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A new stochastic Gompertz diffusion process with threshold parameter: Computational aspects and applications

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Abstract

In this paper we propose a new homogeneous stochastic Gompertz diffusion model with a threshold parameter. This can be considered an extension of the homogeneous three parameter Gompertz process with the addition of a fourth parameter. From the corresponding Kolmogorov equations and Ito's stochastic differential equations, we obtain the transition probability density function and the moments of this process (specifically, the trend functions). The parameters are estimated by considering discrete sampling of the sample path of the model and by using maximum likelihood methodology. Estimation of the threshold parameter requires us to solve a non-linear equation, which is achieved by the Newton–Raphson method. Simulated model data are considered and the methodology in question is applied to estimate the parameters; the values obtained are compared with those used in the simulation. Finally, the model is applied to model the evolution of the trend of the dynamic variable "average monthly salary cost", for all sectors and broken down (construction, industry, services) in Spain, for the period (1985–2005).

Keywords: Gompertz diffusion with threshold parameter; Discrete sampling; Statistical inference; Newton–Raphson method; Simulation; Average monthly salary cost

1. Introduction

In recent decades, the notion of the stochastic diffusion process, defined by means of stochastic differential equations (SDE) or by the Kolmogorov equation, has been used in many fields, including economics, physics, engineering, cybernetics, environmetrics and biology.

The problem of estimating the parameters of the drift coefficient in these models has received considerable attention recently, especially in situations in which the process is observed continuously. The statistical inference is usually based on approximating maximum likelihood methodology. An extensive review of this theory

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can be found in Prakasa Rao [18], and related new work has been done by Bibby and Sorensen [1], Kloeden et al. [16], Singer [19] and others.

A wide variety of stochastic diffusion processes can be found in the literature, both general and specific. One such process is the stochastic Gompertz diffusion process (SGDP). From the point of view of stochastic differential equations, the homogeneous SGDP was introduced by Ricciardi [21] in a theoretical form, and subsequently applied by Ferrante et al. [4] (growth of cancer cells) and by Gutiérrez et al. [12] (consumption of natural gas in Spain). From the perspective of the Kolmogorov equations, the model was defined by Nafidi [17] in a general form, and later applied by Gutiérrez et al. [10] in a study of the stock of motor vehicles in Spain. The non-homogeneous form of the process (with exogenous factors) has been addressed by Nafidi [17] in a very general context. Later, Gutiérrez et al. [13,15] studied the case in which only the growth rate in the drift is affected by exogenous factors in a linear way, and applied this both to the growth in the price of new housing in Spain and to the consumption of electricity in Morocco. Finally, Ferrante et al. [5] considered a non-homogeneous version in which the growth rate is the sum of two exponential functions that are exogenous factors.

As well as these non-homogeneous extensions of the SGDP, it would be useful to possess other extensions for real situations that might arise in various scientific fields. Frank [2] and Frank et al. [3] introduced a Gompertz diffusion process with delay, which was studied on the basis of the generalized Fokker–Planck equations with delays (given by Guillouzic et al. [23]). The present Gompertz model is used in the context of the stochastic system with delays.

With respect to introducing delay into the Gompertz diffusion process, another possibility might be to consider a threshold parameter. Thus, the model described in this paper is an extension of the homogeneous three-parameter SGDP model that is obtained by incorporating a threshold parameter in an analogous way to the procedure for homogeneous lognormal diffusion with two parameters when the threshold parameter is incorporated (see [11,20]).

This parameter influences the dynamic variable under study, as well as its trend functions, and so we can obtain a better fit of the SGDP model to certain real phenomena that naturally present a threshold value in their behaviour pattern, i.e. a non-null minimum value from which the process trajectories evolve in time.

It is a known fact that the transition probability density function (TPDF) of a diffusion process, in general, cannot be expressed in closed form. Fortunately, for the process we propose, this function can be obtained, and thanks to its type (established as the density of a lognormal distribution), it offers the possibility of developing an inferential methodology based on the use of a discrete sampling of its trajectories, which is different from what was studied, for example, by [6,12,14], and which enables us to estimate its trend functions (both conditioned and non-conditioned), which are the necessary tools for fitting and predicting real phenomena.

This paper is organised, henceforth, as follows: in the second section the proposed model is defined on the basis of the corresponding Kolmogorov equations and on that of Ito's stochastic differential equation, to obtain the TPDF and its moments. Subsequently, we estimate the parameters of the model by means of the maximum likelihood model, using discrete sampling of the process. Estimation of the threshold parameter requires us to resolve a non-linear equation, which is done by means of the Newton Raphson method. Section 3 contains a simulation of the exact solution of Ito's SDE which characterises the process, thus illustrating the methodology by the simulation of its trajectories with respect to the theoretical trend function. The simulated process data are used to estimate the parameters of the model using the proposed methodology and these are compared with the true values used for the simulation. In Section 4, we describe an application of the process studied to model the evolution of the trend of the dynamic variable "average monthly salary cost", both overall and by sectors (construction, industry, services) in Spain, using the data base for the period (1985–2005).

2. The model and its characteristics

2.1. The stochastic diffusion Gompertz with threshold parameters

The one-dimensional Gompertz diffusion process with threshold parameter γ can be defined as a Markov process $\{X_t, t \in [t_0, T], t_0 > 0\}$, taking values on $]\gamma, +\infty[$ with almost certainly continuous paths and the infinitesimal moments

$$\begin{aligned} A(x) &= \alpha(x-\gamma) - \beta(x-\gamma)\log(x-\gamma), \\ B(x) &= \sigma^2(x-\gamma)^2, \end{aligned} \tag{1}$$

where $\alpha \in \mathbb{R}$, β, σ and γ are positive real numbers (to be estimated). In the growth population, α is the intrinsic growth rate; β is the deceleration factor and σ is the diffusion coefficient volatility.

Let $f(y, t/x, s)$ be the transition probability density function (TPDF) of this process. The latter, under certain conditions of regularity and with the initial condition $\lim_{t\to s} f(y, t/x, s) = \delta(y-x)$, is the unique solution of the backward (Fokker–Planck) and forward (Kolmogorov) equations

$$\begin{aligned} \frac{\partial f(y,t/x,s)}{\partial t} &= \frac{\partial[A(y)f(y,t/x,s)]}{\partial y} + \frac{1}{2}\frac{\partial^2[B(y)f(y,t/x,s)]}{\partial y^2}, \\ \frac{\partial f(y,t/x,s)}{\partial s} &= -A(x)\frac{\partial f(y,t/x,s)}{\partial x} - \frac{1}{2}B(x)\frac{\partial^2 f(y,t/x,s)}{\partial x^2}. \end{aligned}$$

Alternatively, the above-defined process can be considered as the solution of Itô's stochastic differential equation (SDE)

$$\begin{aligned} \mathrm{d}X_t &= (X_t-\gamma)[\alpha - \beta\log(X_t-\gamma)]\,\mathrm{d}t + \sigma(X_t-\gamma)\,\mathrm{d}W_t, \\ X_{t_0} &= x_0; \quad t \in [t_0, T], \end{aligned} \tag{2}$$

where W_t is a standard Wiener process.

The common solution to the Kolmogorov equations can be obtained using Ricciardi's theorem (see [22]) of the transformation of the diffusion process into the Wiener process. The infinitesimal moments (1) verify the conditions of the theorem cited; therefore such a transform exists and has the following form:

$$\begin{aligned} \Psi(x,t) &= \frac{\beta t}{\sigma}\log(x-\gamma) - \frac{\alpha - \sigma^2/2}{\sigma}\int^t \mathrm{e}^{\beta\tau}\,\mathrm{d}\tau, \\ \Phi(t) &= \int^t \mathrm{e}^{2\beta\tau}\,\mathrm{d}\tau. \end{aligned}$$

From the above, the TPDF for the considered process is

$$f(y,t|x,s) = \frac{1}{y-\gamma}[2\pi\sigma^2\lambda^2(s,t)]^{-1/2}\exp\left(-\frac{[\log(y-\gamma)-\mu(s,t)]^2}{2\sigma^2\lambda^2(s,t)}\right). \tag{3}$$

This transition is the density function of the one-dimensional lognormal distribution: $\Lambda_1(\mu(s,t), \sigma^2\lambda^2(s,t))$ and $\mu(s,t)$ and $\lambda(s,t)$ are given, respectively, by

$$\begin{aligned} \mu(s,t) &= \mathrm{e}^{-\beta(t-s)}\log(x-\gamma) + \frac{\alpha-\sigma^2/2}{\beta}(1-\mathrm{e}^{-\beta(t-s)}), \\ \lambda^2(s,t) &= \frac{1}{2\beta}(1-\mathrm{e}^{-2\beta(t-s)}). \end{aligned}$$

2.2. The moments of the process

Taking into account that the random variable $X_t - \gamma | X_s = x_s$ has the lognormal distribution $\Lambda_1(\mu(s,t), \sigma^2\lambda^2(s,t))$, we have

$$\mathbf{E}((X_t-\gamma)^r | X_s = x) = \exp\left(r\mu(s,t) + \frac{\sigma^2 r^2}{2}\lambda^2(s,t)\right)$$

and by using Newton's binomial expansion, the rth conditional moment of the process is given by

$$\mathbf{E}(X_t^r | X_s = x) = \sum_{k=0}^{r}\binom{r}{k}\gamma^{r-k}\exp\left(k\mu(s,t) + \frac{\sigma^2 k^2}{2}\lambda^2(s,t)\right)$$

from which we obtain the conditional trend function of the process ($r = 1$)

$$\mathbf{E}(X_t|X_s = x_s) = \gamma + \exp\left(\log(x_s - \gamma)\mathrm{e}^{-\beta(t-s)} + \frac{\alpha - \sigma^2/2}{\beta}(1 - \mathrm{e}^{-\beta(t-s)})\right) \times \exp\left(\frac{\sigma^2}{4\beta}\left(1 - \mathrm{e}^{-2\beta(t-s)}\right)\right) \tag{4}$$

and by assuming $P[X_{t_0} = x_0] = 1$, the non-conditional trend function of the process is given by the following expression:

$$\mathbf{E}(X_t) = \gamma + \exp\left(\log(x_0 - \gamma)\mathrm{e}^{-\beta(t-t_0)} + \frac{\alpha - \sigma^2/2}{\beta}(1 - \mathrm{e}^{-\beta(t-t_0)})\right) \times \exp\left(\frac{\sigma^2}{4\beta}(1 - \mathrm{e}^{-2\beta(t-t_0)})\right). \tag{5}$$

2.3. Computation of the parameter estimators

The drift parameters α, β and γ, and the diffusion coefficient σ are estimated by means of the maximum likelihood (ML) method using discrete sampling. Let us consider a discrete sample of the process $(x_1, x_2, \ldots, x_n)$ at the time instants $(t_1, t_2, \ldots, t_n)$, and let us assume, moreover, that the length of the time intervals $[t_{i-1}, t_i]$ $(i = 2, \ldots, n)$ is equal to constant h $(t_i - t_{i-1} = h)$. Under the initial condition $P[X_{t_1} = x_1] = 1$, the associate likelihood function can be obtained by the following expression:

$$L(x_1, \ldots, x_n, a, \beta, \sigma^2, \gamma) = \prod_{j=2}^{n} f(x_j, t_j|x_{j-1}, t_{j-1}).$$

This function tends to infinity when γ tends to $x_{(1)}$, where $x_{(1)} = \inf_{0\leqslant j\leqslant n}(x_j)$.

Using (3), the log-likelihood function can be written as follows:

$$\mathrm{Log}(L) = \frac{-\beta\sum_{j=2}^{n}\left[\log(x_j - \gamma) - \mathrm{e}^{-\beta h}\log(x_{j-1} - \gamma) - \frac{a(1-\mathrm{e}^{-\beta h})}{\beta}\right]^2}{\sigma^2(1 - \mathrm{e}^{-2\beta h})} - \frac{n-1}{2}\log(2\pi) - \frac{n-1}{2}\log(\sigma^2) - \frac{n-1}{2} \times \log\left(\frac{1 - \mathrm{e}^{-2\beta h}}{2\beta}\right) - \sum_{j=2}^{n}\log(x_j - \gamma),$$

where $a = \alpha - \sigma^2/2$.

By applying ML methodology, i.e. deriving with respect to the parameters and setting these derivatives equal to zero, and after various operations, the resulting ML estimators of a and σ^2 and β are found to be

$$\hat{a} = \frac{\hat{\beta}}{(n-1)(1 - \mathrm{e}^{-\hat{\beta}h})}\sum_{j=2}^{n}\left[y_{j,\hat{\gamma}} - \mathrm{e}^{-\hat{\beta}h}y_{j-1,\hat{\gamma}}\right], \tag{6}$$

$$\hat{\sigma}^2 = \frac{2\hat{\beta}}{(n-1)(1 - \mathrm{e}^{-2\hat{\beta}h})}\sum_{j=2}^{n}\left[y_{j,\hat{\gamma}} - \mathrm{e}^{-\hat{\beta}h}y_{j-1,\hat{\gamma}} - \frac{\hat{a}}{\hat{\beta}}(1 - \mathrm{e}^{-\hat{\beta}h})\right]^2, \tag{7}$$

$$\hat{\beta} = \frac{1}{h}\log\left(\frac{\left[\sum_{j=2}^{n}y_{j-1,\hat{\gamma}}\right]^2 - (n-1)\sum_{j=2}^{n}y_{j-1,\hat{\gamma}}^2}{\left[\sum_{j=2}^{n}y_{j,\hat{\gamma}}\right]\left[\sum_{j=2}^{n}y_{j-1,\hat{\gamma}}\right] - (n-1)\sum_{j=2}^{n}y_{j,\hat{\gamma}}y_{j-1,\hat{\gamma}}}\right), \tag{8}$$

where $y_{j,\hat{\gamma}} = \log(x_j - \hat{\gamma})$ and $y_{j-1,\hat{\gamma}} = \log(x_{j-1} - \hat{\gamma})$, for $j = 2, \ldots, n$ and the estimator of the threshold parameter γ is given by the following non-linear equation:

$$\sum_{j=2}^{n}\frac{1}{x_j - \hat{\gamma}} + \frac{2\hat{\beta}}{\hat{\sigma}^2(1 - \mathrm{e}^{-2\hat{\beta}h})}\sum_{j=2}^{n}\left[\frac{1}{x_j - \hat{\gamma}} - \frac{\mathrm{e}^{-\hat{\beta}h}}{x_{j-1} - \hat{\gamma}}\right] \times \left[\log(x_j - \hat{\gamma}) - \mathrm{e}^{-\hat{\beta}h}\log(x_{j-1} - \hat{\gamma}) - \frac{\hat{a}}{\hat{\beta}}(1 - \mathrm{e}^{-\hat{\beta}h})\right] = 0, \tag{9}$$

where $\hat{\gamma} < x(1)$.

3. Simulation studies

The trajectory of the model can be obtained by simulation of the exact solution of the equation (2). This solution can be obtained by means of Itô's formula applied to the transform $e^{\beta t}\log(X_t-\gamma)$, from which we obtain the following equation:

$$d[e^{\beta t}\log(X_t-\gamma)] = (\alpha-\sigma^2/2)e^{-\beta t}\,dt + \sigma e^{-\beta t}\,dW_t.$$

By simplifying and integrating, the solution of the equation (2) leads us to

$$X_t = \gamma + \exp\left(\log(x_s-\gamma)e^{-\beta(t-s)} + \frac{\alpha-\sigma^2/2}{\beta}(1-e^{-\beta(t-s)})\right)\times\exp\left(\sigma\int_s^t e^{-\beta(t-\tau)}\,dW_\tau\right).$$

From this explicit solution, we obtain the simulated trajectories of the process by the following discretising time interval $[t_0, T]$: $t_i = t_0 + ih$, for $i = 0,\ldots,n$ with $h = T/n$ (n is an integer) and taking into account that the random variable in the last expression $\int_s^t e^{-\beta(t-\tau)}\,dW_\tau$ is distributed as a one-dimensional normal distribution $\mathcal{N}_1(0, \int_s^t e^{-2\beta(t-\tau)}\,d\tau)$.

From this simulated process sample, the parameters are estimated by ML using the Newton–Raphson non-linear approach to approximate the value of $\hat{\gamma}$. The parameters of the process are then estimated by applying the method to the simulated data set described previously, which enables us to test the effectiveness of the method.

Table 1 shows the values used in the simulation and the results obtained by estimating the parameters, using the methods described above, implemented using the Mathematica packages, and considering $h = 1$, $n = 30$ and an initial value $x_0 = 0.99$.

Fig. 1 illustrates the process with the behaviour of its theoretical trend function obtained from expression (5) with respect to the trajectories of the process being studied.

4. Application to reals data

In this application, we examine the average salary cost per worker and per month, by sectors (all activities, construction, services and industry) for the period 1986–2005. The following time-dependent random variables are considered:

- $X_1(t)$: average salary cost per worker and per month, for all activities.
- $X_2(t)$: average salary cost per worker and per month, for the construction sector.

Table 1
Estimation based on simulated data

	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\beta}$	$\hat{\gamma}$
Simulation	1	0.0001	0.5	0.5
Estimation	0.999896	0.000130768	0.500062	0.500003

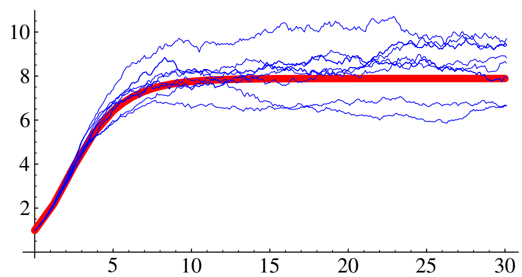

Fig. 1. Simulated trajectories versus the trend function.

- $X_3(t)$: average salary cost per worker and per month, for the industrial sector.
- $X_4(t)$: average salary cost per worker and per month, for the services sector.

The data in Tables 3 and 4 are expressed in euros and were obtained from the Survey of Salaries in Industry and Services for the years 1986–1999, and from the Quarterly Survey of Labour Costs for the years 2000–2005 from the TEMPUS database maintained by the Spanish National Institute of Statistics (INE). This may be consulted at http://www.ine.es/. In the present study, we analyse the trend of the above-mentioned "Average salary cost per worker and per month, by activities" in Spain, by fitting the observed data to the estimated trend function (ETF) and the estimated conditional trend function (ECTF) of a threshold parameter- stochastic Gompertz diffusion model, as introduced in the above paragraphs. The methodology can be summarised in the following two phases:

- Step 1: Use the first 19 data in the series of observations considered to estimate the parameters of the model, using expressions (6)–(9).
- Step 2: For the year 2005, predict the corresponding values for "Average salary cost per worker in Spain" using the ETF and the ECTF, obtained by replacing the parameters with their estimators in expressions (4) and (5), and compare the results with the corresponding observed data for the same year.

A Mathematica program was implemented to carry out the calculations required for this study. The values of the corresponding estimators are given in Table 2.

Table 2
Estimation based on observed data

	$\hat{\mathbf{a}}$	$\hat{\sigma}$	$\hat{\beta}$	$\hat{\gamma}$
$X_1(t)$	1.01257	0.0184793	0.146124	419.901
$X_2(t)$	1.18294	0.0274114	0.173729	428.210
$X_3(t)$	0.89271	0.0194647	0.124331	401.796
$X_4(t)$	1.09802	0.0197937	0.160986	437.634

Table 3
Fit and prediction $X_1(t)$ and $X_2(t)$

Times	$X_1(t)$	ETF	ECTF	$X_2(t)$	ETF	ECTF
1986	546.91	546.91	546.91	469.49	469.49	469.49
1987	589.81	588.50	588.50	495.83	495.73	495.73
1988	634.32	635.25	636.70	529.68	530.32	530.44
1989	675.95	685.97	684.97	569.81	572.76	572.01
1990	736.22	739.31	728.89	628.27	621.82	618.49
1991	799.01	793.94	790.81	693.17	675.72	682.63
1992	864.37	848.61	853.63	748.34	732.47	750.40
1993	921.52	902.24	917.52	801.51	790.12	805.92
1994	967.07	953.95	972.35	843.15	846.94	857.99
1995	1010.82	1003.08	1015.44	887.78	901.54	897.94
1996	1054.68	1049.15	1056.36	929.22	952.91	940.06
1997	1089.75	1091.88	1096.98	964.49	1000.38	978.58
1998	1113.11	1131.13	1129.18	995.80	1043.58	1010.97
1999	1135.88	1166.89	1150.51	1032.76	1082.41	1039.44
2000	1160.15	1199.23	1171.20	1067.43	1116.93	1072.72
2001	1199.97	1228.30	1193.15	1116.61	1147.35	1103.65
2002	1246.61	1254.29	1228.96	1164.62	1173.96	1147.07
2003	1283.62	1277.42	1270.59	1212.36	1197.08	1188.98
2004	1310.13	1297.91	1303.40	1258.59	1217.07	1230.23
Prediction						
2005	1341.26	1316.02	1326.78	1281.21	1234.28	1269.79

Tables 3 and 4 summarise the prediction results, i.e. the observed data, and the values predicted by ETF and ECTF, respectively.

Figs. 2 and 3 show the fits and the predictions made using the ETF and the ECTF, respectively.

Table 4
Fit and prediction $X_3(t)$ and $X_4(t)$

Times	$X_1(t)$	ETF	ECTF	$X_2(t)$	ETF	ECTF
1986	581.08	581.08	581.08	545.20	545.20	545.20
1987	630.05	628.01	628.01	585.95	585.52	585.52
1988	677.86	679.57	681.78	631.89	631.54	632.03
1989	720.39	734.80	732.98	674.62	681.86	682.23
1990	785.01	792.63	777.66	734.83	734.85	727.34
1991	857.00	852.00	844.24	797.02	788.94	788.91
1992	931.23	911.88	916.88	858.59	842.67	850.59
1993	993.77	971.34	990.39	915.38	894.84	910.10
1994	1044.16	1029.60	1051.38	954.85	944.52	963.84
1995	1096.65	1085.98	1099.97	989.78	991.04	1000.63
1996	1149.37	1139.96	1150.12	1027.47	1033.99	1032.84
1997	1204.13	1191.16	1200.04	1053.95	1073.17	1067.25
1998	1233.68	1239.33	1251.46	1077.44	1108.55	1091.24
1999	1258.89	1284.29	1279.04	1098.08	1140.22	1112.38
2000	1284.31	1326.01	1302.47	1131.31	1168.35	1130.87
2001	1331.80	1364.48	1326.02	1168.89	1193.18	1160.46
2002	1393.08	1399.79	1369.81	1212.84	1214.99	1193.66
2003	1432.34	1432.04	1425.92	1248.29	1234.05	1232.17
2004	1470.24	1461.39	1461.66	1269.71	1250.64	1263.00
Prediction						
2005	1508.96	1488.00	1496.01	1303.29	1265.04	1281.53

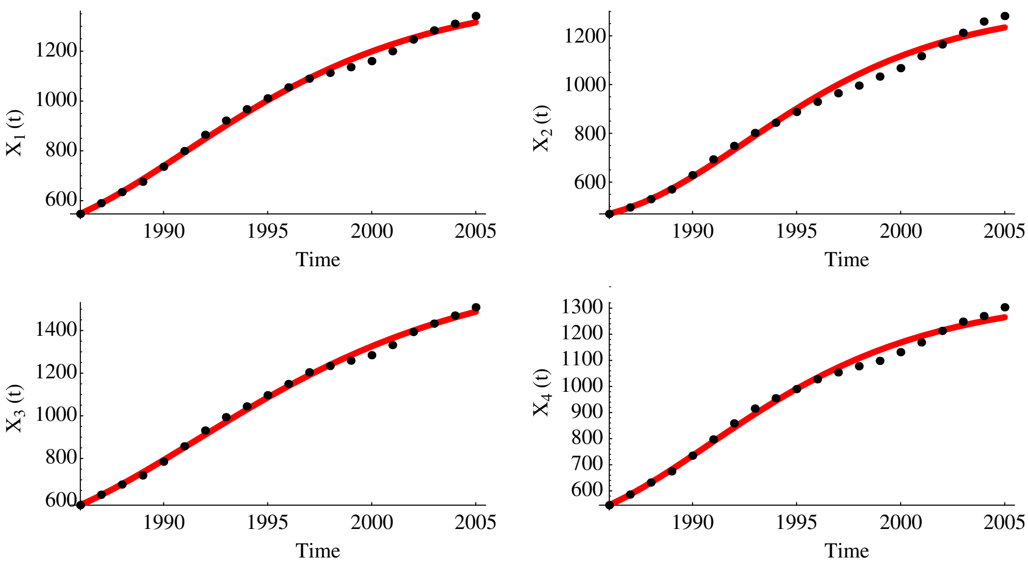

Fig. 2. Fits and forecasts using ETF.

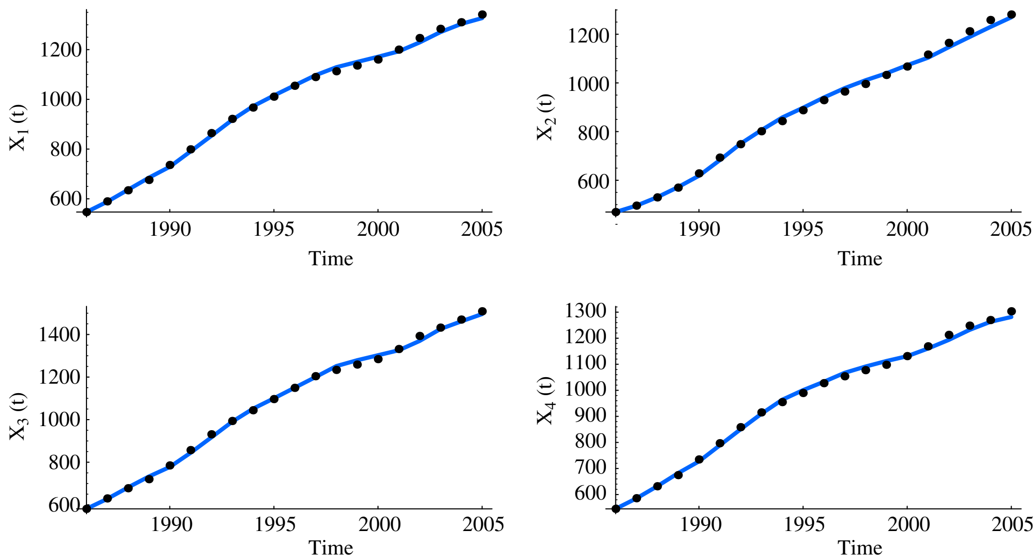

Fig. 3. Fits and forecasts using ECTF.

5. Discussions and conclusions

- From a theoretical standpoint, the main conclusion to be drawn from the present study is that it is possible to utilise a Gompertz homogeneous diffusion process that contains a $\gamma > 0$ parameter in such a way that the process is defined in $]\gamma, \infty[$, and that this extends to the Gompertz process the idea of the existence of a "threshold parameter", which has previously been considered both for probabilistic distributions (for example, in the distributions of extreme values in reliability studies) and in other diffusion processes such as the three-parameter lognormal distribution studied by the present authors (see, for example [11,20]). The process presented in this paper can be studied probabilistically in an explicit way, thus obtaining its transition probability density function and moment functions (trend functions). From the statistical standpoint, we have shown that it is possible to establish an estimation method based on obtaining a conditional likelihood function (Section 2.3) in a natural way, associated with the Markovian nature of the Gompertz process being considered, this function being constructed from a discrete time sampling of the process. We can then derive the maximum likelihood estimators of the four parameters for the model in question, including the threshold parameter. In this respect, we have established a methodology for obtaining the estimators that includes the numerical resolution of the implied non-linear equation.
- From a practical standpoint, we have followed the methodology previously adopted by the present authors for other Gompertz-type diffusion processes with no threshold parameter (see, for example [12,13,15]) for lognormal processes (see [7–9]) and Rayleigh processes (see [14]). The fundamental idea is as follows: on the basis of the discrete observations of the dynamic stochastic variable being considered in a real application, the Gompertz model is fitted using a threshold parameter. This fitting is effected as follows: the four parameters of the model are estimated using the methodology described in Section 2.3. From Zehna's Theorem of the theory of maximum likelihood estimation, the conditional and non-conditional (mean function) trends are estimated; these are given by expressions (4) and (5), with the parameters being substituted by their respective estimators. Finally, these estimated trend functions are used to calculate, for the different time values, the respective fitted values, which are taken as the values assigned to the study variable by the fitted model. The values estimated by the model (the fitted values) are compared with their respective observed values, for the same time periods, for the process being studied.
- In the real case under study, which is of interest for economic studies in countries, like Spain where the Growth Domestic Product increased considerably during the period in question (1985–2005), the following conclusions are drawn: firstly, the consideration of a threshold parameter is found to confer a substantial

advantage over the basic three-parameter Gompertz homogeneous model lacking such a threshold parameter. Thus, the fits obtained (see Figs. 2 and 3 and Tables 3 and 4) present a high degree of matching for all the process variables considered ($X_i: i = 1, 2, 3, 4$) and this is especially so for the fits obtained using ECTF. The fits obtained with the two trend types, ECTF and ETF, can be compared in Tables 3 and 4. In every case, if the same methodology is utilised to obtain fits for the variables $X_i: i = 1, 2, 3, 4$, with the Gompertz model lacking a threshold factor, it is apparent that these fits are not satisfactory. Finally, let us note that, as is the case with other diffusion models, for example the lognormal type, the introduction of exogenous factors (time-continuous factors that affect the drift) into the Gompertz model we are studying, produces non-homogeneous extensions of the model, which may in turn improve the fits for real phenomena, as the possible effects of exogenous variables on the endogenous ones are taken into account.

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