Monografías del Seminario Matemático García de Galdeano **31**, 347–353 (2004)

INFERENCE IN THE STOCHASTIC GOMPERTZ DIFFUSION MODEL WITH CONTINUOUS SAMPLING

R. Gutiérrez, A. Nafidi and R. Gutiérrez Sánchez

Abstract. In the present paper, we approach the stochastic Gompertz diffusion process (SGDP) from the point of view of Itô's stochastic differential equations. The stochastic model is solved analytically by applying Itô's calculus and the mean value of the proposed process is calculated. The parameter estimators are then derived by means of two procedures: the first is used to estimate the parameters in the drift coefficient by the maximum likelihood principle, based on continuous sampling, and the second procedure approximates the diffusion coefficient. Finally, a simulation of the process is presented. Thus, a typical simulated trajectory of the process and its estimators is obtained.

Keywords: Gompertz diffusion process, Stochastic differential equation, Maximum likelihood in diffusion processes, Simulation.

AMS classification: 60J60, 62M05

§1. Introduction

The stochastic diffusion process is of great interest to investigators in many fields, such as biology, physics, demographics and economics,the process generally being defined by means of stochastic differential equations. The problem of estimating parameters of the drift coefficient has received considerable attention in recent years, especially in situations in which the process is observed continuously. In most cases, the statistical inference is based on approximating the maximum likelihood methodology, an extensive review of which can be found in Prakasa Rao [16], while new studies have been published by Bibby and Sorensen [2], Kloeden et al. [9], Singer [14] and others. A wide variety of stochastic diffusion processes have been described, both in general and in specific texts, one such being the stochastic Gompertz diffusion process (SGDP). The deterministic case of this process (the Gompertz growth curve) has been the object of many studies. A stochastic version of this, as a birth and death process, was introduced by Pajenshu [15] and Tan [17], and applied by Troynikov [19], Miller et.al [10]. The SGDP version was applied by Ricciardi [12] in population growth by adding white noise fluctuation to the intrinsic fertility of a population; it has also been used by Dennis and Patil [4] in ecology modelling. Finally, we should mention the recent extension of this process by Frank [5] to the case of SGDP with delay. This paper is organized as follows: in the second section, the analytical expression and the mean value of the SGDP are obtained. In the third, the estimators of the parameters in the drift coefficient are derived by the maximum likelihood principle and the diffusion coefficient estimator is approximated. The final section presents a simulation of the process and its parameter estimators.

§2. The model and the mean value of SGDP

2.1. The model

We consider a one-dimensional stochastic differential equation (SDE):

$$dX(t) = a(t, X(t))dt + b^{1/2}(t, X(t))dW_t\,; \qquad X(0) = x_0 \tag{1}$$

where $\{W_t, t \in [0, T]\}$ is a one-dimensional Wiener process, with an independent increment $W_t - W_s$ normally distributed with mean $\mathbb{E}(W_t - W_s) = 0$ and variance $\mathrm{Var}(W_t - W_s) = t - s$, for $t \geq s$, and x_0 is a fixed real $(x_0 \in \mathbb{R}^*_+)$. We assume that, for all $t \in [0, T]$, $x \in (0, \infty)$; $a(t, x)$ and $b(t, x)$ are functions with values in $\mathbb{R}$, given respectively by

$$\begin{aligned} a(t, x) &= \alpha x - \beta x \log(x) \\ b(t, x) &= \sigma^2 x^2 \end{aligned}$$

where σ, α and β are real parameters. After substitution, we obtain the following SDE:

$$dX(t) = (\alpha X(t) - \beta X(t) \log X(t))\, dt + \sigma X(t) dW_t\,; \qquad X(0) = x_0 \tag{2}$$

Considering the analytical properties of $a(t, x)$ and $b(t, x)$, it follows that the SDE (2) has a unique solution $\{X(t), t \in [0, T]\}$ which is a $(0, \infty)$ - valued diffusion process with an initial value x_0, a drift coefficient $a(t, x)$ and a diffusion coefficient $b(t, x)$ (cf. [1]) The process $\{X(t), t \in [0, T]\}$ is called a one-dimensional SGDP (known in the literature as the Stochastic Gompertz Growth Model, cf. [12] and [5])
For $\beta = 0$, $X(t)$ is the Lognormal process (cf. [18])
If $\alpha = 0$, we obtain the Skiadas et al. [13] version of the SGDP.

2.2. Analytic solution of the SGDP

By means of the appropriate transformation of the form $Y(t) = e^{\beta t} \log(X(t))$, and by using the Itô rule, the SDE (2) becomes

$$\begin{aligned} dY(t) &= (\gamma dt + \sigma dW_t)\, e^{\beta t} \\ Y(0) &= \log x_0 \end{aligned}$$

where $\gamma = \alpha - \dfrac{\sigma^2}{2}$, by evaluation of the integral, and we find that the solution of the last SDE has the following form:

$$Y(t) = \log x_0 + \frac{\gamma}{\beta}\left(e^{\beta t} - 1\right) + \int_0^t \sigma e^{\beta s} dW_s$$

Finally, we deduce that the solution of the original SDE (2) is :

$$X(t) = \exp\left(\log(x_0) e^{-\beta t}\right) \exp\left\{\frac{\gamma}{\beta}\left(1 - e^{-\beta t}\right)\right\} \exp\left\{\sigma \int_0^t e^{-\beta(t-s)} dW_s\right\} \tag{3}$$

2.3. Mean value of the (SGDP)

The mean value of SGDP is given by the following expression:

$$\mathbb{E}(X(t)) = \exp\left\{\log(x_0)e^{-\beta t}\right\} \exp\left\{\frac{\gamma}{\beta}\left(1-e^{-\beta t}\right)\right\} \mathbb{E}\left(\exp\left\{\sigma\int_0^t e^{-\beta(t-s)}dW_s\right\}\right)$$

The random variable in the last expression is normally distributed with mean zero and variance $\sigma^2 \displaystyle\int_0^t e^{-2\beta(t-s)}ds$, and so its expectation can be calculated using the Gardiner [4] relation

$$\mathbb{E}\left(\exp\left\{Z_t\right\}\right) = \exp\left\{(1/2)\mathbb{E}({Z_t}^2)\right\}$$

where Z_t is a zero-Gaussian random process.
Then

$$\mathbb{E}\left(\exp\left\{\sigma\int_0^t e^{-\beta(t-s)}dW_s\right\}\right) = \exp\left\{\frac{\sigma^2}{2}\int_0^t e^{-2\beta(t-s)}ds\right\}$$

After substitution, we obtain the final form of the mean value of SGDP

$$\mathbb{E}\left(X(t)\right) = \exp\left\{\log(x_0)e^{-\beta t}\right\} \exp\left\{\frac{\gamma}{\beta}\left(1-e^{-\beta t}\right)\right\} \exp\left\{\frac{\sigma^2}{4\beta}\left(1-e^{-2\beta t}\right)\right\}$$

§3. Parameter estimation

In this section, two methods are presented to estimate SGDP parameters: the first estimates the drift parameters α and β by the maximum likelihood principle, and the second approximates the diffusion coefficient σ^2 (the white noise). We first provide a brief review of the theory of the equivalence of the Radom - Nikodym probability measure induced by one class of diffusion process.

3.1. Equivalence of the Radom - Nikodym probability measure

Here, we restrict our attention to the one-dimensional diffusion process defined by a class of SDE (1). We assume that this equation can be written in the following form:

$$dX_t = A_t(X_t).\theta dt + B_t(X_t)dW_t\,; \qquad 0 \le t \le T \tag{4}$$

where the parameter $\theta \in \mathbb{R}^k$, A_t is k- dimensional vector and B_t is $\mathbb{R}$- valued depending only on the sample path up to the given instant. We assume that equation (4) has a unique solution for every θ. Let C_T be the set of continuous functions f: $[0,T] \to \mathbb{R}$ and let $\mathcal{B}_T$ be the σ-algebra generated by the sets :

$$A_{B,t} = \{f \in C_T; f(t) \in B;\ t \in [0,T]; B \in \mathcal{B}\}$$

where $\mathcal{B}$ is the σ-algebra of Borel in $\mathbb{R}$

P_θ denotes the probability measure induced in the measurable space $(C_T, \mathcal{B}_T)$ by the diffusion process solution of (4), if we denote the observed trajectory by X_0^T, then the following result ensures that the probability measures P_θ for all θ are equivalent and gives us an expression for the Radom - Nikodym derivative.

Proposition
Let θ_0 denote a fixed value of the parameter; the probability measures P_θ and P_{θ_0} for all θ are equivalent and

$$\frac{dP_\theta}{dP_{\theta_0}}(X_0^T) = \exp\left\{(\theta-\theta_0)^* V_T - \frac{1}{2}(\theta-\theta_0)^* J_T(\theta+\theta_0)\right\}$$

where V_T is the following k- dimensional vector:

$$V_T = \int_0^T A_t^*(X_t)(B_t(X_t)B_t(X_t))^{-1} dX_t \tag{5}$$

J_T is the $k \times k$- matrix:

$$J_T = \int_0^T A_t^*(X_t)(B_t(X_t)B_t(X_t))^{-1} A_t(X_t) dt \tag{6}$$

where the star denotes the transpose.
For the proof we refer reader to Prakasa Rao [16]

3.2. Maximum likelihood principle

Suppose we continuously observe a trajectory of a process which we know solves (5) in the interval $[0, T]$, we seek to infer the true value of the parametric vector θ. For this purpose, let us consider the likelihood function

$$_T(X_0^T, \theta) = \frac{dP_\theta}{dP_{\theta_0}}(X_0^T) = \exp\left\{(\theta-\theta_0)^* V_T - \frac{1}{2}(\theta-\theta_0)^* J_T(\theta+\theta_0)\right\}$$

The estimator obtained by maximizing the equation is the following

$$V_T - J_T\hat{\theta}_T = 0$$

and from the last equation, it is obtained that, when it exists, the desired estimator

$$\hat{\theta}_T = {J_T}^{-1} V_T \tag{7}$$

which can be observed is independent of θ_0.

3.3. Estimation of drift parameters

SDE (2) can be written in the following vectorial form:

$$dX_t = A_t(X_t).\theta dt + B_t(X_t) dW_t$$

where:

$$\theta^* = (\alpha, -\beta) \quad , \quad A_t(X_t) = (X_t, X_t \log(X_t)) \;\text{ and }\; B_t(X_t) = \sigma X_t \tag{8}$$

The random matrix J_T is a 2×2-matrix and can be obtained by inserting equation (8) into (6)

$$J_T = \frac{1}{\sigma^2}\begin{pmatrix} T & \int_0^T \log(X_t)dt \\ \int_0^T \log(X_t)dt & \int_0^T \log^2(X_t)dt \end{pmatrix}$$

The random vector V_T in this case is 2-dimensional and can be obtained by substituting equation (8) into (5)

$$V_T^* = \frac{1}{\sigma^2}\left(\int_0^T \frac{dX_t}{X_t}, \int_0^T \frac{\log(X_t)}{X_t}dX_t\right) \tag{10}$$

After some calculation (not shown), we obtain the expressions of the estimators

$$\hat{\alpha}_T = \frac{\left(\int_0^T \log^2(X_t)dt\right)\left(\int_0^T \frac{dX_t}{X_t}\right) - \left(\int_0^T \log(X_t)dt\right)\left(\int_0^T \frac{\log(X_t)}{X_t}dX_t\right)}{T\int_0^T \log^2(X_t)dt - \left(\int_0^T \log(X_t)dt\right)^2}$$

$$\hat{\beta}_T = \frac{\left(\int_0^T \log(X_t)dt\right)\left(\int_0^T \frac{dX_t}{X_t}\right) - T\left(\int_0^T \frac{\log(X_t)}{X_t}dX_t\right)}{T\int_0^T \log^2(X_t)dt - \left(\int_0^T \log(X_t)dt\right)^2}$$

The Itô integrals in expression (10) can be calculated by using the Itô formula, hence

$$\int_0^T \frac{dX_t}{X_t} = \log\left(\frac{X_T}{X_0}\right) + \frac{\sigma^2}{2}T$$

$$\int_0^T \frac{\log(X_t)}{X_t}dX_t = \frac{1}{2}\left(\log^2(X_T) - \log^2(x_0)\right) - \frac{\sigma^2}{2}T + \frac{\sigma^2}{2}\int_0^T \log(X_t)dt$$

The resulting maximum likelihood estimators then give

$$\hat{\alpha}_T = \frac{\left(\log\left(\frac{X_T}{x_0}\right) + \frac{T\sigma^2}{2}\right)\int_0^T \log^2(X_t)dt - \frac{\sigma^2}{2}\left(\frac{\log^2(X_T)-\log^2(x_0)}{\sigma^2} - T + \int_0^T \log(X_t)dt\right)\int_0^T \log(X_t)dt}{T\int_0^T \log^2(X_t)dt - \left(\int_0^T \log(X_t)dt\right)^2}$$

$$\hat{\beta}_T = \frac{\left(\int_0^T \log(X_t)dt\right)\left(\log(X_T) - \log(x_0)\right) - \frac{T}{2}\left(\log^2(X_T) - \log^2(x_0) - T\sigma^2\right)}{T\int_0^T \log^2(X_t)dt - \left(\int_0^T \log(X_t)dt\right)^2}$$

3.4. Estimation of the noise coefficient

The coefficient σ can be estimated by using an extension of the procedure proposed by Chesney and Elliot [3] for estimating the coefficient diffusion for a linear SDE with multiplicative noise to the case of a non linear SDE with multiplicative noise. The method is the same as in Katsamaki and Skiadas [8] and the resulting estimator has the following form:

$$\hat{\sigma}_T = \frac{1}{T-1}\sum_{t=2}^{T} \frac{\mid X_t - X_{t-1} \mid}{(X_t X_{t-1})^{\frac{1}{2}}}$$

§4. Simulation

In order to obtain an illustration of this model, we simulate the SGDP in $[0, T]$. We consider equidistant time discretizations of the interval $[0, T]$, with time points $t_n = nh$ and step size $h = T/n_T$ for an integer n_T, and $n = 0, 1, \ldots, n_T$. Assuming parameters values $\alpha = 2$, $\beta = 1$, $\sigma = 1$, $T = 2000$, $h = 2^{-4}$, starting at $x_0 = 2$, and approximating the Riemann integral by a trapezoid formula, a typical trajectory of the process is obtained using MATLAB package. Estimates for α, β and σ over $[0, T]$, as well as the trajectory generated restricted to the initial segment $[0, 10]$, are displayed in (Figure 1).

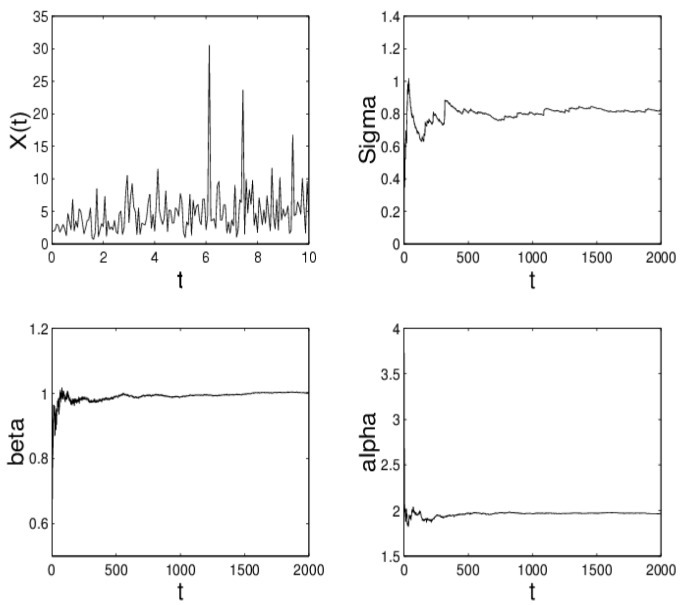

Figure 1: A typical trajectory of the SGDP and evolution of the estimates calculated form it for the parameters α, β and σ.

References

[1] L.Arnold. Stochastic differential equations, John Wiley and Sons. (1973).

[2] B.M.Bibby and M.Sorensen. Martingale estimation functions for discretely observed diffusion processes. *Bernoulli*. **1(1/2)**.(1995).017-039.

[3] M.Chesney and R.J.Elliot. Estimating the Volatility of Exchange Rate in Proceedings of the sixth international symposium on Applied Stochastic Models and Data Analysis. *Wordl Scientific*. Chania, Greece.(1993).131-135,.

[4] B.Dennis and G.P.Patil. Application in Ecology, in Crow and Shimizu. Lognormal Distributions: Theory and Applications, chap.12.(1988).

[5] T.D.Frank. Multivariate Markov processes for stochastic systems with delays: application to the stochastic Gompertz model with delay. *Physical Review E*, **66** (1),011914.(2002).

[6] C.W.Gardiner. Handbook of stochastic methods for physics, chemistry and natural sciences, 2nd edn Springer Verlag, Berling, Germany.(1990).

[7] R.Gutiérrez, J.M.Angulo , A.González and R.Pérez. Inference in lognormal multidimensional diffusion processes with exogenous factors. Application to modelling in economics. *Applied Stochastic Models and Data Analysis*. **7**.(1991).295-316.

[8] A.Katsamaki, C.H.Skiadas. Analytic solution and estimation of parameters on a stochastic exponential model for a technological diffusion process: Applications. *Applied Stochastic Models and Data Analysis*. **11**.(1995).59-75.

[9] P.Kloeden, E.Platen, H.Schurz and M. Sorensen. On effects of discretization on estimators of drift parameters for diffusion processes. *Journal of Applied Probability*, **33**, (1996),1061-1076.

[10] J.H.Miller, R.A.Minard, G.A.Orner and B.BullL. In vivo MRI measurments of tumor growth induced by dichloroacetate: implications for mode of action, *Toxicology* **145**.(2000).115-125.

[11] A.Nafidi. Ph.D. Difusiones Lognormales con factores exogenos, Extensiones a partir del proceso de difusión de Gompertz. Universidad de Granada.(1997).

[12] L.M.Ricciardi. Diffusion process and related topic in biology, Lectures Notes in Biomathematics, 14, Springer Verlag.(1977).

[13] C.H.Skiadas, A.N.Giovani and I. Dimoticales. Investigation of stochastic differential models : The Gompertzian case. Selected Topics on Stochastic Modelling, World Scientific Singapore.(1994).296-310.

[14] H.Singer. Parameter estimation of nonlinear stochastic differential equations: Simulated Maximum likelihood versus extended Kalman filter and Itô- Taylor expansion, *Journal of Computational and Graphical Statistics*. **11**.(2002). 972-995.

[15] Prajneshu. Diffusion approximations for models of population growth with logarithmic interactions. *Stochastic Processes and their Applications* **10**. Issue 1.(1980).087-099.

[16] B.L.S.Prakasa Rao. Statistical inference for diffusion type processes, Indian Statistical Institute, New Delhi.(1999).

[17] W.Y.Tan. A stochastic Gompertz birth-death process. *Statistics and Probability Letters*. **4**.(1986).025-028.

[18] G.Tintner and J. K. Sengupta. Stochastic Economics, Press. New York and London.(1972).

[19] V.S.Troynikov, R.W.Day and A.M.Leorke. Estimation of seasonal growth parameters using a stochastic Gompertz model for tagging data. *Journal of Shellfish Research*, **17** (3). 833-838.(1998).

R. Gutiérrez, A. Nafidi, R. Gutiérrez Sánchez
Departamento de Estadística e Investigacíon Operativa
Universidad de Granada
rgjaimez@ugr.es & nafidi@hotmail.com & ramongs@ugr.es