^3) = e_0 + \left\{e_1 - \frac{n}{n-1}\frac{v_1}{v_0}e_0\right\}t +$

  $\left\{e_2 - \frac{n}{n+1}\frac{v_1}{v_0}e_1 + \frac{n(2n+1)}{2(n+1)^2}\left(\frac{v_1}{v_0}\right)^2 e_0 - \frac{n}{n+1}\left(\frac{v_2}{v_0}\right)e_0\right\}t^2 + \cdots$

- $\left(\frac{d}{dt}\widehat{V}(\Sigma_t^2) = E(\Sigma_t^2) \Rightarrow v_1 = e_0,\ \ 2v_2 = e_1\right)$.

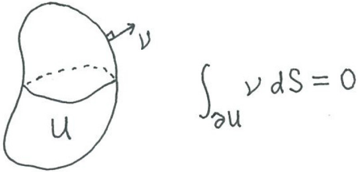
  $E'(0) = 0 \Rightarrow v_0 = \frac{n}{n+1}\frac{e_0^2}{e_1}$.

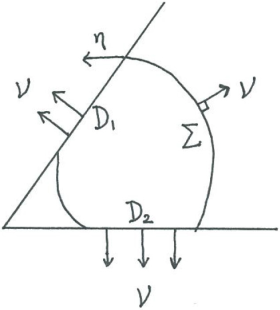
  $E''(0) = \frac{1}{ne_0}\{2ne_0e_2 - (n-1)e_1^2\} \geq 0$.

- $ne_0 E''(0) = 2n(|\Sigma| - \sum_i \cos\theta_i |D_i|)\times$
$\left(\int_\Sigma \sum_{i<j} k_i k_j dS - \frac{1}{2}\sum_i \cos\theta_i \sin^2\theta_i \cdot \int_{\partial D_i}(n-1)\bar{H} dS\right)$
$-(n-1)(nH|\Sigma| - \sum_i \cos\theta_i \sin\theta_i |\partial D_i|)^2.$

- Balancing formula

  $\Delta_\Sigma X = nH\nu$: Integrate over Σ.





$$\begin{cases} \int_\Sigma \nu dS \to -\int_{D_i} \nu dS \to |D_i| \\ \int_\Sigma \Delta_\Sigma X dS = \int_{\partial\Sigma} \eta = \int_{\partial D_i} \eta^\top + \int_{\partial D_i} \eta^\perp \to |\partial D_i| \end{cases}$$

- $\therefore\ nH|D_i| = \sin\theta_i |\partial D_i|$

- $ne_0 E''(0) = (|\Sigma| - \sum_i \cos\theta_i |D_i|) \times \{-\int_\Sigma \sum_{i<j}(k_i - k_j)^2 dS$
  $-(n-1)\sum_i \cos\theta_i \sin^2\theta_i \left(n\int_{\partial D_i} \bar{H} dS - \frac{|\partial D_i|^2}{|D_i|}\right)\}$

  $E''(0) \leq 0.$

  But stability $\Rightarrow E''(0) \geq 0.$

  $\therefore E''(0) = 0$, umbilic everywhere.
- $n\int_{\partial D_i} \bar{H} dS - \frac{|\partial D_i|^2}{|D_i|} \leq 0$??

- Minkowski inequality

  Minkowski sum of $A, B \subset \mathbb{R}^n$:

  $A + B = \{a + b : a \in A, b \in B\}$

- D: convex body in $\mathbb{R}^n$, B: unit ball in $\mathbb{R}^n$.

  Steiner formula: $|D + tB| = \sum_{j=0}^{n} \binom{n}{j} W_j(D)t^j$,

  $W_j(D)$: jth **quermassintegral** of D.

- $\frac{d}{dt}|D + tB| = |\partial(D + tB)|$

$$
\begin{aligned}
\therefore\ W_0(D) &= |D|, \\
nW_1(D) &= |\partial D|, \\
nW_2(D) &= \int_{\partial D} H dS, \\
n(n-1)(n-2)W_3(D) &= 2\int_{\partial D} \sum_{i<j} k_i k_j dS.
\end{aligned}
$$

- **Alexandrov-Fenchel inequality**:

  $W_i(D)^2 \geq W_{i-1}(D) \cdot W_{i+1}(D)$.

  $\therefore W_1(D)^2 \geq W_0(D) \cdot W_2(D)$: $n \int_{\partial D} HdS \leq \frac{|\partial D|^2}{|D|}$: Minkowski

  $W_2(D)^2 \geq W_1(D) \cdot W_3(D)$:

  $\int_{\partial D} \sum_{i<j} k_i k_j dS \leq \frac{(n-1)(n-2)}{2n^2} \frac{|\partial D|^3}{|D|^2}$.

- $D \subset \mathbb{R}^2 : \int_{\partial D} kds = 2\pi$.

  $\therefore$ Minkowski $\Rightarrow 4\pi|D| \leq |\partial D|^2$.

  (Theorem) $D_i \subset \Pi_i \subset \mathbb{R}^3$: embedded, not necessarily convex

  $D \subset \mathbb{R}^3 : \int_{\partial D} k_1 k_2 dS = 4\pi$.

  $\therefore$ Minkowski $\Rightarrow 36\pi|D|^2 \leq |\partial D|^3$.

- $n \int_{\partial D} HdS \leq \frac{|\partial D|^2}{|D|} \Leftrightarrow n\frac{|\partial D_t|'}{|\partial D_t|} \leq (n-1)\frac{|D_t|'}{|D_t|} \Leftrightarrow \left(\frac{|\partial D_t|^n}{|D_t|^{n-1}}\right)' \leq 0$.

  $\Rightarrow D_t$ becomes rounder as t increases.

Capillary surfaces in a slab

- Examples: cylinder, unduloid, nodoid, catenoid
- Wente (1980): An embedded capillary hypersurface in a slab $\subset \mathbb{R}^{n+1}$ is rotationally invariant. ($\therefore$ spherical, Delaunay)
- Ros (2007): A stable capillary surface in a slab $\subset \mathbb{R}^3$ with $\theta = \pi/2$ is a cylinder.
- Ainouz-Souam (2015): An immersed stable capillary surface of genus 0 in a slab $\subset \mathbb{R}^3$ with contact angles θ_1, θ_2 is a Delaunay surface.

- Among embedded rotationally symmetric capillary hypersurfaces in a slab $\subset \mathbb{R}^{n+1}$ with $\theta = \pi/2$ only the circular cylinders are stable for $2 \leq n \leq 7$;
  Some unduloids are also stable for $n \geq 9$.
  ($n = 2$: Athanassenas, Vogel, $n \geq 3$: Pedrosa-Ritoré)
- Ainouz-Souam (2015): If $\Sigma \subset \mathbb{R}^{n+1}$ is an immersed stable capillary hypersurface in a slab with $\theta = \pi/2$ and with embedded boundary, then Σ is rotationally symmetric.

Capillary surfaces in a ball

- Nitsche: A capillary disk in a ball $\subset \mathbb{R}^3$ is a spherical cap.
- Ros-Souam: (i) A stable capillary surface of genus 0 in a ball $\subset \mathbb{R}^3$ is a spherical cap.

  (ii) A stable minimal surface with constant contact angle in a ball $\subset \mathbb{R}^3$ is a flat disk or a surface of genus 1 with at most 3 boundary components.
- Ros-Vergasta: A stable minimal hypersurface in a ball $B \subset \mathbb{R}^n$, $\perp \partial B$, is totally geodesic.

Problems

- If Σ is a minimal annulus in a ball $B \subset \mathbb{R}^3$ and orthogonal to ∂B, is Σ the catenoidal waist?

  Fraser-Schoen (2013): For all $n \geq 3$, $\exists$ minimal surface of genus 0, $\#(\text{ends}) = n$, and with free boundary in B.

  Kapouleas-Li (2015): $\exists$ minimal surface of sufficiently large genus with free boundary in B (3 boundary components).

  Zolotareva et al. (2015): $\exists$ minimal surface of genus 1, sufficiently large number of ends, and with free boundary in B.
- Show that a minimal surface with connected free boundary in B is flat.

- Let $\Sigma \subset \mathbb{R}^3$ be a capillary surface in a solid cylinder C. Show that if the boundaries of Σ are not null homotopic in the surface cylinder ∂C then Σ is part of the Delaunay surface. Do we need to assume that Σ is stable?
- Let $\Sigma \subset \mathbb{R}^3$ be a capillary surface **outside** C whose boundaries are not null homotopic on ∂C. Prove that Σ is also part of the Delaunay surface with (or without) the stability assumption.

- Let Σ be a CMC surface in a half space $\mathbb{H} \subset \mathbb{R}^3$ making a constant contact angle $\neq \pi/2$ with the plane $\partial\mathbb{H}$. Can one extend Σ across $\partial\mathbb{H}$ analytically? In case Σ is minimal, an affirmative answer is obtained in [C]. The Schwarz reflection principle and the Weierstrass representation formula are used in the proof of [C].

  [C] J. Choe, On the analytic reflection of a minimal surface, Pacific J. Math. 157 (1993), 29-35.

- Let Σ_1, Σ_2 be compact minimal surfaces in $\mathbb{S}^3$. Suppose they intersect at a constant angle $\neq \pi/2$. Is it true that they are both great spheres or both Clifford tori?