Capillary Surfaces and Floating Bodies

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Geometric aspects on capillary problems and related topics
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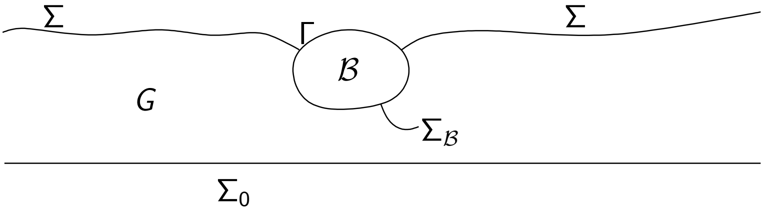
Problem

Rigid body $\mathcal{B}$ floating on a layer of a viscuous, incompressible fluid; upper surface Σ of the fluid domain (which is an unknown of the problem) is governed by surface tension



Two unknowns

(i) Position, orientation and motion of $\mathcal{B}$ and capillary surface Σ

(ii) Velocity v and pressure p in the fluid domain

Existence theorem

Approximation where the unknowns $(\Sigma,\mathcal{B})$ are determined under the assumption that (v, p) are known, as well as the other way around.

Capillarity problem

$G := \Omega \times \mathbb{R}^+,\ \Omega \subset \mathbb{R}^2$ bounded domain; G partly filled with fluid.

$\mathcal{B}(c, R) \subset \Omega \times \mathbb{R}^+$ domain occupied by the body $\mathcal{B}$ after Euclidean motion

$$y = x + c + Rx,$$

where $c =$ translation, $R = R(\alpha) =$ rotation about an axis that contains the center of $\mathcal{B}$.

Position of $\mathcal{B}$

Position of $\mathcal{B}(c, R)$ is determined by the force that the fluid exerts on it, i.e.

$$\int_{\partial\mathcal{B}^-} T(v, p) \cdot n \, \mathrm{d}\sigma,$$

where $\partial\mathcal{B}^-$ is the wetted part of $\partial\mathcal{B}$.

The mean curvature H_Σ of the capillary surface is proportional to the normal component of the stress vector:

$$\sigma H_\Sigma = n \cdot T(v, p) \cdot n$$

In the hydrostatic case the integrand reduces to $p \cdot n$ and the right-hand side in the mean-curvature equation equals p.

Gravitational energies

Gravitational energy of $\mathcal{B}(c, R)$:

$$\rho_0 g \int_{\mathcal{B}(c,R)} x_3 \mathrm{d}x, \ \ \rho_0 \text{ density of } \mathcal{B}$$

Gravitational energy of the fluid:

$$\rho g \int_E x_3 \mathrm{d}x, \ \ \rho \text{ density and } E \text{ domain occupied by fluid}$$

Adhesion and cohesion energy

Adhesion energies:

$$\kappa \int_{(\Omega \times \mathbb{R}^+) \setminus \mathcal{B}(c,R)} \varphi_E \, \mathrm{d}\sigma$$

$$\kappa_0 \int_{\partial \mathcal{B}(c,R)} \varphi_E \, \mathrm{d}x$$

Cohesion energy:

$$\sigma \int_{(\Omega \times \mathbb{R}^+) \setminus \mathcal{B}(c,R)} |D\varphi_E|$$

Variational problem (hydrostatic case $v \equiv 0$)

$$\begin{aligned}\mathcal{E}(c,R;E) :=&\sigma \int_{(\Omega\times\mathbb{R}^+)\setminus\mathcal{B}(c,R)} |D\varphi_E| \\ &+ \kappa \int_{(\Omega\times\mathbb{R}^+)\setminus\mathcal{B}(c,R)} \varphi_E \mathrm{d}\sigma \\ &+ \kappa_0 \int_{\partial\mathcal{B}(c,R)} \varphi_E \mathrm{d}x \\ &+ \rho g \int_E x_3 \mathrm{d}x + \rho_0 g \int_{\mathcal{B}(c,R)} x_3 \mathrm{d}x \longrightarrow \min.\end{aligned}$$

in

$$\begin{aligned}\mathcal{C} := \{ &(c,R;E) \,:\, c \in \mathbb{R}^3,\ R \in SO(3), \\ &\text{such that } \mathcal{B}(c,R) \subseteq \Omega \times \mathbb{R}^+; \\ &E \subset \Omega \times \mathbb{R}^+ \text{ measurable set with } E \cap \mathcal{B}(c,R) \neq \emptyset \text{ and} \\ &\mathscr{L}^3(E) = V_0 \,\}\end{aligned}$$

Existence of a minimizer

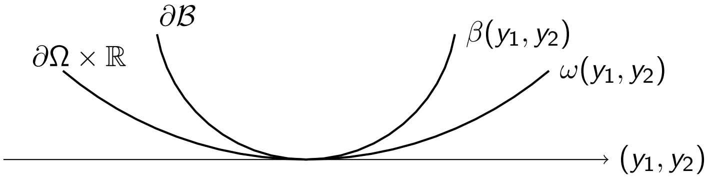
(i) $\mathcal{E}(c, R; E)$ bounded from below on $\mathcal{C}$

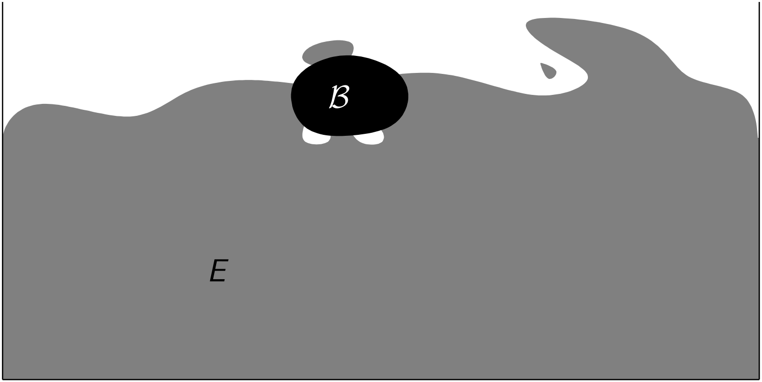
(ii) $\{(c_n, R_n, E_n)\}$ bounded: $|c_n| \leq C_1$, $|R_n| \leq C_2$; $\|\varphi_{E_n}\|_{\mathrm{BV}} \leq C_3$
$\Rightarrow \exists$ subsequence with $\varphi_{E_{n_k}} \to \varphi_{E_0}$ in $L^1(G)$, $k \to \infty$

(iii) $\mathcal{E}$ is lower semicontinuous
with respect to the convergence in (ii)

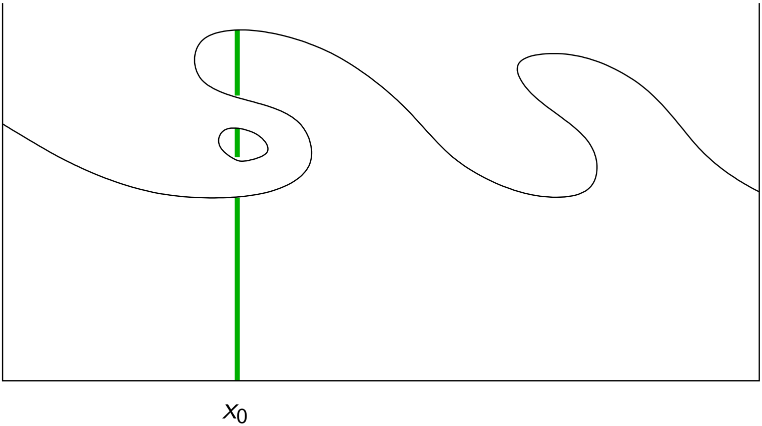
Emmer's Lemma

$$\int_{\partial G} u \, \mathrm{d}\sigma \leq \sqrt{1+L^2} \int_{G(\varepsilon)} |Du| + C_\varepsilon \int_{G(\varepsilon)} |u| \, \mathrm{d}x$$




$\mathcal{B}$
E

Tool from capillarity theory



$$E(x_0) = \{(x, x_3) \in E \ : \ x = x_0\}$$

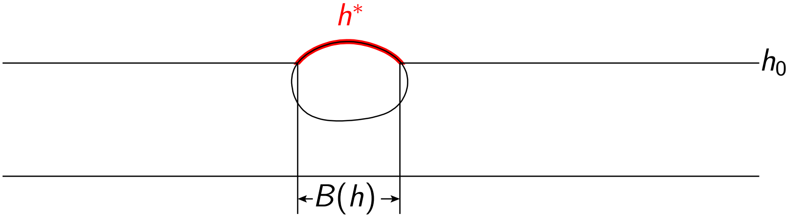
Replace $E(x_0)$ by segment $(0, h(x_0))$ with $h(x_0) = \text{meas}\,(E(x_0))$.
This reduces the energy.

Restriction to the class of graphs

$\mathcal{B}(0,R)$ rigid body; $\exists\ h_0 > 0$, such that

$$\text{meas}\,\{\mathcal{B}(0,R)\cap\{x_3 < h_0\}\,\} = \frac{2}{3}|\mathcal{B}|.$$

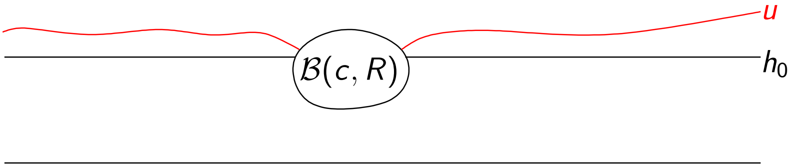
$\mathcal{B}$ strictly convex, hence $\exists$ function $h^* : B(h) \subset \Omega \to \mathbb{R}$ that describes $\partial\mathcal{B}(0,R)\cap\{x_3 > h_0\}$:



Set $h(0,R;x) = \begin{cases} h^*(0,R;x) & \forall x \text{ s. t. } h^*(x) > h_0 \\ h_0 & \text{elsewhere in } \Omega \end{cases}$

Obstacle problem

We then look for graphs u, such that $u(x) \geq h(c, R; x)$:



Properties of u

a) $u \in \mathrm{BV}(\Omega)$

b) $u(x) \geq h(c, R; x) \ \forall \ x \in \Omega$ for some c and R

c) $\int_\Omega u(x)\mathrm{d}x = V_0 + |\mathcal{B}|$

Energy

$$\mathcal{E}(c,R;u) := \sigma \int_{\Omega \cap \{u>h\}} \sqrt{1+|Du|^2} + \kappa \int_{\partial\Omega} u \, \mathrm{d}\sigma$$
$$+ (\rho - \rho_0) g \int_{\mathcal{B}(c,R)} x_3 \mathrm{d}x_3 \mathrm{d}x + \kappa_0 \Big\{ |\mathcal{B}| - \int_{B(h_0)\setminus\{u>h\}} \sqrt{1+|Dh|} \mathrm{d}x \Big\}$$

in

$$\mathcal{C} := \{(c,R;u) \ : \ c, R \text{ such that } \mathcal{B}(c,R) \subseteq \Omega \times \mathbb{R}^+;$$
$$u \text{ satisfies a), b), c)}\}$$

$$\int_{\Omega\setminus\{u>h\}} \sqrt{1+|Du|^2} = \int_{\Omega} \sqrt{1+|Du|^2} + \int_{B(h_0)\cap\{u>h\}} \sqrt{1+|Dh|^2}\ \mathrm{d}x' - \int_{B(h_0)} \sqrt{1+|Dh|^2}\ \mathrm{d}x'$$

Properties of the solution $(c_0, R_0; u_0)$

(i) $\exists\ C$, such that $|u_0(x)| \leq C$

(ii) u_0 is regular in the set $\{x\ :\ u(x) > h(x)\}$

(iii) u_0 meets the obstacle h in a smooth curve $C(u)$ that is contained in $B(h)$, in particular: $u(x) > h_0$, i.e. u_0 never meets the "artificial" obstacle h_0

(iv) u_0 meets h under a constant angle θ with $\cos\theta = -\frac{\kappa_0}{\sigma}$

(v) The projection of the part of $\partial\mathcal{B}$ that is not in contact with the fluid is a simply connected set.

First Variation of the Energy Functional

The first variation of $\mathcal{F}(c, R; u)$ with respect to u gives the Euler-Lagrange equations

$$\sigma \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = \rho g u + \lambda \quad \text{in } \Omega \setminus \overline{B(h)} \tag{1}$$

and the boundary conditions

$$\frac{Du \cdot n}{\sqrt{1 + |Du|^2}} = -\frac{\kappa}{\sigma} \quad \text{on } \partial\Omega \tag{2}$$

$$\frac{1 + Du \cdot Dh}{\sqrt{1 + |Du|^2} \cdot \sqrt{1 + |Dh|^2}} = -\frac{\kappa_0}{\sigma} \quad \text{on } \gamma\ . \tag{3}$$

First variation with respect to motions of $\mathcal{B}$

For capillary surfaces Σ that are given by parametric surfaces $x : B_2(0) \setminus \overline{B_1(0)} \to \mathbb{R}^3$ and for general deformations of the body the first variation of the enrgy has been calculated by J. McCuan. Here we present a much shorter proof for the case that is discussed before; in particular we write the (infinitesimal) Euclidean motions of $\mathcal{B}$ as perturbations of the real function h that describes the upper boundary of $\mathcal{B}$.

For $\mathcal{B}_\varepsilon = \mathcal{B} + \varepsilon \cdot e_3,\ e_3 = (0,0,1)$, we clearly have

$$\varphi(x') = 1\,; \tag{4.1}$$

for $\mathcal{B}_\varepsilon = \mathcal{B} + \varepsilon \cdot e,\ e_3 = (e',0),\ \|e'\| = 1$, we get

$$\varphi(x') = -Dh(x') \cdot e' + \mathcal{O}(\varepsilon) \tag{4.2}$$

because

$$h_\varepsilon(x') = h(x' - \varepsilon e') = h(x') - \varepsilon Dh(x') \cdot e' + o(\varepsilon)\,.$$

For a general rotation about an axis with direction $d = (d_1, d_2, d_3)\,,\ \|d\| = 1$, we have

$$\mathcal{B}_\varepsilon = \{x^\varepsilon \in \mathbb{R}^3 : x^\varepsilon = \cos(\varepsilon)x + (1 - \cos(\varepsilon))(d\cdot x)d + \sin(\varepsilon)d \wedge x, x \in \mathcal{B}\}$$

which gives

$$x^\varepsilon = x + \varepsilon d \wedge x + o(\varepsilon)\,,$$

in particular,

$$\begin{cases} x_1 - \varepsilon d_3 x_2 & = x_1^\varepsilon - \varepsilon\{d_2 h(x') - d_3 x_2\} + o(\varepsilon)\,, \\ \varepsilon d_3 x_1 + x_2 & = x_2^\varepsilon + \varepsilon d_1 h(x') + o(\varepsilon) \end{cases}$$

$$h^\varepsilon(x_1, x_2) = h(x_1 - \varepsilon[d_2 h(x') - d_3 x_2], x_2 - \varepsilon[d_1 h(x') - d_3 x_1])$$

$$\varphi(x') = (d_1 x_2 - d_2 x_1) + Dh(x') \cdot \{(-d_2, d_1)h(x') - (d_3 x_2, d_3 x_1)\}$$

Some domains of integration contain the set $\{u > h_\varepsilon\}$, hence we must write

$$\gamma_\varepsilon = \partial\{u(x_1, x_2) > h(x_1, x_2) + \varepsilon\varphi(x_1, x_2)\}$$

as a perturbation of

$$\gamma = \partial\{u(x_1, x_2) > h(x_1, x_2)\}.$$

Set

$$x' = \xi + t n_\gamma(\xi) \quad \xi \in \gamma, |t| < \varepsilon_0$$

for all x' from a neighborhood of γ, and for $\gamma_\varepsilon = \{\xi + \delta(\xi, \varepsilon) n_\gamma(\xi),\ \xi \in \gamma\}$, we obtain

$$u(\xi + \delta(\xi, \varepsilon) n_\gamma(\xi)) = h(\xi + \delta(\xi, \varepsilon) n_\gamma(\xi)) + \varepsilon\varphi(\xi + \delta(\xi, \varepsilon) n_\gamma(\xi)) \,.$$

From this, δ can be determined:

$$u(\xi) + \delta(\xi, \varepsilon) Du(\xi) \cdot n_\gamma(\xi) = h(\xi) + \delta(\xi, \varepsilon) Dh(\xi) \cdot n_\gamma(\xi) + \varepsilon\varphi(\xi) + o(\varepsilon) \,.$$

Hence,

$$\delta = \delta(\xi, \varepsilon) = \varepsilon \frac{\varphi(\xi)}{(Du(\xi) - Dh(\xi)) \cdot n_\gamma(\xi)} + o(\varepsilon)$$

First Variation of Surface Energies

$$\mathcal{I} = \oint_\gamma \left(-\frac{\sqrt{1+|Du|^2}}{(Du - Dh)\cdot n_\gamma} + \kappa \frac{\sqrt{1+|Dh|^2}}{(Du - Dh)\cdot n_\gamma} \right) \varphi \, \mathrm{d}s \,.$$

$$\mathcal{I} = \oint_\Gamma E \cdot N_0 \, \mathrm{d}s$$

with $E = (0, 0, \varphi(x'))$ and N_0 being the unit vector that is normal to the contact line Γ and lies in the tangent plane to $\Sigma = \mathrm{graph}(u)$.

Equilibrium Condition

$$\sigma \oint_{\Gamma} E \cdot N_0 \, \mathrm{d}s + \rho g \int_{\Sigma_{\mathcal{B}}} -E \cdot N x_3 \, \mathrm{d}\sigma$$
$$- \rho_0 g (e + d \wedge x_s)_3 |\mathcal{B}| = 0 \,,$$

where N is the normal to the floating body $\mathcal{B}$, and $\Sigma_{\mathcal{B}}$ denotes its wetted part.

Literature

J. McCuan: A variational formula for floating bodies. Pac. J. Math. 231 (2007) 167-191

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