Cybernetics and Systems: An International Journal, 36: 203–216, 2005

ISSN: 0196-9722 print/1087-6553 online
DOI:10.1080/01969720590897233

INFERENCE IN GOMPERTZ-TYPE NONHOMOGENEOUS STOCHASTIC SYSTEMS BY MEANS OF DISCRETE SAMPLING

R. GUTIÉRREZ, R. GUTIÉRREZ-SANCHEZ, A. NAFIDI, P. ROMÁN, AND F. TORRES

Department of Statistics and Operations Research, University of Granada, Granada, Spain

We consider an extension of the Gompertz homogeneous diffusion process by introducing time functions (exogenous factors) that affect its trend. After obtaining its transition probability density function, the inference on the parameters of the process is obtained by considering discrete sampling of the sample paths. Finally, we apply this stochastic process to model housing price in Spain.

INTRODUCTION

The study of stochastic systems by using Markovian processes has become of great interest to investigators in many disciplines (biology, physics, demography, economics, cybernetics, etc.). Among these processes, diffusions have been widely considered and its study has covered several areas such the inference (especially the estimation of the parameters of the drift and the diffusion coefficient) and first passage times through varying boundaries.

Diffusions can be introduced from stochastic differential equations, this being the natural way when the process is observed continuously. An extensive review of this theory can be found in the literature (Prakasa Rao 1999) and related recent work has been done by Kloeden et al. (1996) and Singer (2002), among others. On the other hand, the statistical

Address correspondence to R. Gutiérrez, Departamento de Estadística e I.O. Facultad de Ciencias, Avda, Fuente Nueva, s/n, 18071 Granada, Spain. E-mail: rgjaimez@ugr.es

inference for diffusion processes realized by discrete sampling and the first-passage-times problem need the formulation of the Kolmogorov's forward and backward equations whose unique solution, under certain analytical conditions, is the transition probability density function (pdf). This approach is an alternative formulation to that given by the corresponding Itô stochastic equation. There are many works in the context of inference by discrete sampling. Among others, we can cite those related to particular processes such as the lognormal diffusion process (Gutiérrez et al. 1991, 1997b, 1999, 2001a,b, 2003; Tintner and Sengupta 1972) as well as in a general context (Bibby and Sørensen 1995). For results about first passage times, we can see the works of Gutiérrez et al. (1995, 1997a) and references therein.

There is a wide variety of diffusion processes (and Markovian processes in general) in the literature. One of these is the Gompertz diffusion process, whose deterministic antecedent, the Gompertz growth curve, has been extensively treated. A stochastic version of this, as a birth and death process, has been introduced by Tan (1986) and has been applied by Troynikov and Gorfine (1998), and Miller et al. (2000) in animal and tumor population growth. On the other hand, the stochastic Gompertz diffusion process has also been treated by Ricciardi (1977) in population growth by adding white noise fluctuation to intrinsic fertility of the population and by Dennis and Patil (1988) in ecology modeling. Recently, and in the context of the *stochastic systems with delays* modeling by using univariate and multivariate Markov processes, Frank (2002) applies the generalized Fokker–Planck equations with delays (given by Guillouzic et al. 2000) to a Gompertz model.

However, most of the aforementioned applications have been treated in a homogeneous context. In this paper, we consider an extension of the Gompertz homogeneous diffusion process by introducing exogenous factors in the stochastic model. Such exogenous factors are time functions that affect the drift of the process. Their time behavior is assumed known and they must contribute to the description of the process evolution and its external control.

First we introduce the Gompertz diffusion process with exogenous factors from the point of view of stochastic differential equations as the Kolmogorov equations. From the second approach, we obtain the transition PDF. Later, the estimation of the parameters (and consequently of the trend) is realized by using discrete sampling. Finally, we consider an application of the theoretical results developed in the previous section. This application treats one of the most important economic variables: the housing price.

THE GOMPERTZ DIFFUSION PROCESS WITH EXOGENOUS FACTORS

The Model

Let $\{X(t); t \in [t_0, T], t_0 > 0\}$ be the one-dimensional diffusion process taking values on $\mathbb{R}^+$ and with infinitesimal moments

$$\begin{aligned} a(x,t) &= h(t)x - \beta x \log(x) \\ b(x,t) &= \sigma^2 x^2 \end{aligned} \tag{1}$$

where $\sigma > 0$, $\beta \in \mathbb{R}$, $h(t) = \alpha_0 + \sum_{i=1}^{q} \alpha_i g_i(t)$, and where $\alpha_i \in \mathbb{R}$ and g_i are time-continuous functions in $[t_0, T]$.

This process can be studied from the point of view of the partial differential equations. The starting point for this is the forward equation,

$$\frac{\partial f(x,t|x_0,t_0)}{\partial t} = -\frac{\partial}{\partial x}[a(x,t)f(x,t|x_0,t_0)] + \frac{\partial^2}{\partial x^2}[b(x,t)f(x,t|x_0,t_0)] \tag{2}$$

and the backward equation,

$$\frac{\partial f(x,t|x_0,t_0)}{\partial t_0} + a(x_0,t_o)\frac{\partial f(x,t|x_0,t_0)}{\partial x_0} + b(x_0,t_o)\frac{\partial^2 f(x,t|x_0,t_0)}{\partial x_0^2} = 0 \tag{3}$$

with initial condition $\lim_{t\to t_0} f(x,t|x_0,t_0) = \delta(x - x_0)$ and from which the transition PDF can be obtained. (This will be the way followed in the next section.)

Alternatively, we can consider the stochastic differential equation given by

$$\mathrm{d}X(t) = a(X(t),t)\mathrm{d}t + \sqrt{b(X(t),t)}\mathrm{d}W(t) \qquad X(t_0) = x_0 \tag{4}$$

where $\{W(t) : t \in [t_0, T]\}$ is a one-dimensional Wiener process and x_0 is a fixed real number belonging to $\mathbb{R}^+$. Considering the analytical properties of $a(x,t)$ and $b(x,t)$, it follows that Eq. (4) has unique solution that will be the $\mathbb{R}^+$-valued diffusion process with initial value x_0 and infinitesimal moments given by Eq. (1).

We note that this process generalizes that treated, in homogenous context, by Skiadas et al. (1994) and Capocelli and Ricciardi (1974). The actual process includes a time term affecting the drift. This term allows introduction of an external influence on the endogenous variable of the process that can contribute to explain some alterations in the observed trend beside the trend of the homogenous process. Because of the introduction of these external variables, we will refer to this process in the future as the Gompertz diffusion process with exogenous factors.

Also we have to point out that, when β vanishes, this process leads to the lognormal diffusion process with exogenous factors. We will see, throughout this paper, that the limit situation $\beta \to 0$ leads us to results already obtained for the lognormal process.

Transition PDF

The transition PDF of the process can be obtained by looking for a transformation

$$t' = \phi(t)$$
$$x' = \psi(x, t)$$

that changes its Kolmogorov equation into that of the Wiener process. Indeed, the infinitesimal moments (1) verify the conditions of theorem 1 of Ricciardi (1976), so that such transformation exists. Concretely,

$$\phi(t) = \frac{k_1}{2\beta}\left[e^{2\beta(t-t_0)} - e^{2\beta(t_1-t_0)}\right] + k_3$$

$$\psi(x,t) = \frac{k_1^{\frac{1}{2}}}{\sigma}\left(e^{\beta(t-t_0)}\log(x/z) - \int_{t_2}^{t} h(\tau)e^{\beta(\tau-t_0)}\mathrm{d}\tau\right) + \frac{k_1^{\frac{1}{2}}}{\beta\sigma}\left(\frac{\sigma^2}{2} + \beta\log(z)\right)\left(e^{\beta(t-t_0)} - e^{\beta(t_2-t_0)}\right) + k_2$$

where $z \in \mathbb{R}^+$, $t_i > 0$, and the k_i are arbitrary constants with $k_1 > 0$. Note that this transformation changes the state-space $\mathbb{R}^+$ into $\mathbb{R}$ and allows us to obtain the transition PDF for the considered process, resulting in

$$
\begin{aligned}
&f(x,t|y,s)\\
&=\frac{1}{x}\left[\frac{\pi\sigma^2}{\beta}\left(1-e^{-2\beta(t-s)}\right)\right]^{-\frac{1}{2}}\\
&\times\exp\left(-\frac{\beta\left[\log(x)-e^{-\beta(t-s)}\log(y)+\frac{\sigma^2}{2\beta}\left(1-e^{-\beta(t-s)}\right)-\int_s^t h(\tau)e^{-\beta(t-\tau)}d\tau\right]^2}{\sigma^2(1-e^{-2\beta(t-s)})}\right)
\end{aligned}
\tag{5}
$$

which corresponds with a lognormal distribution.

From Eq. (5) and by considering the initial distribution of the process, we obtain the rth moment of the endogenous variable. In particular, by assuming $P[X(t_0)=x_0]=1$,

$$
\begin{aligned}
E[X(t)^r]=\exp\Bigg(&re^{-\beta(t-t_0)}\log(x_0)+r\int_{t_0}^t h(\tau)e^{-\beta(t-\tau)}\mathrm{d}\tau\\
&-\frac{r\sigma^2}{2\beta}\left(1-e^{-\beta(t-t_0)}\right)+\frac{r^2\sigma^2}{4\beta}\left(1-e^{-2\beta(t-t_0)}\right)\Bigg)
\end{aligned}
\tag{6}
$$

from which the trend and variance functions follow.

Inference on the Model

Now we give a brief summary of the inferential procedure whose main aim is to obtain the estimation of the moments of the endogenous variable and, in particular, the estimation of the trend of the process that will be used in the practical application.

The inference on the process can be achieved by means of

1. *continuous sampling*; that is, from the sample paths of the process and using the methodology derived of Itô's calculus from stochastic differential equation (4), or
2. *discrete sampling*, that is, for fixed times $t_1,\ldots,t_n$, we observe the variables $X(t_1),\ldots,X(t_n)$, whose values will be the basic sample from which we carry out the inferential process. The joint distribution (likelihood function) of the sample is built from transition PDF (5), taking into account the initial condition $P[X(t_1)=x_1]=1$ from which the maximum likelihood method is used to obtain the estimation of the parameters.

In this paper, the employed method has been the maximum likelihood using discrete sampling. For simplicity we assume that the length of the time intervals $[t_{i-1}, t_i]$ $(i = 2, \ldots, n)$ is equal to one. This assumption is not restrictive in practice because there are many data in such a situation, for example, those extracted from time series as considered in the next section.

Let $x_1, \ldots, x_n$ be the observed values of the sampling. Now we transform these values by means of $v_1 = x_1$ and $v_{i,\beta} = \log(x_i) - e^{-\beta}\log(x_{i-1})$ $(i = 2, \ldots, n)$. From Eq. (5), the likelihood function for the transformed sample is

$$L_{v_{2,\beta},\ldots,v_{n,\beta}}(\mathbf{a}, \beta, \sigma^2) = \left[\frac{\beta}{\pi\sigma^2(1 - e^{-2\beta})}\right]^{\frac{n-1}{2}} \times \exp\left(-\frac{\beta(\mathbf{v}_\beta - \gamma_\beta \mathbf{U}'_\beta \mathbf{a})'(\mathbf{v}_\beta - \gamma_\beta \mathbf{U}'_\beta \mathbf{a})}{\sigma^2(1 - e^{-2\beta})}\right) \tag{7}$$

where $\mathbf{v}_\beta = (v_{2,\beta}, \ldots, v_{n,\beta})'$, $\mathbf{a} = (\alpha_0 - \sigma^2/2, \alpha_1, \ldots, \alpha_q)'$, $\gamma_\beta = \left(1 - e^{-\beta}\right)/\beta$, and $\mathbf{U}_\beta$ is the $(q + 1) \times (n - 1)$ matrix, whose rank is assumed to be $q + 1$, given by $\mathbf{U}_\beta = (\mathbf{u}_{2,\beta}, \ldots, \mathbf{u}_{n,\beta})$ with

$$\mathbf{u}_{i,\beta} = \left(1, \frac{1}{\gamma_\beta}\int_{t_{i-1}}^{t_i} g_1(\tau)e^{-\beta(t_i-\tau)}\mathrm{d}\tau, \ldots, \frac{1}{\gamma_\beta}\int_{t_{i-1}}^{t_i} g_q(\tau)e^{-\beta(t_i-\tau)}\mathrm{d}\tau\right)' \quad i = 2, \ldots, n$$

Also, we consider $\mathbf{l}_x = (\log(x_1), \ldots, \log(x_{n-1}))'$, which we are going to use in the following expressions.

After calculating the derivatives of the log-likelihood function with respect to $\mathbf{a}$ and σ^2, we have the following equations:

$$\mathbf{U}_\beta \mathbf{v}_\beta = \gamma_\beta \mathbf{U}_\beta \mathbf{U}'_\beta \mathbf{a} \tag{8}$$

$$(n - 1)\left(1 - e^{-2\beta}\right)\sigma^2 = 2\beta(\mathbf{v}_\beta - \gamma_\beta \mathbf{U}'_\beta \mathbf{a})'(\mathbf{v}_\beta - \gamma_\beta \mathbf{U}'_\beta \mathbf{a}) \tag{9}$$

And from the derivative with respect to β, making use of Eqs. (8) and (9), the third equation is obtained, resulting in

$$\left(e^{-\beta}\mathbf{l}'_x + \gamma_\beta \mathbf{a}' \frac{\partial \mathbf{U}_\beta}{\partial \beta}\right)\left(\mathbf{v}_\beta - \gamma_\beta \mathbf{U}'_\beta \mathbf{a}\right) = 0 \tag{10}$$

where $\partial \mathbf{U}_\beta / \partial \beta$ is the matrix whose elements are the derivatives, with respect to β, of the components of $\mathbf{U}_\beta$.

Taking into account Eq. (8), this last expression can be expressed as

$$\left(e^{-\beta}\mathbf{l}_x' + \mathbf{v}_\beta'\mathbf{U}_\beta'(\mathbf{U}_\beta\mathbf{U}_\beta')^{-1}\frac{\partial \mathbf{U}_\beta}{\partial\beta}\right)\mathbf{H}_{\mathbf{U},\beta}\mathbf{v}_\beta = 0 \tag{11}$$

where $\mathbf{H}_{\mathbf{U},\beta}$ is the symmetric and idempotent matrix given by $\mathbf{H}_{\mathbf{U},\beta} = \mathbf{I}_{n-1} - \mathbf{U}_\beta'(\mathbf{U}_\beta\mathbf{U}_\beta')^{-1}\mathbf{U}_\beta$.

To obtain the maximum likelihood estimators of the parameters it is necessary to solve Eq. (11) in the unknown quantity β. Nevertheless, this equation depends on the functional form of the g_j functions and so its resolution will be dependent on the practical situations in which we use this process. In the next section, we consider a type of exogenous factor that is related to the considered data.

Once Eq. (11) is solved, the maximum likelihood estimators of $\mathbf{a}$ and σ^2 are

$$\hat{\mathbf{a}} = \frac{\hat\beta}{1-e^{-\hat\beta}}\left(\mathbf{U}_{\hat\beta}\mathbf{U}_{\hat\beta}'\right)^{-1}\mathbf{U}_{\hat\beta}\mathbf{v}_{\hat\beta} \tag{12}$$

$$\hat{\sigma^2} = \frac{2\hat\beta}{(n-1)\left(1-e^{-2\hat\beta}\right)}\mathbf{v}_{\hat\beta}'\mathbf{H}_{\mathbf{U},\hat\beta}\mathbf{v}_{\hat\beta} \tag{13}$$

Remarks 1: By Zehna's theorem, the predicted trend of the process can be obtained from Eq. (6) (taking $r = 1$) by replacing the parameters by its estimators. Then we have

$$E\left[\hat X(t)\right] = \exp\left(e^{-(t-t_1)\hat\beta}\log(x_1) + \frac{\hat{\alpha_0} - \hat{\sigma^2}/2}{\hat\beta}\left(1-e^{-(t-t_1)\hat\beta}\right)\right.$$
$$\left. + \frac{\hat{\sigma^2}}{4\hat\beta}\left(1-e^{-2(t-t_1)\hat\beta}\right)\right) \times \exp\left(\sum_{i=1}^{q}\hat\alpha_i\int_{t_1}^{t} e^{-\hat\beta(t-\tau)}g_i(\tau)\mathrm{d}\tau\right) \quad t \geq t_1$$

Remarks 2: If $\beta \to 0$, as mentioned earlier, we obtain the lognormal diffusion process with exogenous factors. Then the maximum likelihood estimators of the parameters of this process (in the case of $t_i - t_{i-1} = 1$) can be obtained from those of the Gompertz process by taking limits, as $\hat\beta \to 0$, in Eqs. (12) and (13). In this sense, we have

$$\lim_{\hat{\beta}\to 0} \hat{\mathbf{a}} = (\mathbf{U}\mathbf{U}')^{-1}\mathbf{U}\mathbf{v} \tag{14}$$

$$\lim_{\hat{\beta}\to 0} \hat{\sigma^2} = \frac{1}{n-1}\mathbf{v}'\mathbf{H}_{\mathbf{U}}\mathbf{v} \tag{15}$$

where, now, $\mathbf{v} = (v_2, \ldots, v_n)'$ with $v_i = \log(x_i/x_{i-1})$ $(i = 2, \ldots, n)$ and the columns of the matrix $\mathbf{U}$ are given by

$$\mathbf{u}_i = \left(1, \int_{t_{i-1}}^{t_i} g_1(\tau)\mathrm{d}\tau, \ldots, \int_{t_{i-1}}^{t_i} g_q(\tau)\mathrm{d}\tau\right) \qquad i = 2, \ldots, n$$

We have to note that these estimators have been already obtained by Gutiérrez et al. (1997b, 1999, 2001a).

APPLICATION TO HOUSING PRICE IN SPAIN

Econometric models have been widely used to treat some economic aspects about housing price (see, for instance, Jaffee and Rosen 1979). In particular, the housing investment equations included in some macroeconometrics models have been reviewed by Egebo and Lienert (1988). In this review, the authors compare some models applied to several developed countries (concretely the seven major OECD countries). Other models that can be cited are those of Nellis and Longbotton (1981) and Meen (1990). These last are multiplicative models that consider supply-and-demand equations connecting the housing price with some economic variables such as family income, demographic factors, mortgage credit availability, interest rate, retail price index, and so on.

In this application, we consider the evolution of housing price—concretely, new housing price in Spain. This subject is, actually, one of the most important social problems and has great repercussions in the global economy of Spain. (Note, for example, that the housing price per m^2 has increased by a 80% in the past four years.)

Our aim is to describe the development and behavior of this variable by applying the methodology explained in the previous section about the Gompertz diffusion process and by considering external influences (exogenous factors). With this treatment, some problems related to the application of the multiplicative econometric models are removed (for example, the problems that arise when the exogenous variables take negative values).

The first question that arises in this application is the choice of the exogenous factors. In this case we have considered the retail price index, the gross national product per inhabitant, and the long-term interest rate.

Nevertheless, two interesting questions need to be pointed out:

1. First, it is not common to take the real value of these variables in each observation time but the increment relative to the previous instant. We denote these values by y_{ij} $(i = 2, \ldots, n;\ j = 1, 2, 3)$ $\big(y_{1j} = 0,\ j = 1, 2, 3\big)$.
2. Second, it is normal to consider such variables to be constant functions in each time interval to be used later, for example, with forecasting aims. Such an approach is common in practice but it is not valid for our model, and so we have taken another one. Concretely, we have built the exogenous factors $g_j(t)$ $\quad (j = 1, 2, 3)$ from the observed values of the aforementioned economic variables by considering the polygonal functions

$$\begin{aligned} g_j(t) = {} & y_{i-1,j} + \big(y_{ij} - y_{i-1,j}\big)(t - t_{i-1}) \\ & t_{i-1} \leq t \leq t_i \quad i = 2, \ldots, n \quad j = 1, 2, 3 \end{aligned} \tag{16}$$

From this approach, and by denoting

$$z_{ij}(\beta) = y_{i-1,j} + (y_{ij} - y_{i-1,j}) \frac{\beta - 1 + e^{-\beta}}{\beta(1 - e^{-\beta})} \qquad i = 2, \ldots, n \quad j = 1, 2, 3$$

we have

$$\int_{t_{i-1}}^{t_i} g_j(\tau) e^{-\beta(t_i - \tau)} \mathrm{d}\tau = \gamma_\beta z_{ij}(\beta) \qquad i = 2, \ldots, n \quad j = 1, 2, 3$$

and, consequently,

$$\mathbf{u}_{i,\beta} = (1, z_{i1}(\beta), z_{i2}(\beta), z_{i3}(\beta))' \qquad i = 2, \ldots, n \tag{17}$$

Table 1 shows the observed values, from 1976 to 2001, of the endogenous and exogenous variables from which the exogenous factors have been built.

Taking into account polygonal (16) and that the columns of the matrix $\mathbf{U}_\beta$ are given by Eq. (17), the estimation procedure described in the previous section can be used by solving Eq. (11) and then

Table 1. Observed values: constant prices, base 1995

Time	Housing price (€/m^2)	Retail price index	Gross national product/inhab. (€)	Long-term interest rate
1976	102.54	16	7976	0.11
1977	122.43	19.9	8049	0.12
1978	162.09	23.9	8049	0.13
1979	211.98	27.6	8026	0.13
1980	218.29	31.9	8092	0.14
1981	230.18	36.6	8037	0.16
1982	239.24	41.8	8096	0.17
1983	248.25	46.9	8203	0.18
1984	265.27	52.2	8317	0.18
1985	290.65	56.8	8481	0.16
1986	328.88	61.8	8731	0.15
1987	407.92	65.1	9193	0.15
1988	509.97	68.2	9640	0.17
1989	628.50	72.9	10086	0.16
1990	727.63	77.7	10454	0.17
1991	830.69	82.4	10704	0.16
1992	820.51	87.2	10778	0.15
1993	816.84	91.2	10644	0.14
1994	823.06	95.5	10878	0.1
1995	851.38	100	11161	0.11
1996	866.98	103.6	11417	0.09
1997	880.63	105.6	11856	0.07
1998	921.10	107.6	12337	0.06
1999	987.58	110	12791	0.05
2000	1166.83	113.8	13214	0.06
2001	1346.27	117.9	13465	0.06

Source: I.N.E.

substituting its solution into Eqs. (12) and (13). We have to point out that Eq. (11) cannot be solved explicitly and numerical methods must be used, for example, by means of numerical packages. In this case we have used Mathematica, resulting in $\hat{\beta} = 0.0313376157$. From this value, one has

$$\hat{\mathbf{a}} = \begin{pmatrix} 0.14669244572143308 \\ 0.7718302129347099 \\ 4.167358788915776 \\ 0.2255066224128105 \end{pmatrix} \quad \text{and} \quad \hat{\sigma^2} = 0.003278028409722636$$

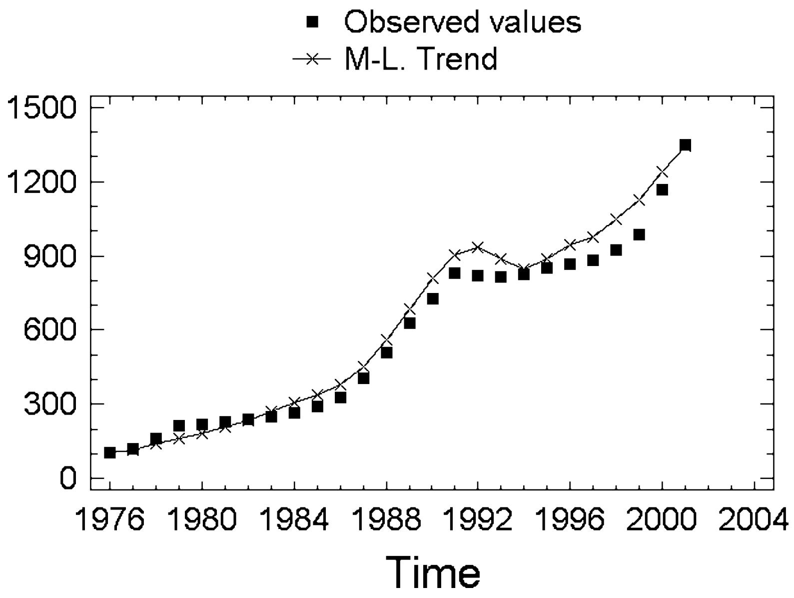

Figure 1. Estimated maximum likelihood trend versus observed values of the endogenous variable (€).

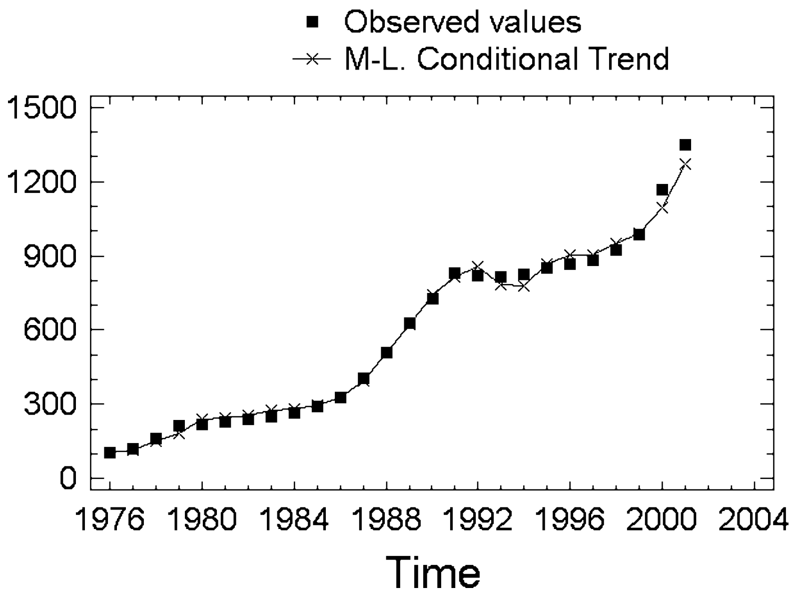

Figure 2. Estimated conditional maximum likelihood trend versus observed values of the endogenous variable (€).

Once the estimations of the parameters have been obtained, the following step is to estimate the trend of the process for the time interval considered. Alternatively, one can use the conditional trend (Gutiérrez et al. 2003) that can be calculated from Eq. (5). Figures 1 and 2 show these functions. Obviously, these functions can be use with forecasting aims if future values of the exogenous variables are available.

ACKNOWLEDGMENT

This work was supported in part by the "Ministerio de Ciencia y Tecnología", Spain, under Grant BFM 2002-03633.

REFERENCES

Bibby, B. M. and M. Sørensen. 1995. Martingale estimation functions for discretely observed diffusion processes. *Bernoulli*, 1(1/2):17–39.

Capocelli, R. M. and L. M. Ricciardi. 1974. Growth with regulation in random environment. *Kybernetik*, 15:147–157.

Dennis, B. and G. P. Patil. 1988. Application in ecology. In *Lognormal Distributions: Theory and Applications*, pp. 287–302, edited by E. L. Crow and K. Shimizu. New York: Marcel Dekker.

Frank, T. D. 2002. Multivariate Markov processes for stochastic systems with delays: Application to the stochastic Gompertz model with delays. *Physical Review E*, 66:1–8.

Egebo, T. and I. Lienert. 1988. *Modelling housing investment for seven major OECD countries*. Working paper 63, OECD Department of Economic and Statistics, Paris.

Guillouzic, S., I. L'Heureux, and A. Longtin. 2000. Rate processes in a delayed stochastically driven and overdamped system. *Physical Review E*, 61(5): 4906–4914.

Gutiérrez, R., J. M. Angulo, A. González, and R. Pérez. 1991. Inference in lognormal multidimensional diffusion process with exogenous factors: application to modelling in economics. *Applied Stochastic Models and Data Analysis*, 7:295–316.

Gutiérrez, R., P. Román, and F. Torres. 1995. A note on the Volterra integral equation for the first-passage-time density. *Journal of Applied Probability*, 32(3):635–648.

Gutiérrez, R., L. M. Ricciardi, P. Román, and F. Torres. 1997a. First-passage-time densities for time-non-homogeneous diffusion processes. *Journal of Applied Probability*, 34(3):623–631.

Gutiérrez, R., A. González, and F. Torres. 1997b. Estimation in multivariate lognormal diffusion process with exogenous factors. *Applied Statistics*, 46(1):140–146.

Gutiérrez, R., P. Román, and F. Torres. 1999. Inference and first-passage-time for the lognormal diffusion process with exogenous factors: Application to modelling in economics. *Applied Stochastic Models in Business and Industry*, 15(4):325–332.

Gutiérrez, R., P. Román, and F. Torres. 2001a. Inference on some parametric functions in the univariate lognormal diffusion process with exogenous factors. *Test*, 10(2):357–373.

Gutiérrez, R., P. Román, D. Romero, and F. Torres. 2001b. UMVU estimation of the trend and covariance function for the univariate lognormal diffusion process with exogenous factors. In *Proceedings of the Tenth International Symposium on A.S.M.D.A.*, pp. 516–521, edited by J. Jansen and C. H. Skiadas, Université de Technologie de Compiègne, Compiègne, France.

Gutiérrez, R., P. Román, D. Romero, and F. Torres. 2003. Application of the univariate lognormal diffusion process with exogenous factors in forecasting. *Cybernetics and Systems*, 34(8):709–724.

Jaffee, D. M. and K. T. Rosen. 1979. Mortgage credit availability and residential construction. *Brookings Papers on Economic Activity*, 2:333–376.

Kloeden, P. E., E. Platen, H. Schurz, and M. Sørensen. 1996. On effect of discretization on estimators of drift parameters for diffusion processes. *Journal of Applied Probability*, 33(4):1061–1076.

Meen, G. 1990. The removal of mortgage market constraints and the implications for econometrics modelling of U.K. house price. *Oxford Bulletin of Economics and Statistics*, 52(1):1–23.

Miller, J. H., K. R. A. Minard, G. A. Wind, L. B. Orner, Sasser, and R. J. Bull. 2000. In vivo MRI measurements of tumor growth induced by dichloroacetate: implication for mode action. *Toxicology*, 145(2–3):115–125.

Nellis, S. G. and J. A. Longbotton. 1981. An empirical analysis of the determination of housing prices in U.K. *Urban Studies*, 18.

Prakasa Rao, B. L. S. 1999. *Statistical inference for diffusion type processes*. London: Arnold.

Ricciardi, L. M. 1976. On the transformation of diffusion processes into the Wiener processes. *Journal of Mathematical Analysis and Applications*, 54:185–199.

Ricciardi, L. M. 1977. *Diffusion processes and related topics in biology*. Lectures Notes in Biomathematics, 14. Berlin: Springer-Verlag.

Singer, H. 2002. Parameter estimation of nonlinear stochastic differential equations: Simulated maximum likelihood versus extended Kalman filter and Itô-Taylor expansion. *Journal of Computational and Graphical Statistics*, 11(4):972–995.

Skiadas, C. H., A. N. Giovani, and I. Dimoticales. 1994. Investigation of stochastic differential models: The Gompertzian case. In *Selected topics on stochastic modelling*, pp. 296–310, edited by R. Gutiérrez and M. Valderrama, Singapore: World Scientific.

Tan, W. Y. 1986. A stochastic Gompertz birth-death process. *Statistics and Probability Letters*, 4:25–28.

Tintner, G. and J. K. Sengupta. 1972. *Stochastic economics*. New York: Academic Press.

Troynikov, V. S. and H. K. Gorfine. 1998. Alternative approach for establishing legal minimum lenghts for abalone based on stochastic growth models for length increment data. *Journal of Shellfish Research*, 17:827–831.