A Dirichlet problem involving the mean curvature operator in Minkowski space

Pedro J. Torres
(joint work with C. Bereanu and P. Jebelean)

Departamento de Matemática Aplicada,
Universidad de Granada (Spain)

Trieste 2013
On the occasion of the 60th birthday of Fabio Zanolin
The problem

We consider the existence and multiplicity of radial positive solutions of the problem

\[ \mathcal{M} v + f(|x|, v) = 0, \quad x \in B_R, \quad (1) \]
\[ v = 0, \quad x \in \partial B_R, \quad (2) \]

where \( B_R = \{ x \in \mathbb{R}^N : |x| < R \} \) and \( f : [0, R] \times [0, \alpha) \to \mathbb{R} \) is a continuous function, which is positive on \((0, R] \times (0, \alpha)\), and

\[ \mathcal{M} v = \text{div} \left( \frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right). \]
The problem

Setting, as usual, \( r = |x| \) and \( v(x) = u(r) \), the Dirichlet problem (1) reduces to the mixed boundary value problem

\[
(r^{N-1} \phi(u'))' + r^{N-1} f(r, u) = 0, \quad u'(0) = 0 = u(R),
\]

(3)

Standing hypotheses:

\((H_\phi)\) \( \phi : (-a, a) \to \mathbb{R} \) \((0 < a < \infty)\) is an odd, increasing homeomorphism with \( \phi(0) = 0 \);

\((H_f)\) \( f : [0, R] \times [0, \alpha) \to [0, \infty) \) is a continuous function with \( 0 < \alpha \leq \infty \) and \( f(r, s) > 0 \) for all \( (r, s) \in (0, R] \times (0, \alpha) \).
Existence result I

**Theorem 1**

Assume that

$$\lim_{s \to 0^+} \frac{f(r, s)}{\phi(s)} = +\infty \quad \text{uniformly with} \quad r \in [0, R] \quad (4)$$

$$\limsup_{s \to 0} \frac{\phi(\tau s)}{\phi(s)} < +\infty \quad \text{for all} \quad \tau > 0. \quad (5)$$

Then problem (3) has at least one positive solution if either $aR < \alpha$ or $\alpha = a$, $R = 1$ and

$$\lim_{s \to a^-} \frac{f(r, s)}{\phi(s)} = 0 \quad \text{uniformly with} \quad r \in [0, 1], \quad (6)$$

holds true.
Examples

Fix $0 \leq q < 1$ and let $\mu : [0, R] \rightarrow (0, \infty)$, $h : [0, R] \times [0, \infty) \rightarrow [0, \infty)$ be continuous functions.

(i) The Dirichlet problem

$$\mathcal{M} v + \mu(|x|) v^q + h(|x|, v) = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R,$$

has at least one positive classical radial solution for any $R > 0$. 
According to Theorem 3.2 and Remark 3.1 in A. Capietto, W. Dambrosio, F. Zanolin, Infinitely many radial solutions to a boundary value problem in a ball, Ann. Mat. Pura Appl. 179 (2001), 159-188.

the problem

$$\mathcal{M} v + \mu(|x|)|v|^{q-1} v = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R$$

has infinitely many radial solutions with prescribed number of nodes (the positive case is not covered), provided that $0 < q < 1$ and $\mu$ is continuously differentiable.
Examples

(ii) If $0 \leq q < 1 \leq p$ and $\lambda > 0$, then problem

$$\mathcal{M} v + \lambda v^q + v^p = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R$$

has at least one positive classical radial solution for any $R > 0$. 
Examples

(ii) If $0 \leq q < 1 \leq p$ and $\lambda > 0$, then problem

$$\mathcal{M}v + \lambda v^q + v^p = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R$$

has at least one positive classical radial solution for any $R > 0$.

In the classical case, using the upper and lower solutions method, it has been proved by Ambrosetti, Brezis and Cerami that problem

$$\Delta v + \lambda v^q + v^p = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R$$

has a positive solution iff $0 < \lambda \leq \Lambda$ for some $\Lambda > 0$ ($0 < q < 1 < p$).
Examples

(iii) The Dirichlet problems

\[ \mathcal{M}v + \frac{\mu(|x|)v^q}{\sqrt{\alpha^2 - v^2}} = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R, \]

and

\[ \mathcal{M}v + \frac{\mu(|x|)v^q}{(\alpha - v)^\gamma} = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R, \]

have at least one positive classical radial solution for any \( R < \alpha \).

(iv) If, in addition, \( \gamma < \frac{1}{2} \), then the Dirichlet problem

\[ \mathcal{M}v + \frac{\mu(|x|)v^q}{(1 - v^2)^\gamma} = 0 \quad \text{in} \quad B(R), \quad v = 0 \quad \text{on} \quad \partial B(R), \]

has at least one positive classical radial solution for any \( R \leq 1 \).
Sketch of the proof

We use the compact linear operators

\[ S : C \rightarrow C, \quad Su(r) = \frac{1}{r^{N-1}} \int_0^r t^{N-1} u(t) \, dt, \quad Su(0) = 0; \]

\[ K : C \rightarrow C, \quad Ku(r) = \int_r^R u(t) \, dt, \quad (r \in [0, R]) \]

and the Nemytskii type operator

\[ N_f : B_{\alpha} \rightarrow C, \quad N_f(u) = f(\cdot, |u(\cdot)|). \]
Lemma 1

A function $u \in C$ is a solution of (3) if and only if it is a fixed point of the continuous nonlinear operator

$$\mathcal{N} : B_\alpha \to C, \quad \mathcal{N} = K \circ \phi^{-1} \circ S \circ N_f.$$ 

Moreover, $\mathcal{N}$ is compact on $\overline{B}_\rho$ for all $\rho \in (0, \alpha)$. 

Sketch of the proof

Proposition 1
Assume (4) and (5). Then there exists $0 < \rho_0 < \alpha$ such that

$$d_{LS}[I - \mathcal{N}, B_{\rho}, 0] = 0 \quad \text{for all} \quad 0 < \rho \leq \rho_0.$$
Sketch of the proof

Let $\mathcal{H}(\lambda, \cdot) : B_\alpha \to C$ be the fixed point operator associated to
$$(r^{N-1} \phi(u'))' + r^{N-1}[f(r, |u|) + \lambda] = 0, \quad u'(0) = 0 = u(R), \quad (7)$$
where $\lambda \in [0, 1]$. 
Sketch of the proof

Let $\mathcal{H}(\lambda, \cdot) : B_\alpha \to C$ be the fixed point operator associated to

$$(r^{N-1} \phi(u'))' + r^{N-1}[f(r, |u|) + \lambda] = 0, \quad u'(0) = 0 = u(R), \quad (7)$$

where $\lambda \in [0, 1]$. We can prove

$$u \neq \mathcal{H}(\lambda, u) \quad \text{for all} \quad (\lambda, u) \in [0, 1] \times \partial B_\rho$$

This implies

$$d_{LS}[I - \mathcal{H}(0, \cdot), B_\rho, 0] = d_{LS}[I - \mathcal{H}(1, \cdot), B_\rho, 0].$$
Sketch of the proof

Let $\mathcal{H}(\lambda, \cdot) : B_\alpha \to C$ be the fixed point operator associated to

$$(r^{N-1}\phi(u'))' + r^{N-1}[f(r, |u|) + \lambda] = 0, \quad u'(0) = 0 = u(R), \quad (7)$$

where $\lambda \in [0, 1]$. We can prove

$$u \neq \mathcal{H}(\lambda, u) \quad \text{for all} \quad (\lambda, u) \in [0, 1] \times \partial B_\rho$$

This implies

$$d_{LS}[I - \mathcal{H}(0, \cdot), B_\rho, 0] = d_{LS}[I - \mathcal{H}(1, \cdot), B_\rho, 0].$$

Besides

$$u \neq \mathcal{H}(1, u) \quad \text{for all} \quad u \in \overline{B_\rho},$$

implying that

$$d_{LS}[I - \mathcal{H}(1, \cdot), B_\rho, 0] = 0.$$

Consequently,

$$d_{LS}[I - \mathcal{N}, B_\rho, 0] = d_{LS}[I - \mathcal{H}(0, \cdot), B_\rho, 0] = d_{LS}[I - \mathcal{H}(1, \cdot), B_\rho, 0] = 0.$$
Proposition 2

If $aR < \alpha$, then one has

$$d_{LS}[I - \mathcal{N}, B_{aR}, 0] = 1.$$
Sketch of the proof

**Proposition 2**

If $aR < \alpha$, then one has

$$d_{LS}[I - \mathcal{N}, B_{aR}, 0] = 1.$$ 

Consider the compact homotopy

$$\mathcal{H} : [0, 1] \times \overline{B}_{aR} \rightarrow C, \quad \mathcal{H}(\lambda, u) = \lambda \mathcal{N}(u).$$
Proposition 2

If \( aR < \alpha \), then one has

\[
d_{LS}[I - \mathcal{N}, B_{aR}, 0] = 1.
\]

Consider the compact homotopy

\[
\mathcal{H} : [0, 1] \times \overline{B}_{aR} \to C, \quad \mathcal{H}(\lambda, u) = \lambda\mathcal{N}(u).
\]

Let \((\lambda, u) \in [0, 1] \times \overline{B}_{aR}\) be such that \(\mathcal{H}(\lambda, u) = u\). It follows immediately that \(\|u'\| < a\), implying that \(\|u\| < aR\). So,

\[
u \neq \mathcal{H}(\lambda, u) \quad \text{for all} \quad (\lambda, u) \in [0, 1] \times \partial B_{aR},
\]
Sketch of the proof

**Proposition 2**

If $aR < \alpha$, then one has

$$d_{LS}[I - \mathcal{N}, B_{aR}, 0] = 1.$$ 

Consider the compact homotopy

$$\mathcal{H} : [0, 1] \times \overline{B}_{aR} \to C, \quad \mathcal{H}(\lambda, u) = \lambda \mathcal{N}(u).$$

Let $(\lambda, u) \in [0, 1] \times \overline{B}_{aR}$ be such that $\mathcal{H}(\lambda, u) = u$. It follows immediately that $||u'|| < a$, implying that $||u|| < aR$. So,

$$u \neq \mathcal{H}(\lambda, u) \quad \text{for all} \quad (\lambda, u) \in [0, 1] \times \partial B_{aR},$$

which implies that

$$d_{LS}[I - \mathcal{H}(0, \cdot), B_{aR}, 0] = d_{LS}[I - \mathcal{H}(1, \cdot), B_{aR}, 0].$$
Sketch of the proof

Proposition 2

If \( aR < \alpha \), then one has

\[
d_{LS}[I - \mathcal{N}, B_{aR}, 0] = 1.
\]

Consider the compact homotopy

\[
\mathcal{H} : [0, 1] \times \overline{B}_{aR} \to C, \quad \mathcal{H}(\lambda, u) = \lambda \mathcal{N}(u).
\]

Let \((\lambda, u) \in [0, 1] \times \overline{B}_{aR}\) be such that \(\mathcal{H}(\lambda, u) = u\). It follows immediately that \(\|u'\| < a\), implying that \(\|u\| < aR\). So,

\[
u \neq \mathcal{H}(\lambda, u) \quad \text{for all} \quad (\lambda, u) \in [0, 1] \times \partial B_{aR},
\]

which implies that

\[
d_{LS}[I - \mathcal{H}(0, \cdot), B_{aR}, 0] = d_{LS}[I - \mathcal{H}(1, \cdot), B_{aR}, 0].
\]

Consequently,

\[
d_{LS}[I - \mathcal{N}, B_{aR}, 0] = d_{LS}[I, B_{aR}, 0] = 1
\]
If $\alpha = a$, then in Proposition 2 one has that $R < 1$. We consider now the case $R = 1$, assuming that $f$ is sublinear with respect to $\phi$ at $a$.

**Proposition 3**

Assume that $a = \alpha$ and $R = 1$. If

$$\lim_{s \to a^-} \frac{f(r, s)}{\phi(s)} = 0$$

uniformly with $r \in [0, 1]$, then there exists $0 < \delta_1 < a$ such that

$$d_{LS}[I - N, B_{\delta}, 0] = 1$$

for all $\delta_1 \leq \delta < a$. 
A second existence result

Now, we study the problem

\[
\left( r^{N-1} \frac{u'}{\sqrt{1 - u'^2}} \right)' + r^{N-1} \mu(r)p(u) = 0, \quad u'(0) = 0 = u(R), \tag{8}
\]

under the standing hypotheses:

\((H_\mu)\) \(\mu : [0, R] \to \mathbb{R} \) is continuous and \(\mu(r) > 0 \) for all \( r > 0 \);

\((H_p)\) \(p : [0, \infty) \to \mathbb{R} \) is a continuous function such that \(p(0) = 0\) and \(p(s) > 0\) for all \( s > 0 \).
A second existence result

**Theorem 2**

Let $P$ be the primitive of $p$ with $P(0) = 0$. If

$$R^N < N \int_0^R r^{N-1} \mu(r) P(R - r) dr,$$

(9)

then problem (12) has at least one solution $u$ such that $u > 0$ on $[0, R)$ and $u$ is strictly decreasing.
Given \( m \geq 0 \) and \( q > 0 \), let us consider the Hénon type problem

\[
\mathcal{M}v + \lambda |x|^m v^q = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R. \tag{10}
\]

It is easy to see that in this case inequality (9) becomes

\[
1 < \lambda \frac{NR^{m+q+1} \Gamma(q+2) \Gamma(N+m)}{(q+1) \Gamma(N+m+q+2)}. \tag{11}
\]

Consequently, if (11) holds then problem (10) has at least one classical positive radial solution.
Sketch of the proof

We follow a VARIATIONAL APPROACH. The problem is

\[
\left( r^{N-1} \frac{u'}{\sqrt{1 - u'^2}} \right)' + r^{N-1} \mu(r)p(u) = 0, \quad u'(0) = 0 = u(R),\tag{12}
\]
We follow a VARIATIONAL APPROACH. The problem is

$$\left( r^{N-1} \frac{u'}{\sqrt{1 - u'^2}} \right)' + r^{N-1} \mu(r)p(u) = 0, \quad u'(0) = 0 = u(R), \quad (12)$$

Define $K_0 := \{ v \in W^{1,\infty} : \| v' \| \leq a, v(R) = 0 \}$. The associated energy functional $I : C \to (-\infty, +\infty]$ is

$$I(v) = \frac{R^N}{N} - \int_0^R r^{N-1} \sqrt{1 - v'^2} \, dr - \int_0^R r^{N-1} \mu(r)P(v) \, dr \quad (v \in K_0)$$

and $I \equiv +\infty$ on $C \setminus K_0$. 
Step 1: Each critical point of \( I \) is a solution of (12). Moreover, (12) has a solution which is a minimum point of \( I \) on \( C \).
Sketch of the proof

**Step 1:** Each critical point of $I$ is a solution of (12). Moreover, (12) has a solution which is a minimum point of $I$ on $C$.

**Step 2:** Assume $\inf_{K_0} I < 0$. Then problem (12) has at least one solution $u$ such that $u > 0$ on $[0, R)$ and $u$ is strictly decreasing.
Sketch of the proof

**Step 1:** Each critical point of $I$ is a solution of (12). Moreover, (12) has a solution which is a minimum point of $I$ on $C$.

**Step 2:** Assume $\inf_{K_0} I < 0$. Then problem (12) has at least one solution $u$ such that $u > 0$ on $[0, R)$ and $u$ is strictly decreasing.

**Step 3:** Consider the function $v_R \in K_0$ given by

$$v_R(r) = R - r \quad \text{for all} \quad r \in [0, R].$$

Using (9), one gets

$$I(v_R) = \frac{R^N}{N} - \int_0^R r^{N-1} \mu(r) P(R - r) dr < 0.$$
A multiplicity result for the Hénon problem

Given $m \geq 0$ and $q > 1$, let us consider the Hénon type problem

$$
\mathcal{M} v + \lambda |x|^m v^q = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R.
$$
(13)
Given $m \geq 0$ and $q > 1$, let us consider the Hénon type problem

$$\mathcal{M} v + \lambda |x|^m v^q = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R. \quad (14)$$

It is interesting to compare with the classical Hénon equation

$$\Delta v + \lambda |x|^m v^q = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R.$$

If $N \geq 3$, it has a unique positive radial solution if $1 < q < \frac{N+2m+2}{N-2}$ and no solution if $q \geq \frac{N+2m+2}{N-2}$ (Pohozaev identity).
Theorem 3

Define $\mu_M := \max_{[0,R]} |\mu|$. There exists $\Lambda > 2N/(\mu_M R^{q+1})$ such that problem (13) has zero, at least one or at least two positive solutions according to $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$ or $\lambda > \Lambda$. Moreover, $\Lambda$ is strictly decreasing with respect to $R$. 
Sketch of the proof

**Step 1:** No solution for small \( \lambda \).
Sketch of the proof

**Step 1:** No solution for small $\lambda$.

**Step 2:**

$$d_{LS}[I - \mathcal{N}, B_\rho, 0] = 1 \quad \text{for all } \rho \geq a(R + 1).$$
Sketch of the proof

Step 1: No solution for small $\lambda$.

Step 2:

$$d_{LS}[I - N, B_\rho, 0] = 1$$
for all $\rho \geq a(R + 1)$.

Step 3:

$$d_{LS}[I - N_f, B_\rho, 0] = 1$$
for all $0 < \rho \leq \rho_0$.

We conclude by a simple excision argument.
Sketch of the proof

Step 1: No solution for small $\lambda$.

Step 2:

$$d_{LS}[l - \mathcal{N}, B_\rho, 0] = 1 \quad \text{for all } \rho \geq a(R + 1).$$

Step 3:

$$d_{LS}[l - \mathcal{N}_f, B_\rho, 0] = 1 \quad \text{for all } 0 < \rho \leq \rho_0.$$

Step 4:

$$d_{LS}[l - \mathcal{N}, B(u_0, \rho), 0] = 1 \quad \text{for all } 0 < \rho \leq \rho_0,$$

where $u_0$ is the solution found by the variational approach.

We conclude by a simple excision argument.
Sketch of the proof

Step 1: No solution for small $\lambda$.

Step 2:

$$d_{LS}[I - \mathcal{N}, B_\rho, 0] = 1 \quad \text{for all } \rho \geq a(R + 1).$$

Step 3:

$$d_{LS}[I - \mathcal{N}_f, B_\rho, 0] = 1 \quad \text{for all } 0 < \rho \leq \rho_0.$$ 

Step 4:

$$d_{LS}[I - \mathcal{N}, B(u_0, \rho), 0] = 1 \quad \text{for all } 0 < \rho \leq \rho_0,$$

where $u_0$ is the solution found by the variational approach.

We conclude by a simple excision argument.
THANKS
AND
CONGRATULATIONS FABIO!!