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A bivariate stochastic Gompertz diffusion model: statistical aspects and application to the joint modeling of the Gross Domestic Product and CO_2 emissions in Spain

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SUMMARY

In this paper we propose a bivariate stochastic Gompertz diffusion model as the solution for a system of two Itô stochastic differential equations (SDE) that are similar as regards the drift and diffusion coefficients to those considered in the univariate Gompertz diffusion model, which has been the object of much study in recent years. We establish the probabilistic characteristics of this model, such as the bivariate transition density, the bidimensional moment functions, the conditioned trend functions and in particular, the correlation function between each of the components of the model. We then go on to study the maximum likelihood estimation of the bidimensional drift and the diffusion matrix of the diffusion in question, proposing a computational statistical methodology for this purpose based on discrete observations over time, for both components of the model. By these means we are able to achieve the maximum likelihood estimation of the trend and correlation functions and thus establish a method for trend analysis, which we apply to the real case of two dependent variables, Gross Domestic Product (GDP) and CO_2 emission in Spain, the joint dynamic evolution of which is modeled by the proposed Gompertz bidimensional model. This implementation is carried out on the basis of annual observations of the variables over the period 1986–2003. The application is a new methodology in environmental and climate change studies, and provides an alternative to other approaches of a more econometric nature, or those corresponding to the methodology of secular trends in Time Series. Copyright © 2008 John Wiley & Sons, Ltd.

KEY WORDS: bivariate Gompertz diffusion process; marginal trend and correlation functions; likelihood estimation; climate change; CO_2 emission and GDP in Spain

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1. INTRODUCTION

1.1. Background: the univariate stochastic Gompertz diffusion model

Stochastic diffusion processes are of great interest to investigators in many fields, such as biology, physics, demography, economics and environmental sciences, and these processes are generally defined

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by means of stochastic differential equations. The problem of estimating the parameters of the drift coefficient has received considerable attention in recent years, especially in situations in which the process is observed continuously. In most cases, the statistical inference is based on approximating the maximum likelihood methodology, an extensive review of which can be found in Rao (1999), while new studies have been published by Bibby and Sorensen (1995), Kloeden *et al*. (1996), and Singer (2002).

In recent decades, various diffusion-type stochastic models have been developed and successfully applied to the fitting and prediction of real phenomena. These models include stochastic diffusion processes such as lognormal (Gutiérrez *et al*., 1991, 1997), Bass (Skiadas and Giovani, 1997), Rayleigh (Gutiérrez *et al*., 2006c), Gompertz (Gutiérrez *et al*., 2006a, 2005b), Logistic (Giovanis and Skiadas, 1999), cubic (Gutiérrez *et al*., 2007a) and inverse CIR (Gutiérrez *et al*., in press b), among others. In the present paper, we examine, in particular, the stochastic Gompertz case. The deterministic case of this process (the Gompertz growth curve) has been the object of many studies and applications. A stochastic version, as a birth and death process, was introduced by Prajneshu (1980) and Tan (1986), and applied by Troynikov *et al*. (1998) and by Miller *et al*. (2000).

From the point of view of stochastic differential equations, the homogeneous stochastic Gompertz diffusion process (SGDP) was introduced by Ricciardi (1977) in a theoretical form, and then applied to the question of population growth by adding white noise fluctuation to the intrinsic fertility of a population. Subsequently, this diffusion was also considered by Ferrante *et al*. (2000) (growth of cancer cells) and by Gutiérrez *et al*. (2005a) (consumption of natural gas in Spain). From the perspective of the Kolmogorov equations, the model was defined by Nafidi (1997) in a general form, and later applied by Gutiérrez *et al*. (2004b) in a study of the stock of motor vehicles in Spain. The non-homogeneous form of the process (with exogenous factors) has been addressed by Nafidi (1997) in a very general context. Later, Gutiérrez *et al*. (2006a, 2005b) studied the case in which only the growth rate in the drift is affected by exogenous factors in a linear way, and applied this both to the increase in the price of new housing in Spain and to the consumption of electricity in Morocco. Also, Ferrante *et al*. (2005) considered a non-homogeneous version in which the growth rate is the rate, which is the sum of two exponential functions that are exogenous factors.

More recently, Albano and Giorno (2006) considered a stochastic model of solid tumor growth, of a univariate Gompertz type, generalizing the deterministic Gompertz growth models that are widely used in cell growth studies. The drift of the basic stochastic Gompertz model they propose is considered to be affected by a deterministic time function that models the therapeutic effect (reduction of growth) achieved by a given treatment. This model is a particular case of the non-homogeneous univariate Gompertz models (with exogenous factors) discussed in Gutiérrez *et al*. (2006a, 2005b). Albano and Giorno (2006) expressly cite the statistical methodology established for the stochastic Gompertz process by Gutiérrez *et al*. in their studies.

Other recent contributions concerning the univariate Gompertz process include a probabilistic and statistical study of a Gompertz model with a threshold parameter (see Gutiérrez *et al*., 2006b) and its application to occupational data. See also Frank (2002), who considered a Gompertz process with delay, modeling typical situations in population theory, which was studied using a methodology based on multidimensional Kolmogorov (Fokker–Planck) equations associated with a process with delay, such that the cited univariate Gompertz case was found to be one of the few cases of diffusions for which the methodology proposed produces explicit results. Subsequently, Patanarapeelert *et al*. (2005) expanded the results of Frank (2002), expressly citing the contributions of statistical inference to the non-homogeneous Gompertz made by Gutiérrez *et al*. (2005b). Finally, let us also acknowledge the recent study of a type of Gompertz diffusion in which the bound can depend on the initial value, a situation that is not provided by the models considered to date (see Gutiérrez *et al*., 2007b).

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Statistical inference for the (stochastic diffusion) univariate Gompertz model has been comprehensively studied in Gutiérrez *et al*. (2008), with a discrete and continuous sampling. Various extensions of univariate Gompertz diffusion, to the non-homogeneouscase, have been discussed (see Gutiérrez *et al*., 2005a, 2005b). These results concerning inference and its application to real cases are discussed in the study by Meade and Islam (2006).

1.2. Gross Domestic Product and CO_2 emissions: the case of Spain

In recent years, many papers on climate change have been published, discussing the relation between economic growth (measured in terms of Gross Domestic Product—GDP) and either CO_2 emissions or energy consumption. Recent investigations have been made into the relations, at a worldwide level, between the three factors: CO_2 emissions, GDP, and energy consumption. For example, Ramanathan (2006), with data for the period 1980–2001, and using Data Envelopment Analysis (DEA) methodology, made a macrolevel analysis of the interrelations between these three variables. In this respect, the following comment by this author (work cited, p. 492) is noteworthy: "While the analysis at the world level will be of interest to research bodies such as the United Nations or the IPCC (Intergovernmental Panel on Climate Change), analysis at various geographic levels could provide information of interest to individual nations/regions." Such is the case of Spain, as addressed in the present paper. Spain is the EU country that fails by the greatest margin to comply with the Kyoto Protocol, while with respect to economic growth, as a developed country (in the EU, it has recorded the highest sustained rate of growth in recent years, with 3.9% in 2006 and a forecast of 4% for 2007), this country constitutes a paradigmatic case for examining the relation between GDP and emissions of CO_2 (and of greenhouse gases in general). A more detailed description of the particular features of the situation of Spain, in this respect, can be found in Gutiérrez *et al*. (in press a, b).

There is an undeniable correlation between economic development and CO_2 emission (see, e.g., Aldy, 2006, p. 534). One of the questions we address is whether it is possible to quantify the dynamic correlation between GDP and CO_2 emission in Spain, on the basis of prior modeling, and using a stochastic two-dimensional model (GDP–CO_2 emission).

The problem of the possible interdependence between the global emission of CO_2 and GDP in Spain was recently addressed, to some extent, in Gutiérrez *et al*. (in press a) and the case of the variable CO_2, considered alone, also in Gutiérrez *et al*. (in press b), by using a cubic diffusion process. In the first of these references, the two dynamic variables are considered, with the study focusing on modeling the variable CO_2, taken as an endogenous variable, by means of a non-homogeneous univariate Gompertz model, such that the GDP acts as an exogenous variable that varies outside the system defined by the endogenous variable CO_2 which it, in turn, affects in the course of its own evolution over time. Technically, the above-mentioned effect is modeled via the inclusion in the drift of the baseline endogenous diffusion of a deterministic function that describes the evolution in time of the exogenous variable in question. This leads us to consider a stochastic, non-homogeneous diffusion model (the drift being affected by a time function), which in Gutiérrez *et al*. (in press a) is, in particular, taken as a Gompertz-type model. For this non-homogeneous Gompertz model, Gutiérrez *et al*. (in press a) established results based on statistical inference that were based, in particular, on time-discrete observations, both of CO_2 emission (the exogenous variable) and of GDP (the endogenous variable). These results enabled us to obtain maximum likelihood estimators of the parameters of this diffusion, together with a com, and as suggested in the above-mentioned studies, it is necessary to carry out a two-dimensional analysis of the joint evolution, within a random environment, of the GDP and the CO_2 emissions, considering both of these as endogenous variables. This constitutes a more realistic approach

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to the situation, because in the above study, the evolution of CO_2 emissions is modeled with respect to the dynamic behavior of GDP. One might, on the contrary, propose modeling GDP in terms of CO_2. In our opinion, each of these possibilities only represents a partial view of reality. Therefore, we propose a joint, bidimensional modeling, addressing the two variables as interrelated components, such that they vary in a correlated way in their stochastic evolution.

1.3. Aims of the study

In the theoretical–practical context defined in Subsections 1.1 and 1.2, we may stipulate the fundamental goal of the present paper, which is to propose a homogeneous bivariate Gompertz diffusion model and to determine the corresponding results on statistical inference to enable it to be fitted to real data, particularly to CO_2–GDP bidimensional data. In the case of lognormal diffusion, which is technically a special case of the Gompertz model, we have made an extensive study of the multivariate version and of the corresponding statistical inference (estimation and test of hypotheses), see, for example, Gutiérrez *et al*. (1991, 1997, 2004a). As regards the multivariate Gompertz model, we are unaware of any such theoretical studies, or of applications to real cases.

2. BIVARIATE STOCHASTIC GOMPERTZ DIFFUSION PROCESS AND ITS CHARACTERISTICS

2.1. The model and its analytical expression

Let $\{x(t) = (x_1(t), x_2(t))'; t \in [t_0, T]; t_0 \geq 0\}$ be a two-dimensional stochastic process that satisfies the following Ito's SDE:

$$\mathrm{d}x(t) = a(t, x(t))\mathrm{d}t + b(t, x(t))\mathrm{d}w(t); \quad P\left[x(t_0) = x_{t_0}\right] = 1 \tag{1}$$

where $\{w(t); t \in [t_0, T]\}$ is a two-dimensional standard Wiener process, x_{t_0} is a fixed vector belonging to $(0, \infty)^2$, and for $x = (x_1, x_2)' \in (0, \infty)^2$, the vector $a(t, x)$ and the matrix $b(t, x)$ are given as follows:

$$\begin{aligned} a(t, x) &= (a_1 x_1 - \beta x_1 \log(x_1); a_2 x_2 - \beta x_2 \log(x_2))' \\ b(t, x) &= D(x) B^{1/2} \end{aligned} \tag{2}$$

in which $D(x)$ is a diagonal matrix where the elements of the principal diagonal are x_1 and x_2 and where $B = (b_{ij})_{i,j}$ is a 2×2 symmetric non negative definite matrix. The parameters a_1, a_2, β, and $b_{i,j}$ for $1 \leq i, j \leq 2$ are real and to be estimated.

The existence and the uniqueness of the strong solution of the SDE (1) (for a discussion on strong and weak solutions, see, e.g., Chung and Willians, 1990 pp. 243–246) with the infinitesimal moments specified in Equation (2) can be justified as follows: by a state transform of the proposed process, the SDE given can be transformed into one that confirms the conditions of existence and of uniqueness of the strong solution to the SDE (see, e.g., Arnold, 1973) and the sole strong solution to which is the two-dimensional Wiener process. Specifically, this is a transform of the type $y(t) = \mathrm{e}^{\beta t} \log(x(t))$, and therefore the SDE (1) (with moments given in Equation (2)) has a unique strong solution $x(t) =$

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$\exp(\mathrm{e}^{-\beta t}Y(t))$ and is a bivariate Gompertz diffusion process with a drift vector $A(x)$ that is given by

$$A(x) = (a_1 x_1 - \beta x_1 \log(x_1); a_2 x_2 - \beta x_2 \log(x_2))'$$

and the matrix diffusion $B(x)$ is given by

$$B(x) = \left(D(x)B^{1/2}\right)\left(D(x)B^{1/2}\right)' = D(x)BD(x) = \begin{pmatrix} b_{11}x_1^2 & b_{12}x_1x_2 \\ b_{12}x_1x_2 & b_{22}x_2^2 \end{pmatrix}$$

The analytical expression of the process $\{x(t), t \in [t_0, T]\}$ can be obtained by using Ito's formula and which we present below for the multidimensional case (see, e.g., Arnold, 1973 and Lamberton and Lapeyre, 2007).

Let $u = u(t, x)$ denote a continuous function on $[t_0, T] \times \mathbb{R}^m$ with values in $\mathbb{R}^k$ and with the following continuous partial derivatives:

$$\frac{\partial u(t,x)}{\partial t} = u_t, \ \frac{\partial u(t,x)}{\partial x_i} = u_{x_i}, \text{ and } \frac{\partial^2 u(t,x)}{\partial x_i x_j} = u_{x_i x_j} \quad \text{for} \quad 1 \le i, j \le m.$$

If the m-dimensional stochastic process $\{X_t; t \in [t_0, T]\}$ defined by the stochastic differential

$$\mathrm{d}X_t = A(t, X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t; \quad W_t \ q\text{-dimensional Wiener process}$$

where $A(t, x)$ is an $m \times 1$ and $B(t, x)$ is an $m \times q$ matrix.

Then the k-dimensional process $Y_t = u(t, X_t)$ also possesses a stochastic differential with respect to the same Wiener process, and we have

$$\mathrm{d}Y_t = \left[u_t(t, X_t) + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} u_{x_i x_j}(t, X_t)(B(t, X_t)B'(t, X_t))_{ij} + u_x(t, X_t)A(t, X_t)\right]\mathrm{d}t$$
$$+ u_x(t, X_t)B(t, X_t)\,\mathrm{d}W_t$$

where $u_x = (u_{x_i})_{1\le i\le k}$ is a $(k \times m)$ matrix.

By applying this formula in the particular bivariate case to a transform of the type $y(t) = \mathrm{e}^{\beta t}\log(x(t)) = (\mathrm{e}^{\beta t}\log(x_1(t), \mathrm{e}^{\beta t}\log(x_2(t))))'$, we obtain

$$\mathrm{d}y(t) = \mathrm{e}^{\beta t}\left(a - \frac{b}{2}\right)\mathrm{d}t + \mathrm{e}^{\beta t}B^{1/2}\mathrm{d}w(t), \quad P\left[y(t_0) = \mathrm{e}^{\beta t_0}\log(x_{t_0})\right] = 1$$

where $a = (a_1, a_2)'$ and $b = (b_{11}, b_{22})'$, and then by integration we have

$$y(t) = y(t_0) + \left(\int_{t_0}^{t} \mathrm{e}^{\beta\theta}\mathrm{d}\theta\right)(a - b/2) + B^{1/2}\left(\int_{t_0}^{t}\mathrm{e}^{\beta\theta}\mathrm{d}w(\theta)\right)$$

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from which we can deduce the strong solution of our original SDE Equation (1) with the coefficient given in Equation (2) being

$$x(t) = \exp\left(\mathrm{e}^{-\beta(t-t_0)}\log(x_{t_0}) + \frac{1-\mathrm{e}^{-\beta(t-t_0)}}{\beta}(a-b/2)\right)$$
$$\exp\left(\mathrm{e}^{-\beta t}B^{1/2}\int_{t_0}^{t}\mathrm{e}^{\beta\theta}\mathrm{d}W_\theta\right)$$

2.2. The ptdf and moments of the model

Taking into account that the random vector $\int_s^t \mathrm{e}^{\beta\theta}\mathrm{d}W_\theta$ has a bivariate normal distribution $\mathcal{N}_2(0, \int_s^t \mathrm{e}^{2\beta\theta}\mathrm{d}\theta I_2)$ (where I_2 denotes the 2×2 identity matrix), we can deduce that $x(t)\,|\,x(s)=x_s$ has a bivariate lognormal distribution $\Lambda_2(\mu(s,t,x_s),\ \Sigma(s,t))$ with

$$\mu(s,t,x) = \mathrm{e}^{-\beta(t-s)}\log(x) + \frac{1-\mathrm{e}^{-\beta(t-s)}}{\beta}(a-b/2) \tag{3}$$

$$\Sigma(s,t) = \frac{1}{2\beta}\left(1-\mathrm{e}^{-2\beta(t-s)}\right)B \tag{4}$$

and therefore the transition density function of the process $f(y,t\,|\,x,s)$ (for $y=(y_1,y_2)'$ and $x=(x_1,x_2)'$) has the form

$$f(y,t\,|\,x,s) = [2\pi]^{-1}|\Sigma(s,t)|^{-\frac{1}{2}}(y_1y_2)^{-1}\exp\left\{-\frac{Q}{2}\right\} \tag{5}$$

where $|\Sigma(s,t)|$ is the determinant of the matrix $\Sigma(s,t)$, and Q is a quadratic form that is given by

$$Q = (\log(y)-\mu(s,t,x))'\,[\Sigma(s,t)]^{-1}\,(\log(y)-\mu(s,t,x))$$

where $\mu(s,t,x)$ and $\Sigma(s,t)$ are given respectively in Equations (3) and (4).

The marginal conditional and non-conditional moments of order r (r is a non negative integer) are obtained from the function generating of the random vector $Z(t)=\log[x(t)\,|\,x(s)=x_s]$, which follows the law $\mathcal{N}_2(\mu(s,t,x_s);\ \Sigma(s,t))$, and is expressed as follows, for $\lambda\in\mathbb{R}^2$

$$\mathbb{E}\left(\mathrm{e}^{\lambda' Z(t)}\right) = \exp\left\{\lambda'\mu(s,t,x_s) + \frac{1}{2}\lambda'\Sigma(s,t)\lambda\right\}$$

For particular values of the vector $\lambda=(r,0)'$ or $\lambda=(0,r)'$ ($r\in\mathbb{N}^*$), we obtain, for example, the marginal conditional trend functions of order r of the process and which have the following form, for $i=1,2$

$$\mathbb{E}\left(x_i^r(t)\,|\,x_i(s)=x_{s,i}\right) = \exp\left(r\mu_i(s,t,x_s) + \frac{r^2}{4\beta}(1-\mathrm{e}^{-2\beta(t-s)})b_{ii}\right) \tag{6}$$

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and for $\lambda = (r_1, r_2)' \ (r_1, r_2 \in \mathbb{N}^*)$, we obtain the joint conditional trend of the process:

$$\mathbb{E}\left(x_1^{r_1}(t)x_2^{r_2}(t) \mid x(s) = x_s\right) = \exp\left(r_1\mu_1(s,t,x_s) + r_2\mu_2(s,t,x_s) + \frac{1}{4\beta}\left(1 - \mathrm{e}^{-2\beta(t-s)}\right)\left(r_1^2 b_{11} + r_2^2 b_{22} + 2r_1 r_2 b_{12}\right)\right) \tag{7}$$

Using Equation (6) in the particular case $r = 1$, we obtain the marginal conditional trend function of the process:

$$\mathbb{E}\left(x_i(t) \mid x_i(s) = x_{s,i}\right) = \exp\left(\mu_i(s,t,x_s) + \frac{1}{4\beta}\left(1 - \mathrm{e}^{-2\beta(t-s)}\right)b_{ii}\right) \tag{8}$$

By assuming the initial condition $\mathrm{P}(x(t) = x_{t_0}) = 1$, and using Equation (8) then the non-conditional marginal trend functions are

$$\mathbb{E}\left[x_i(t)\right] = \exp\left(\mu_i(t_0,t,x_{t_0}) + \frac{1}{4\beta}\left(1 - \mathrm{e}^{-2\beta(t-t_0)}\right)b_{ii}\right) \tag{9}$$

From Equations (6) and (8), we can deduce that the marginal variance function of the process, for $i = 1, 2$ is

$$\mathrm{Var}\left(x_i(t)\right) = \exp\left(2\mu_i(t_0,t;x_{t_0}) + \frac{b_{ii}}{2\beta}\left(1 - \mathrm{e}^{-2\beta(t-t_0)}\right)\right)\left(\exp\left[\frac{b_{ii}}{2\beta}\left(1 - \mathrm{e}^{-2\beta(t-t_0)}\right)\right] - 1\right)$$

and the covariance function at the same instant is

$$\mathrm{Cov}\left(x_1(t), x_2(t)\right) = \exp\left(\mu_1(t_0,t;x_{t_0}) + \mu_2(t_0,t;x_{t_0}) + \frac{1}{4\beta}\left(1 - \mathrm{e}^{-2\beta(t-t_0)}\right)(b_{11} + b_{22})\right)\left(\exp\left[\frac{b_{12}}{2\beta}(1 - \mathrm{e}^{-2\beta(t-t_0)})\right] - 1\right)$$

The correlation function of the process at the same instant $\varrho(x_1(t), x_2(t)) \equiv \varrho(t)$ is given by

$$\varrho(t) = \left(\exp\left[\frac{b_{12}}{2\beta}\left(1 - \mathrm{e}^{-2\beta(t-t_0)}\right)\right] - 1\right)\left(\exp\left[\frac{b_{11}}{2\beta}\left(1 - \mathrm{e}^{-2\beta(t-t_0)}\right)\right] - 1\right)^{-1/2}\left(\exp\left[\frac{b_{22}}{2\beta}\left(1 - \mathrm{e}^{-2\beta(t-t_0)}\right)\right] - 1\right)^{-1/2} \tag{10}$$

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3. STATISTICAL INFERENCE ON THE MODEL

3.1. Parameter likelihood estimation

Let us now obtain the maximum likelihood estimators of the parameters corresponding to the model β, a, and B, using discrete sampling. To construct the likelihood function associated with the process, we used the following discrete sampling: $\{x(t_1) = x_{t_1}; x(t_2) = x_{t_2}; \ldots, x(t_n) = x_{t_n}\}$ at the instants $t_1, t_2; \ldots; t_n$ (with $t_i - t_{i-1} = 1$ for $i = 2, \ldots, n$), in which each $x(t_\alpha)$ represents the bidimensional vector $x(t_\alpha) = (x_1(t_\alpha), x_2(t_\alpha))'$, which for the sake of simplicity we shall denote as $x_{t_\alpha} = x_\alpha$. We also considered the initial condition $\mathrm{P}[x(t_1) = x_1] = 1$; by applying the Markov property and making use of Equation (5), the likelihood function associated with the sample considered, of size $(n-1)$ is given by

$$\mathbb{L}(x_1, \ldots, x_n; \beta; \gamma; B) = (2\pi)^{-(n-1)} {\nu_\beta}^{-(n-1)} |B|^{-\frac{(n-1)}{2}} \prod_{\alpha=2}^{n} (x_{\alpha,1} x_{\alpha,2})^{-1}$$
$$\exp\left\{-\frac{1}{2}\left[\log(x_\alpha) - \mathrm{e}^{-\beta}\log(x_{\alpha-1}) - \left(1-\mathrm{e}^{-\beta}\right)\frac{\gamma}{\beta}\right]'\right.$$
$$\left.\nu_\beta^{-2} B^{-1}\left[\log(x_\alpha) - \mathrm{e}^{-\beta}\log(x_{\alpha-1}) - \left(1-\mathrm{e}^{-\beta}\right)\frac{\gamma}{\beta}\right]\right\}$$

where $\nu_\beta^2 = \frac{1-\mathrm{e}^{-2\beta}}{2\beta}$ and $\gamma = a - \frac{b}{2}$.

By carrying out the following change of variable $v_1 = x_1$ and $\mathbf{v}_\alpha(\beta) \equiv \mathbf{v}_\alpha = {\nu_\beta}^{-1}(\log(x_\alpha) - \mathrm{e}^{-\beta}\log(x_{\alpha-1}))$ for $\alpha = 2, \ldots, n$, in terms of $\mathbf{v}_\alpha$, the likelihood function is given by

$$\mathbb{L}_{\mathbf{v}_1,\ldots,\mathbf{v}_n}(\beta; \gamma; B) \equiv \mathbb{L} = (2\pi)^{-(n-1)} \nu_\beta^{-(n-1)} |B|^{-\frac{(n-1)}{2}}$$
$$\exp\left\{-\frac{1}{2}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)' B^{-1}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)\right\}$$

where $\xi_\beta = \nu_\beta^{-1}\frac{(1-\mathrm{e}^{-\beta})}{\beta}$.

By taking the logarithm, we obtain

$$\log(\mathbb{L}) = -(n-1)\log(2\pi) - (n-1)\log(\nu_\beta) - \frac{n-1}{2}\log|B|$$
$$-\frac{1}{2}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)' B^{-1}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)$$

Then, calculating the differential of this function, and making use of the fact that $\mathrm{d}(\mathrm{tr}(B)) = \mathrm{tr}(\mathrm{d}B)$ and $\mathrm{d}(\log|B|) = \mathrm{tr}(B^{-1}\mathrm{d}B)$, we have

$$\mathrm{d}\log(\mathbb{L}) = -(n-1)\nu_\beta^{-1}\frac{\partial\nu_\beta}{\partial\beta}\mathrm{d}\beta - \frac{n-1}{2}\mathrm{tr}\left(B^{-1}\mathrm{d}B\right)$$

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$$
\begin{aligned}
&-\frac{1}{2}\sum_{\alpha=2}^{n}\left[-\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)' B^{-1}(\mathrm{d}B)B^{-1}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)\right] \\
&+\xi_\beta\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)' B^{-1}(\mathrm{d}\gamma) \\
&-\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)' B^{-1}\left(\frac{\partial\mathbf{v}_\alpha}{\partial\beta}-\frac{\partial\xi_\beta}{\partial\beta}\gamma\right)\mathrm{d}\beta
\end{aligned}
$$

And as $\mathbf{v}_\alpha=\nu_\beta^{-1}(\log(x_\alpha)-\mathrm{e}^{-\beta}\log(x_{\alpha-1}))$, then

$$
\frac{\partial\mathbf{v}_\alpha}{\partial\beta}=\left(-\nu_\beta^{-1}\frac{\partial\nu_\beta}{\partial\beta}\mathbf{v}_\alpha+\nu_\beta^{-1}\mathrm{e}^{-\beta}\log(x_{\alpha-1})\right)
$$

By applying trace properties, the above differential can be written as follows:

$$
\begin{aligned}
\mathrm{d}\log(\mathbb{L}) = {} & \frac{1}{2}\mathrm{tr}\left\{\sum_{\alpha=2}^{n}\left[B^{-1}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)'-I_2\right]B^{-1}\mathrm{d}B\right\} \\
&+\xi_\beta\mathrm{tr}\left\{\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)' B^{-1}(\mathrm{d}\gamma)\right\} \\
&+\left\{\nu_\beta^{-1}\frac{\partial\nu_\beta}{\partial\beta}\left[-(n-1)+\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)' B^{-1}\mathbf{v}_\alpha\right]\right. \\
&-\nu_\beta{}^{-1}\mathrm{e}^{-\beta}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)' B^{-1}\log(x_{\alpha-1}) \\
&\left.+\frac{\partial\xi_\beta}{\partial\beta}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)' B^{-1}\gamma\right\}\mathrm{d}\beta
\end{aligned}
$$

From the relations $\mathrm{tr}(AB)=\mathrm{Vec}'(A')\mathrm{Vec}(B)$ and $\mathrm{d}\,\mathrm{Vec}(A)=\mathrm{Vec}(\mathrm{d}A)$, where Vec denotes the matrix vectorization (given an $n\times m$ matrix X, the $\mathrm{Vec}(X)$ is the vector of dimension $nm\times 1$ that stacks the columns of X), we obtain

$$
\begin{aligned}
\mathrm{d}\log(\mathbb{L}) = {} & \mathrm{Vec}'\left\{\sum_{\alpha=2}^{n}\left[B^{-1}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)'-I_2\right]B^{-1}\right\}\mathrm{d}\,\mathrm{Vec}(B) \\
&+\xi_\beta\mathrm{Vec}'\left\{B^{-1}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)\right\}\mathrm{d}\,\mathrm{Vec}(\gamma)
\end{aligned}
$$

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$$+\left\{\nu_\beta^{-1}\frac{\partial\nu_\beta}{\partial\beta}\left[-(n-1)+\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)'B^{-1}\mathbf{v}_\alpha\right]\right.$$
$$-\nu_\beta{}^{-1}\mathrm{e}^{-\beta}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)'B^{-1}\log(x_{\alpha-1})$$
$$\left.+\frac{\partial\xi_\beta}{\partial\beta}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)'B^{-1}\gamma\right\}\mathrm{d}\beta$$

Then, making this differential equal to zero, with respect to the estimators of B and γ, we obtain

$$B^{-1}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)=0 \tag{11}$$

$$\sum_{\alpha=2}^{n}\left[B^{-1}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)'-I_2\right]B^{-1}=0 \tag{12}$$

The estimator of the parameter β is given by

$$\nu_\beta^{-1}\frac{\partial\nu_\beta}{\partial\beta}\left[-(n-1)+\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)'B^{-1}\mathbf{v}_\alpha\right]$$
$$-\nu_\beta^{-1}\mathrm{e}^{-\beta}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)'B^{-1}\log(x_{\alpha-1})$$
$$+\frac{\partial\xi_\beta}{\partial\beta}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)'B^{-1}\gamma=0 \tag{13}$$

From Equations (11) and (12), we obtain the maximum likelihood estimators of the vector γ and of the diffusion matrix B, given by

$$\hat{\gamma}=\frac{1}{(n-1)\xi_{\hat{\beta}}}\sum_{\alpha=2}^{n}\mathbf{v}_\alpha \tag{14}$$

$$\hat{B}=\frac{1}{n-1}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_{\hat{\beta}}\hat{\gamma}\right)\left(\mathbf{v}_\alpha-\xi_{\hat{\beta}}\hat{\gamma}\right)' \tag{15}$$

Taking into account Equations (14) and (15) and after various operations in Equation (13) (not shown), the estimator of β is given by

$$\sum_{\alpha=2}^{n}\log(x_{\alpha-1})'\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)=0$$

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From the latter equation, we conclude that the estimator of β is given by the following expression:

$$\hat{\beta} = \log \left\{ \frac{\left(\sum_{\alpha=2}^{n} y'_{\alpha-1}\right)\left(\sum_{\alpha=2}^{n} y_{\alpha-1}\right) - (n-1)\sum_{\alpha=2}^{n} y'_{\alpha-1} y_{\alpha-1}}{\left(\sum_{\alpha=2}^{n} y'_{\alpha-1}\right)\left(\sum_{\alpha=2}^{n} y_{\alpha}\right) - (n-1)\sum_{\alpha=2}^{n} y'_{\alpha-1} y_{\alpha}} \right\} \tag{16}$$

where $y_{.} = \log(x_{.})$.

3.2. *Estimated trend and correlation functions*

We shall now define some functions that enable us to fit real data; making use of the model described in this paper, we consider the estimated marginal trend function (EMTF) and the estimated conditional marginal trend function (EMCTF), together with the estimated correlation function (ECF). Thus we are able to analyze the correlations between the components of the vector process in question. These functions can be obtained by Zehna's theorem (see Zehna, 1966 and corollary 3.2.1 in Anderson, 1984); the EMTF and EMCTF of the process are obtained by replacing the parameters in Equations (8) and (9) by their estimators given in Equations (14), (15), and (16). The EMCTF is then given by the following expression:

$$\widehat{\mathbb{E}}\left(x_i(t) \mid x_i(s) = x_s\right) = \exp\left(\hat{\mu}_i(s, t, x_{s,i}) + \frac{1}{4\hat{\beta}}\left(1 - \mathrm{e}^{-2\hat{\beta}(t-t_0)}\right)\hat{b}_{ii}\right) \tag{17}$$

and the resulting EMTF in this case has the following form:

$$\widehat{\mathbb{E}}\left[x_i(t)\right] = \exp\left(\hat{\mu}_i(t_0, t, x_{t_0}) + \frac{1}{4\hat{\beta}}\left(1 - \mathrm{e}^{-2\hat{\beta}(t-s)}\right)\hat{b}_{ii}\right) \tag{18}$$

We obtain the ECF by replacing the parameters in Equation (10) by their estimators, and thus

$$\begin{aligned} \widehat{\varrho}(t) = {} & \left(\exp\left[\frac{\hat{b}_{12}}{2\hat{\beta}}\left(1 - \mathrm{e}^{-2\hat{\beta}(t-t_0)}\right)\right] - 1\right)\left(\exp\left[\frac{\hat{b}_{11}}{2\hat{\beta}}\left(1 - \mathrm{e}^{-2\hat{\beta}(t-t_0)}\right)\right] - 1\right)^{-1/2} \\ & \left(\exp\left[\frac{\hat{b}_{12}}{2\hat{\beta}}\left(1 - \mathrm{e}^{-2\hat{\beta}(t-t_0)}\right)\right] - 1\right)^{-1/2} \end{aligned} \tag{19}$$

4. APPLICATION TO THE JOINT MODELING OF THE GDP AND CO_2 EMISSIONS IN SPAIN

We applied the methodology presented in this paper to the random vector $x(t) = (x_1(t), x_2(t))'$, where

- "$x_1(t)$" is the global CO_2 emission in Spain observed annually for the period 1986–2003, measured in metric tons of carbon; this may be consulted at http://www.ediac.esd.orl.gov.
- "$x_2(t)$" is the gross domestic product in Spain observed annually for the period 1986–2003, measured in 10^6; it may be consulted at http://www.ine.es (National Statistic Institute of Spain).

Table 1. The ECF versus time

Time	1986	1987	1988	1989	1990	1991	1992	1993	1994
ECF	0	0.7023	0.7021	0.7018	0.7016	0.7014	0.7012	0.7010	0.7009
Time	1995	1996	1997	1998	1999	2000	2001	2002	2003
ECF	0.7008	0.7006	0.7005	0.7004	0.7003	0.7002	0.7001	0.7000	0.6999

The data set of observations considered are used to estimate the parameters of the model, using Equations (14), (15) and (16).

A MatLab program was implemented to carry out the calculation required for this study. The value of the likelihood estimator of β is $\hat{\beta} = 0.03435$ and the estimators of the vector γ and of the matrix diffusion are

$$\hat{\gamma} = \begin{pmatrix} 0.17317 \\ 0.23640 \end{pmatrix} \quad \text{and} \quad \hat{B} = \begin{pmatrix} 0.00286 & 0.00169 \\ 0.00169 & 0.00202 \end{pmatrix}$$

Table 1 shows the ECF between the global CO_2 and GDP in Spain.

Table 2 summarizes the results of this application, that is, the observed data vector $(CO_2, DGP)'$, the EMTF and EMCTF, respectively.

Figure 1 shows the fit of global CO_2 emissions in Spain using the EMCTF of the proposed process.

Figure 2 shows the fit of GDP in Spain using the EMCTF of the proposed process.

Table 2. CO_2, GDP, EMTF, and EMCTF

Years	Real data		EMTF		EMCTF	
	CO_2	GDP	CO_2	GDP	CO_2	GDP
1986	47.610	336.643	47.610	336.643	47.610	336.643
1987	48.570	355.317	49.611	349.288	49.611	349.288
1988	49.856	373.418	51.622	361.945	50.577	367.992
1989	56.841	391.443	53.640	374.599	51.871	386.090
1990	57.814	406.245	55.662	387.235	58.876	404.083
1991	59.096	416.588	57.687	399.838	59.850	418.838
1992	61.657	420.459	59.712	412.396	61.132	429.137
1993	56.504	416.122	61.732	424.895	63.690	432.990
1994	59.090	426.036	63.748	437.322	58.539	428.673
1995	63.672	437.792	65.757	449.665	61.126	438.537
1996	63.732	448.456	67.756	461.914	65.700	450.224
1997	66.961	466.513	69.744	474.059	65.760	460.816
1998	68.645	486.742	71.718	486.088	68.977	478.732
1999	75.230	506.849	73.677	497.994	70.652	498.776
2000	77.099	527.613	75.620	509.768	77.190	518.671
2001	77.470	542.166	77.544	521.401	79.042	539.187
2002	82.998	556.651	79.448	532.888	79.410	553.551
2003	84.340	570.556	81.331	544.220	84.878	567.834

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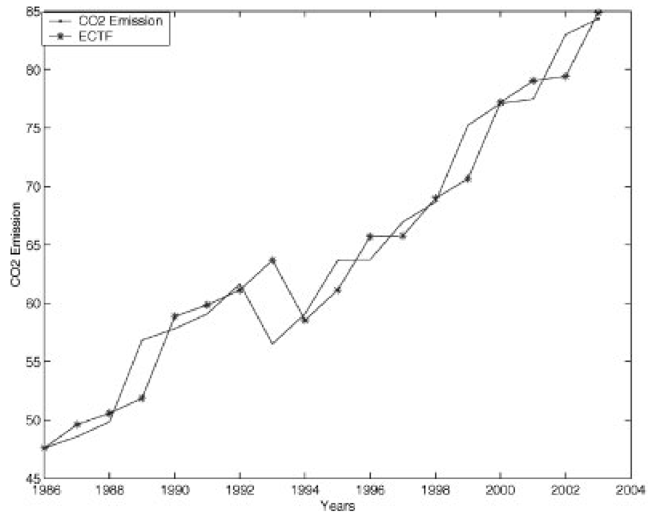

Figure 1. CO_2 emission versus the ECTF

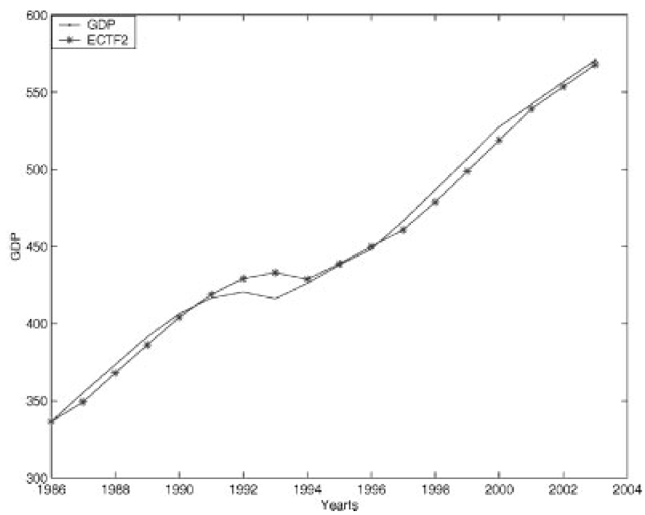

Figure 2. GDP versus the ECTF

5. CONCLUSIONS AND DISCUSSION

- The bidimensional structure (the bivariate Gompertz diffusion process), statistically fitted to the evolution of the pair of dynamic variables, that is, CO_2 emission and GDP in Spain, on the basis of observations for the period 1986–2003, enables us, for the first time, to evaluate the dynamic correlation between the two variables in Spain, using Equation (10), of the correlation function of the Gompertz bivariate process; this was found to be of the order of 0.7, which is clearly significant with respect to the corresponding degree of dependence.

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- The bidimensional approach applied for the CO_2-GDP relationship, based on the Gompertz bidimensional process, enables us to obtain more precise statistical fits of the marginal trends, both of CO_2 and of the GDP, than does the method consisting of modeling the variable CO_2 with GDP as an exogenous variable (as was adopted in Gutiérrez *et al*., in press a). In terms of econometric regression, an analogous interpretation would be that the random regression of the CO_2–GDP pair is better suited for modeling the joint evolution of the two variables than is a CO_2 regression in terms of the GDP regressor.
- Both the CO_2 variable and the GDP, considered marginally, are well modeled by means of a Gompertz-type stochastic growth process, with finite asymptotes, which can indeed be calculated, using a methodology very similar to that used by Gutiérrez *et al*. (2007a).
- In forthcoming years, the situated of the Spanish case considered in the present study will have to be adapted to polities in which the percentage of GDP dedicated to controlling CO_2 emissions is increased.
- Other possible bidimensional stochastic models have also been fitted to the CO_2–GDP pair on the basis of the same data observed (for the period 1986–2003), for example a bivariate lognormal diffusion model, following the methodology described in Gutiérrez *et al*. (1991). The fits obtained are less accurate than those corresponding to the Gompertz model. Taking into account that, technically, the lognormal case is the particular case of the bidimensional Gompertz with a null β parameter in Equation (2), it is proven, indirectly, that the effect of this parameter on Ito's stochastic Equation (1) is significant; this effect has been estimated as taking the following form: $\hat{\beta} = 0.03435$, calculated using Equation (16).
- In the present study we propose a method for analyzing the bidimensional trends present in a real phenomenon that evolves stochastically following a Gompertz model described by Itô's SDE Equation (1). This methodology provides a viable alternative, in both statistical and computational terms, to other possible methodological approaches (e.g., models of bivariate chronological series with nonlinear trends) that are of little on no practical viability regarding the continuous stochastic phenomena observed discretely in time, as is the case of CO_2–GDP.

  Indeed, by adopting the Gompertz bivariate model and calculating the trend and conditional trend functions (see Equations (8) and (9)), both marginal in CO_2 and GDP, and joint (surface trend), it is possible to estimate them using the maximum likelihood method we propose. Moreover, this methodology is computationally viable, as shown by the results presented in Section 3.

  When these trends have been fitted, we calculate the fitted values at the same instants of time in which the two variables have been observed. These variables of t coincide with the annual data (taken on 31 December each year) for the CO_2 and GDP variables considered, being "natural" values of generalized consideration (thus, nobody considers CO_2 and GDP values at other instants of time, such as 7 September of each year). So, we have a methodology describing the evolution of the two variables that, moreover, enables us to obtain the least squares statistical prediction associated with the bivariate lognormal defined in Equations (3) and (4), which is valid for predicting annual values according to the fitted trends.

  In the future, the fitted model could be used for the statistical prediction phase, with a methodology similar to that established for the univariate Gompertz model (both the homogeneous and the non-homogeneous types), see the Gutiérrez *et al*. (2005b). For this purpose, it is necessary to establish the exact or asymptotic distributions of the two-dimensional parameters of the bivariate Gompertz process introduced in the present study.

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REFERENCES

Albano G, Giorno V. 2006. A stochastic model in tumor growth. *Journal of Theoretical Biology* **242**(2): 329–336.

Aldy JE. 2006. Per capita carbon dioxide emissions: convergence or divergence? *Environmental and Resources Economics* **33**: 533–555.

Anderson TW. 1984. *An Introduction to Multivariate Statistical Analysis* (2nd edn). Wiley: New York.

Arnold L. 1973. *Stochastic Differential Equations*. John Wiley and Sons: New York.

Bibby BM, Sorensen M. 1995. Martingale estimation functions for discretely observed diffusion processes. *Bernoulli* **1**(1/2): 017–039.

Chung KL, Willians RJ. 1990. *Introduction to Stochastic Integration*. Birkhäuser: Boston.

Ferrante L, Bompade S, Possati L, Leone L. 2000. Parameter estimation in a Gompertzian stochastic-model for tumor growth. *Biometrics* **56**: 1076–1081.

Ferrante L. Bompade S, Possati L, Leone L, Montanari MP. 2005. A stochastic formulation of the Gompertzian growth model for in vitro bactericidad kinetics: parameter estimation and extinction probability. *Biometrical Journal* **47**(3): 309–318.

Frank TD. 2002. Multivariate Markov processes for stochastic systems with delays: application to the stochastic Gompertz model with delay. *Physical Review E* **66**(1): 011914.

Giovanis AN, Skiadas CH. 1999. A stochastic logistic innovation diffusion-model studying the electricity consumption in Greece and the United States. *Technological Forecasting and Social Change* **61**: 253–264.

Gutiérrez R, Angulo JM, González A, Pérez R. 1991. Inference in lognormal multidimensional diffusion process with exogenous factors: application to modeling in economics. *Applied Stochastics Model and Data Analysis* **7**: 295–316.

Gutiérrez R, González A, Torres F. 1997. Estimation in multivariate lognormal diffusion process with exogenous factors. *Applied Statistics* **4**(1): 140–146.

Gutiérrez R, Gutiérrez-Sánchez R, Nafidi A. 2004a. Maximum likelihood estimation in multivariate lognormal diffusion process with a vector of exogenous factors. *Monografías del Seminario Matemático García de Galdeano* **31**: 337–346.

Gutiérrez R, Gutiérrez-Sánchez R, Nafidi A. 2006a. Electricity consumption in Morocco: stochastic Gompertz diffusion analysis with exogenous factors. *Applied Energy* **83**: 1139–1151.

Gutiérrez R, Gutiérrez-Sánchez R, Nafidi A. 2006b. Ramos, E. A new stochastic Gompertzian process: computational inference and application in modelling. *Applied Mathematics and Computation* **183**: 738–747.

Gutiérrez R, Gutiérrez-Sánchez R, Nafidi A. 2006c. The Stochastic Rayleigh diffusion model: statistical inference and computational aspects. Applications to modelling of real cases. *Applied Mathematics and Computation* **175**: 628–644.

Gutiérrez R, Gutiérrez-Sánchez R, Nafidi A. 2008. Trend analysis using nonhomogeneoud stochastic diffusion processes. Emission of CO_2; Kypto protocol in Spain. *Stochastic Environmetal Research and Risk Assesment.* **22**: 57–66.

Gutiérrez R, Gutiérrez-Sánchez R, Nafidi A. In Press. Emissions of Greenhouse gases attributable to the activities of the land transport: modeling and analysis using I-CIR stochastics diffusion. The case of Spain. *Environmetrics*. DOI:10.1002/env.862.

Gutiérrez R, Gutiérrez-Sánchez R, Nafidi A, Ramos E. 2004b. Studying the vehicle park in Spain using the lognormal and Gompertz diffusion processes. In *Proceedings od SEIO'04*, Spain, 171–172.

Gutiérrez R, Gutiérrez-Sánchez R, Nafidi A, Ramos E. 2007a. A diffusion model with cubic drift: statistical and computational aspects and application to modelling the global CO_2 emission in Spain. *Environmetrics* **18**: 55–69.

Gutiérrez R, Nafidi A, Gutiérrez-Sánchez R. 2005a. Forecasting total natural-gas consumption in Spain by using the stochastic Gompertz innovation diffusion model. *Applied Energy* **80**(2): 115–124.

Gutiérrez R, Nafidi A, Gutiérrez-Sánchez R. 2006d. A non homogeneous Gompertz diffusion process: inference and application. *Monografías del Seminario Matemático García de Galdeano* **33**: 273–280.

Gutiérrez R, Nafidi A, Gutiérrez-Sánchez R, Román P, Torres F. 2005b. Inference in Gompertz type non homogeneous stochastic systems by means of discrete sampling. *Cybernetics and Systems* **36**: 203–216.

Gutiérrez R, Román P, Romero D, Serrano JJ, Torres F. 2007b. A new Gompertz-type diffusion process with application to random growth. *Mathematical Biosciences* **208**: 147–165.

Kloeden P, Platen E, Schurz H, Sorensen M, 1996. On effects of discretization on estimators of drift parameters for diffusion processes. *Journal of Applied Probability* **33**: 1061–1076.

Lamberton D, Lapeyre B. 2007. *Introduction to Stochastic Calculus Applied to Finance* (2nd edn). Chapman & Hall: London.

Meade M, Islam T. 2006. Modelling and forecasting the diffusion of innovation. A 25-years review. *International Journal of Forecasting* **22**(3): 519–545.

Miller JH, Minard RA, Orner GA, Bull B. 2000. In vivo MRI measurments of tumor growth induced by dichloroacetate: implications for mode of action. *Toxicology* **145**: 115–125.

Nafidi A. 1997. Lognormal diffusion with exogenous factors, extensions from the Gompertz diffusion process. *PhD Thesis*. Granada University [in Spanish].

Patanarapeelert K, Frank TD, Friedrich R, Tang IM. 2005. On reducible nonlinear time-delayed stochastic systems: fluctuation-dissipation relations, transitions to bistability, and secondary transitions to non-stationarity. *Journal of Physics A: Mathematical and General* **38**: 10069–10083.

Prajneshu M. 1980. Diffusion approximations for models of population growth with logarithmic interactions. *Stochastic Processes and their Applications* **10**(1): 087–099.

Prakasa-Rao BLS. 1999. *Statistical Inference for Diffusion Type Processes*, Arnold, London and Oxford University Press: New York.

Ramanathan R. 2006. A multi-factor efficiency perspective to the relationships among World GDP, energy consumption and carbon dioxide emissions. *Technological Forecasting and Social Change* **73**(5): 483–494.

Ricciardi L. 1977. Diffusion processes and related topics in biology. *Lecture Notes in Biomathematics*. Springer Verlag: Berlin.

Singer H. 2002. Parameter estimation of nonlinear stochastic differential equations: simulated maximum likelihood versus extended Kalman filter and Itô-Taylor expansion. *Journal of Computational and Graphical Statistics* **11**: 972–995.

Skiadas C, Giovani A. 1997. A stochastic bass innovation diffusion model for studying the growth of electricity consumption in Greece. *Applied Stochastic Models and Data Analysis* **13**: 85–101.

Tan WY. 1986. A stochastic Gompertz birth-death process. *Statistics and Probability Letters* **4**: 025–028.

Troynikov VS, Day RW, Leorke AM. 1998. Estimation of seasonal growth parameters using a stochastic Gompertz model for tagging data. *Journal of Shellfish Research* **17**(3): 833–838.

Zehna PW. 1966. Invariance of maximum likelihood estimators. *Annals of Mathematical Statistics* **37**: 744.

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