Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Physica D 221 (2006) 31–36
PHYSICA D
www.elsevier.com/locate/physd

Similarity transformations for nonlinear Schrödinger equations with time-dependent coefficients

Víctor M. Pérez-García[a,*], Pedro J. Torres[b], Vladimir V. Konotop[c,d]

[a] Departamento de Matemáticas, E. T. S. de Ingenieros Industriales, Universidad de Castilla-La Mancha, 13071 Ciudad Real, Spain
[b] Departamento de Matemática Aplicada, Universidad de Granada, 18071 Granada, Spain
[c] Centro de Física Teórica e Computacional, Universidade de Lisboa, Complexo Interdisciplinar, Av. Prof. Gama Pinto 2, Lisbon 1649-003, Portugal
[d] Departamento de Física, Universidade de Lisboa, Campo Grande, Ed. C8, Piso 6, Lisboa 1749-016, Portugal

Received 21 December 2005; received in revised form 19 June 2006; accepted 10 July 2006
Available online 8 August 2006
Communicated by A. Doelman

Abstract

In this paper we obtain a similarity transformation connecting nonlinear Schrödinger equations with time-varying coefficients with the autonomous cubic nonlinear Schrödinger equation. As applications we construct exact breathing solutions to multidimensional non-autonomous nonlinear Schrödinger equations and discuss how to construct time-dependent coefficients leading to solutions which collapse weakly in three-dimensional scenarios. Our results are applicable to the study of the dynamics of Bose–Einstein condensates in the mean-field limit and dispersion-managed optical systems.
© 2006 Elsevier B.V. All rights reserved.

Keywords: Nonlinear Schrödinger equations; Similarity transformations

1. Introduction

Nonlinear Schrödinger (NLS) equations appear in the modelling of many physical phenomena [1] such as propagation of laser beams in nonlinear media [2,3], plasma dynamics [4], mean field dynamics of Bose–Einstein condensates [5], condensed matter [6], etc.

The interest on NLS equations with cubic nonlinearities whose coefficients depend on the evolution variable has increased in the last few years driven by their applications in different fields. If the evolution variable is denoted as t, many of those problems are special cases of the general equations

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{\alpha(t)}{2}\triangle\psi + \frac{1}{2}\Omega(t)r^2\psi + g(t)|\psi|^2\psi - \mathrm{i}\gamma(t)\psi, \quad (1)$$

whose (complex) solutions are considered on $\mathbb{R}^d$, i.e. $x \in \mathbb{R}^d$, with initial data $\psi(x,0) = \psi_0(x)$.

Eq. (1) arises as a model for the dynamics of Bose–Einstein condensates in the mean field approximation when the nonlinear coefficient $g(t)$ is physically controlled by acting on the so-called Feschbach resonances. The term depending quadratically on the spatial variables ($r = |x|$) allows for the consideration of an external time-modulated trapping potential. Finally, the dispersion coefficient $\alpha(t)$ can account for the effect of an additional lattice potential described in the effective mass approximation.

Another field of application where the coefficients of the NLS model depend on the evolution variable is that of dispersion-managed optical solitons when $d = 1$. In that case it is the term with second derivatives the one which is modulated along the longitudinal direction in order to compensate the nonlinear effects of loss and gain (see e.g. the review [7]).

When more than one spatial dimension and dispersion management are taken into consideration, the situation is more complicated as has been studied in the last few years [8–10]. In those situations typically $p < d$ of the spatial variables in Eq. (1) are modulated, i.e. the term with spatial derivatives is of the form $\alpha(t)\sum_{j=1}^{p}\partial^2/\partial x_j^2 + \sum_{j=p+1}^{d}\partial^2/\partial x_j^2$. In this

* Corresponding author. Tel.: +34 926295435; fax: +34 926295361.
E-mail address: victor.perezgarcia@uclm.es (V.M. Pérez-García).

0167-2789/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.physd.2006.07.002

case, the asymmetry between the spatial variables complicates the application of the method to be presented later and this is why in this paper we will restrict ourselves to the symmetric modulation of the spatial derivatives as in Eq. (1).

The possibility of managing the nonlinear coefficient $g(t)$ has led to the discovery of *stabilized solitons*, which appear when $d = 2$, $\alpha(t)$ is constant and $g(t)$ oscillates periodically between negative values corresponding to compression of initial data and values corresponding to expanding solutions. These breathing solutions were predicted to exist [11] and very recently observed [12] in optical applications, but it was soon realized that a similar phenomenon can be induced in Bose–Einstein condensates [13–16]. A somewhat related problem is that of *enhancement of the collapse* by time-dependent nonlinearities [17]. The case $d = 1$ is also very important, in fact Ref. [18] stimulated many studies including temporal modulation of various coefficients [20,21].

The aim of this paper is to provide information on the dynamics of solutions of equations of the form (1) by using similarity transformations which connect Eq. (1) with models for which the behavior of solutions is better known such as the standard cubic nonlinear Schrödinger equation.

2. Similarity transformations for non-autonomous NLS equations

2.1. Equations for the transformation parameters

Let us consider the similarity transformation

$$\psi(x,t) = \frac{1}{\ell(t)} \mathrm{e}^{\mathrm{i} f(t) r^2} u\left(\frac{x}{L(t)}, \tau(t)\right), \tag{2}$$

where $\ell(t)$, $L(t)$, $f(t)$ and $\tau(t)$ are real scaling functions to be obtained later.

This transformation is analogous to the so-called *lens transformation* [22] which has been also used in other BEC problems [23,24] and differs (mainly because of the extra freedom provided by the definition of the new time τ) from the transformations which are frequent in the context of dispersion-managed solitons [18]. For simplicity and without loss of generality we choose $t_0 = 0$, $\ell(0) = 1$, $L(0) = 1$ and $\tau(0) = 0$. In the linear case, a similar transformation has been used in Ref. [25] to transform the linear Schrödinger equation into a time-dependent problem whose asymptotic behavior provides information on the autonomous one via rescaling. Other different similarity transformations have been used in different contexts [23,19].

In this paper we will use the nonlinear rescaling group given by Eq. (2) to transform the time dependent problem given by Eq. (1) into a simpler, time-independent one by choosing appropriately the transformation parameters. To determine the scaling functions we define $\eta = x/L(t)$ and require the function $u = u(\eta, \tau)$ to solve the equation

$$\mathrm{i}\frac{\partial u}{\partial \tau} = -\frac{1}{2}\triangle_\eta u, +\sigma |u|^2 u \tag{3}$$

where $\sigma = \pm 1$ (the choice of σ may imply a redefinition of the sign of g) and initial data given by $u(\eta, 0) = \mathrm{e}^{-\mathrm{i} f_0 r^2}\psi_0(\eta)$, being $\eta = x$ and $f_0 = f(0)$. Inserting Eq. (2) into Eq. (1) leads to the following equations for the scaling parameters

$$\frac{\mathrm{d}\ell}{\mathrm{d}t} = \alpha(t) f \mathrm{d}\ell + \gamma(t)\ell, \tag{4a}$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = -2\alpha(t) f^2 - \frac{1}{2}\Omega(t), \tag{4b}$$

$$\frac{\mathrm{d}L}{\mathrm{d}t} = 2\alpha(t) f L, \tag{4c}$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\alpha(t)}{L^2}, \tag{4d}$$

$$\frac{\alpha(t)}{L^2} = \sigma\frac{g(t)}{\ell^2}. \tag{4e}$$

Since Eq. (3) is the autonomous cubic NLS equation the transformation given by Eqs. (4) allows us to study the properties of solutions of Eq. (1) from those of the cubic nonlinear Schrödinger equation given by Eq. (3) for which many results are available (see e.g. [22]), provided the coefficients are such that they satisfy Eqs. (4).

An obvious application of our ideas is to start with stationary solutions of Eq. (3) which are known to exist for $\sigma = -1$ [22] and from them construct breathing or blowing up solutions of Eq. (1). In this case, all the time dynamics of the solutions, which are self-similar, is described by Eqs. (4). We will present explicit examples of this methodology later.

Another application for the case $d = 2$ is to use the method of moments in the framework of Eq. (3) to compute the evolution of some relevant integral quantities in the non-autonomous case.

Finally, explicit single and multi-soliton solutions of Eq. (3) for $d = 1$ can be used to construct solutions of Eq. (1) with more complex dynamics in the case of coefficients varying in time, a fact which has been partially explored in the field of dispersion-managed solitons.

These are only a few illustrations of what can be expected from the application of our transformations to specific situations.

In what follows we focus on describing the dynamics which can be obtained from Eqs. (4) with the understanding that the full dynamics in the framework of Eq. (1) will be a combination of the dynamics of u given by Eq. (3) and the dynamics of the scaling parameters given by Eq. (4).

2.2. Solution of the equations for the scaling parameters

The solution of Eqs. (4a) and (4c) is immediate

$$\ell(t) = \Gamma(t)\exp\left(d\int_0^t \alpha(t')f(t')\mathrm{d}t'\right), \tag{5a}$$

$$L(t) = \exp\left(2\int_0^t \alpha(t')f(t')\mathrm{d}t'\right), \tag{5b}$$

where

$$\Gamma(t) = \exp\left(\int_0^t \gamma(t')\mathrm{d}t'\right), \tag{6}$$

and thus obviously $\ell(t) = \Gamma(t)L(t)^{d/2}$ which, together with Eq. (4e) implies that

$$g(t) = \sigma\alpha(t)\Gamma(t)^2 L(t)^{d-2}. \tag{7}$$

This means that once two of the functions $\alpha(t)$, $g(t)$, and $\gamma(t)$ are chosen, the third one is fixed by Eq. (7). This restriction limits the number of situations in which the method of scaling transformations can provide information on the dynamics of the solutions.

The next step is to solve Eq. (4b) which using Eqs. (5), can be used to obtain $L(t)$ and $\ell(t)$. Unfortunately, the general solution of Eq. (4b) cannot be written explicitly.

In the next section we will show how it is possible to provide much information on the solutions of Eq. (4b) in specific cases.

3. Particular cases

3.1. Dissipationless case without external potentials $\Omega(t) = \Gamma(t) = 0$

Let us first consider the case when $\Omega(t) = 0$. Eq. (4b) can be integrated to get

$$f(t) = \frac{f_0}{2f_0\int_0^t \alpha(t')\mathrm{d}t' + 1}. \tag{8}$$

Substituting this expression into Eqs. (5) we get the solution in the form of quadratures.

A typical problem in the context of dispersion-managed systems is the existence of stable solitons, which in our language would correspond to periodic solutions. Limiting the consideration to the case where $|2f_0\int_0^t \alpha(t')\mathrm{d}t'| < 1$ and using the explicit forms of the solutions obtained from Eq. (8) we get that in order to satisfy the periodicity condition $L(T) = L(0)$ the following condition must hold

$$\int_0^T \frac{\alpha(t)}{1 + 2f_0\int_0^t \alpha(t')\mathrm{d}t'}\mathrm{d}t = 0. \tag{9}$$

Using the change of variables $\beta(t) = f_0\int_0^t \alpha(t')\mathrm{d}t'$ this condition is found to be equivalent to $\log(1 + 2\beta(T)) = 0$, which implies $\beta(T) = 0$ and thus

$$\int_0^T \alpha(t')\mathrm{d}t' = 0. \tag{10}$$

In particular, this result implies that under the assumption $\Omega(t) = 0$ any function $\alpha(t)$ periodic and with zero average will lead to periodic $L(t)$, $\ell(t)$ and $f(t)$. From now on, we assume that T is the smallest period of $\alpha(t)$.

Using Eq. (8) we can integrate Eq. (4d) to obtain

$$\tau(t) = \frac{\int_0^t \alpha(t')\mathrm{d}t'}{2f_0\int_0^t \alpha(t')\mathrm{d}t' + 1}. \tag{11}$$

Obviously, for the specific case in which (10) holds we have that $\tau(T) = 0$ and $\tau(t)$ is bounded, i.e. $\tau_m \le \tau(t) \le \tau_M$ with $\tau_m\tau_M < 0$.

3.2. Systems without dispersion management $\alpha(t) \equiv 1$

Eq. (4b) in this case reduces to

$$\frac{\mathrm{d}f}{\mathrm{d}t} = -2f^2 - \frac{1}{2}\Omega(t) \tag{12}$$

valid for arbitrary dimension d. In general, it is not possible to find explicit solutions to this Ricatti equation: however some general information on the solutions can be obtained from the classical tools from the qualitative theory of ordinary differential equations. For instance, if Ω is T-periodic and negative, then there exists two (real) periodic solutions of Eq. (4b), one of them positive and the other negative. This is easily deduced from the classical theory of upper and lower solutions [26]. This sign information can be used to get the qualitative asymptotic behaviour of ℓ, L from Eq. (5). For instance, if f is chosen positive then

$$\lim_{t\to+\infty} \ell(t) = \lim_{t\to+\infty} L(t) = +\infty, \tag{13a}$$

whereas if f is negative then

$$\lim_{t\to+\infty} \ell(t) = \lim_{t\to+\infty} L(t) = 0. \tag{13b}$$

Additional information can be provided by writing $\Omega(t) = \lambda + \tilde{\Omega}(t)$, where $\int_0^T \tilde{\Omega}(t)\mathrm{d}t = 0$ and λ is considered as a parameter. Then, a typical Ambrosetti–Prodi result holds, that is, there exists a saddle–node bifurcation for a critical value $\lambda_0 < 0$, such for $\lambda < \lambda_0$ there exist two periodic solutions, while for $\lambda > \lambda_0$ no periodic solutions exist. Finally, for the critical value $\lambda = \lambda_0$, there is exactly a periodic solution.

When Ω is a general function (not necessarily T-periodic), the following reasoning is useful. If x is a real solution of the linear equation

$$\ddot{x} + \Omega(t)x = 0, \tag{14}$$

then it is a simple substitution to check that $f(t) = \dot{x}(t)/2x(t)$ is a (real) solution of Eq. (12). Note that from Eq. (5) we get $\ell(t) = [x(t)/x(0)]^{d/2}$, $L(t) = x(t)/x(0)$. Eq. (14) is the well-known Hill equation, which has been widely studied [27]. If Ω is T-periodic, then Eq. (14) is oscillatory (i.e., every solution has an infinite number of zeroes going to $\pm\infty$) if and only if the mean value of Ω is positive. In this case, the solution $f(t) = \dot{x}(t)/2x(t)$ is only defined in a finite interval (t_0, t_1) where t_0, t_1 are two consecutive zeroes of x. As a consequence,

$$\lim_{t\to t_1} \ell(t) = \lim_{t\to t_1} L(t) = 0, \tag{15}$$

i.e. there is finite time collapse.

3.3. Case of sign definite $\alpha(t)$

In this subsection we go back to the general case under the restriction of sign definite $\alpha(t) \neq 0$ for any t. Again, upper and lower solutions together with standard arguments from the theory of ordinary differential equations provide some useful information.

If α, Ω are periodic of the same period and $\alpha(t)\Omega(t) < 0$ for all t (that is, the respective signs are constant and opposite) then there exist two (real) periodic solutions of Eq. (4b), one of them positive and the other negative. Also a saddle–node bifurcation like in the previous case holds, where the sign of α determines the direction of such a bifurcation.

Moreover, by using the following change of variables

$$\varphi = -\frac{\dot{\alpha}}{4\alpha} + \alpha(t) f, \tag{16}$$

the Riccati equation (4b) is transformed into the simpler one

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -2\varphi^2 - \frac{1}{2}\hat{\Omega}(t), \tag{17}$$

with

$$\hat{\Omega}(t) = \alpha(t)\Omega(t) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\alpha}}{\alpha}\right) - \frac{1}{4}\left(\frac{\dot{\alpha}}{\alpha}\right)^2. \tag{18}$$

This transformation allows us to reduce the analysis to the case of constant dispersion $\alpha(t) = 1$ which was discussed in Section 3.2.

Finally, we want to discuss how, given an arbitrary T-periodic coefficient $\alpha(t)$, we can design $\Omega(t)$, $g(t)$ in order to get periodic responses such as those obtained in the system without external potential. Looking to Eq. (4d), one realizes that a necessary condition for the existence of a T-periodic $\tau(t)$ is that $\alpha(t)$ must change its sign. This case cannot be treated with the previous arguments, but we can use an inverse procedure. Take L to be a T-periodic function such that

$$\int_0^T \frac{\alpha(t)}{L^2(t)}\mathrm{d}t = 0. \tag{19}$$

Then $\tau(t)$ is T-periodic. Once L is fixed, f is determined by Eq. (4c) and finally Ω is given by Eq. (4b). Of course, g is always fixed by the restriction Eq. (7).

Finally, if we are not interested in the periodicity of $\tau(t)$ but only in the rest of the coefficients, a similar trick is at hand. Given an arbitrary α, take f such that $\int_0^T \alpha(t) f(t)\mathrm{d}t = 0$. This guarantees the existence of periodic solutions ℓ, L of Eqs. (4a) and (4b) (in fact all the solutions of these equations are periodic) and again Ω is determined by Eq. (4b).

3.4. Quasiperiodic coefficients

Let us suppose that the coefficients α, Ω are periodic with different period or, more generally, quasiperiodic functions. Assuming again that $\alpha(t) \neq 0$ for any t, the change given by (16) leads again to the Riccati equation (17) with a quasiperiodic coefficient $\hat{\Omega}(t)$. Then, the general theory of almost-periodic equations [28] can be used in order to prove a saddle–node bifurcation as in Section 3.2. The analogous of the mean value for a quasiperiodic function $p(y)$ is the generalized mean value

$$M[p] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} p(t)\mathrm{d}t. \tag{20}$$

Again, we parametrize $\hat{\Omega}(t) = \lambda + p(t)$ with $M[p] = 0$ and then there exists $\lambda_0 < 0$, such that there exist two quasiperiodic solutions for $\lambda < \lambda_0$ and no quasiperiodic solutions for $\lambda > \lambda_0$. In this situation the information on the critical value $\lambda = \lambda_0$ is lost, and the only result remaining is the existence of a bounded solution.

4. Applications

In this section we will present a few specific applications of the formulae presented in the previous sections. We do not intend to be exhaustive but only to present a few examples showing the many possibilities opened by Eqs. (4).

4.1. Explicit construction of a pulsating two-dimensional breather

Let us consider a system ruled by the equations

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{1}{2}\Delta\psi + \frac{1}{2}\Omega(t)r^2\psi - |\psi|^2\psi, \tag{21}$$

which may correspond to a quasi-two-dimensional Bose–Einstein condensate in an oscillating trap. Eqs. are a particular case of Eq. (1) corresponding to the case discussed in Section 3.2. To fix ideas let us consider

$$\Omega(t) = m\left(1 - 2\,\mathrm{sn}^2(t,m)\right), \tag{22}$$

where $\mathrm{sn}(x,m)$ is a Jacobi elliptic function, which is periodic. For this particular choice of $\Omega(t)$ the solution of Eq. (14) can be constructed explicitly in the form

$$x(t) = \mathrm{dn}(t,m), \tag{23}$$

thus $x(t)$ is a positive periodic function.

Let U be a solution of

$$\frac{1}{2}\triangle_\eta U - U + U^3 = 0, \tag{24}$$

with $U(\eta) \to 0$ when $|\eta| \to \infty$ and $u(\eta,\tau) = U(\eta)\mathrm{e}^{\mathrm{i}\tau}$. Such kinds of solutions have been studied in many papers, the most famous of them being the so-called Townes soliton [22], which is a nodeless solution decaying exponentially at infinity. Then, the function

$$\psi = \frac{1}{\mathrm{dn}(t,m)}U\left(\frac{x}{\mathrm{dn}(t,m)},\tau(t)\right)\mathrm{e}^{\mathrm{i}\tau(t)} \times \exp\left(-\mathrm{i}r^2\frac{m\,\mathrm{cn}(t,m)\,\mathrm{sn}(t,m)}{2\,\mathrm{dn}(t,m)}\right), \tag{25}$$

where $\tau(t) = \int_0^t \mathrm{dn}(t',m)^{-2}\mathrm{d}t'$, is a periodically pulsating soliton solution of Eq. (21).

4.2. Weak collapse with asymptotically vanishing nonlinearity in three dimensions

It is well known that initial data ruled by nonlinear Schrödinger equations with attractive nonlinearities (i.e. equations of the form (3) with $\sigma = -1$) may suffer blow-up in finite time [29,22]. In two spatial dimensions the collapsing part of the solution has a finite L^2 norm, a phenomenon known as weak collapse, while in three spatial dimensions we have the so-called super-strong collapse which can happen even with solutions with arbitrarily small L^2 norm.

Our similarity transformations can be used to construct time-dependent coefficients which lead to solutions displaying weak collapse in three-dimensional scenarios. The most obvious solutions displaying collapse are those for which $L(t) \to 0$ when $t \to t_*$ (with $t_* > t_0 \in \mathbb{R}$).

To fix ideas let us consider a situation with $\Gamma = 0$ and $\alpha(t) = 1$ (although the same result holds for more general situations), and $d = 3$.

Let us take $\Omega(t) = 1 + \epsilon \sin \omega t$ with $0 < \epsilon < 1$, which has positive average, and initial data corresponding to a Townes soliton of Eq. (24) then the discussion after Eq. (14) holds and we can ensure that there is a finite time t_1 in which the solutions blows up as indicated by Eq. (15).

But from Eq. (7) we see $L(t) \to 0$ implies $g(t) \to 0$ when $d = 3$, i. e. the nonlinearity vanishes asymptotically close to the collapse point.

Thus we obtain solutions of Eq. (1) collapsing as a whole (i.e. weakly) at $t = t_1$ when $d = 3$ even when $g(t) \to 0$ when $t \to t_1$. This situation is characteristic of two-dimensional scenarios for which the dynamics is "less singular" than the usual strong collapse scenarios happening in three-dimensional problems with cubic nonlinearities. The smoother behavior of the situation is induced by the decay of the nonlinearity near the singularity.

5. Conclusions

In this paper we have constructed similarity transformations connecting some families of nonlinear Schrödinger equations with time-varying coefficients with the autonomous cubic nonlinear Schrödinger equation. Although we have restricted our attention to the case of cubic nonlinearity in Eq. (1) however, the extension to more general power-type nonlinearities would be technically straightforward.

These similarity transformations hold when specific conditions linking the modulation of the dispersion, potential and nonlinearity are satisfied. In those cases they can provide valuable information since they allow us to apply all known results for the autonomous cubic nonlinear Schrödinger equation to the non-autonomous case.

We have discussed several cases in which information on the dynamics of the transformations can be obtained and provided two explicit examples of application. The first one is based on using stationary solutions of the autonomous nonlinear Schrödinger equation in which we have discussed how to construct exact breathing solutions to the non-autonomous nonlinear Schrödinger equation. A second application is the explicit construction of time dependent coefficients leading to solutions displaying weak collapse in three-dimensional scenarios, which is a very unexpected result. i.e. we obtain a family of collapsing solutions with a behavior which is characteristic of two-dimensional scenarios in a three-dimensional setting by taking a time-dependent nonlinear coefficient which vanishes at the collapse point.

Our results can find physical applicability in specific examples of mean field models of Bose–Einstein condensates and in the field of dispersion-managed optical systems.

Acknowledgements

This work has been supported by grants BFM2003-02832 and MTM2005-03483 (Ministerio de Educación y Ciencia, Spain), PAI-05-001 (Consejería de Educación y Ciencia de la Junta de Comunidades de Castilla-La Mancha, Spain) and POCI/FIS/56237/2004 (the Fundação para a Ciência e a Tecnologia, Portugal and European program FEDER).

References

[1] L. Vázquez, L. Streit, V.M. Pérez-García (Eds.), Nonlinear Klein–Gordon and Schrödinger Systems: Theory and Applications, World Scientific, Singapore, 1997.
[2] Y. Kivshar, G.P. Agrawal, Optical Solitons: From Fibers to Photonic Crystals, Academic Press, 2003.
[3] A. Hasegawa, Optical Solitons in Fibers, Springer-Verlag, Berlin, 1989.
[4] R.K. Dodd, J.C. Eilbeck, J.D. Gibbon, H.C. Morris, Solitons and Nonlinear Wave Equations, Academic Press, New York, 1982.
[5] L.P. Pitaevskii, S. Stringari, Bose–Einstein Condensation, Oxford, 2003.
[6] A. Scott, Nonlinear Science: Emergence and dynamics of coherent structures, in: Oxford Appl. and Eng. Mathematics, vol. 1, Oxford, 1999.
[7] S.K. Turitsyn, E.G. Shapiro, S.B. Medvedev, M.P. Fedoruk, V.K. Mezentsev, Physics and mathematics of dispersion-managed optical solitons, C. R. Physique 4 (2003) 145.
[8] M. Matuszewski, M. Trippenbach, B.A. Malomed, E. Infeld, A.A. Skorupski, Two-dimensional dispersion-managed light bullets in Kerr media, Phys. Rev. E 70 (2004) 016603.
[9] M. Matuszewski, E. Infeld, G. Rowlands, M. Trippenbach, Stability analysis of three-dimensional breather solitons in a Bose–Einstein condensate, Proc. R. Soc. A 461 (2005) 3561.
[10] M.I. Rodas-Verde, G.D. Montesinos, H. Michinel, V.M. Perez-Garcia, Stable propagation of pulsed beams in Kerr focusing media with modulated dispersion, J. Opt. Soc. Am. B 23 (2006) 56.
[11] L. Berge, V.K. Mezentsev, J.J. Rasmussen, P.L. Christiansen, Y.B. Gaididei, Self-guiding light in layered nonlinear media, Opt. Lett. 25 (2000) 1037.
[12] M. Centurion, M.A. Porter, P.G. Kevrekidis, D. Psaltis, Nonlinearity management in optics: Experiment, theory, and simulation, http://arxiv.org/nlin.PS/0604031.
[13] F. Abdullaev, J.G. Caputo, R.A. Kraenkel, B.A. Malomed, Controlling collapse in Bose–Einstein condensates by temporal modulation of the scattering length, Phys. Rev. A 67 (2003) 013605.
[14] H. Saito, M. Ueda, Dynamically stabilized bright solitons in a two-dimensional Bose–Einstein condensate, Phys. Rev. Lett. 90 (2003) 040403.
[15] G.D. Montesinos, V.M. Pérez-García, P. Torres, Stabilization of solitons of the multidimensional nonlinear Schrödinger equation: matter-wave breathers, Physica D 191 (2004) 193.
[16] F.K. Abdullaev, A. Gammal, A. Kamchatnov, L. Tomio, Dynamics of bright matter wave solitons in a Bose–Einstein condensate, Int. J. Mod. Phys. B 19 (2005) 3415.
[17] V.V. Konotop, P. Pacciani, Collapse of solutions of the nonlinear Schrödinger equation with a time-dependent nonlinearity: Application to Bose–Einstein condensates, Phys. Rev. Lett. 94 (2005) 240405.
[18] V.N. Serkin, A. Hasegawa, Novel soliton solutions of the nonlinear Schrödinger equation model, Phys. Rev. Lett. 85 (2000) 4502.
[19] U. Niederer, The maximal kinematical invariance group of the harmonic oscillator, Helv. Phys. Acta 46 (1973) 191.
[20] S. Chen, L. Yi, Chirped self-similar solutions of a generalized nonlinear Schrödinger equation model, Phys. Rev. E 71 (2005) 016606.

[21] Z.X. Liang, Z.D. Zhang, W.M. Liu, Dynamics of a bright soliton in Bose–Einstein condensates with time-dependent atomic scattering length in an expulsive parabolic potential, Phys. Rev. Lett. 93 (2004) 152301.
[22] C. Sulem, P. Sulem, The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse, Springer, Berlin, 2000.
[23] A.V. Rybin, G.G. Varzugin, M. Lindberg, J. Timonen, R.K. Bullough, Similarity solutions and collapse in the attractive Gross–Pitaevskii equation, Phys. Rev. E 62 (2000) 6224.
[24] G. Theocharis, Z. Rapti, P.G. Kevrekidis, D.J. Frantzeskakis, V.V. Konotop, Modulational instability of Gross–Pitaevskii-type equations in $1 + 1$ dimensions, Phys Rev. A 67 (2003) 063610.
[25] F. Frommlet, J.L. López, J. Soler, G. Toscani, Nonlinear rescalings, dispersion lemmas and conservation laws for some linear kinetic and quantum kinetic problems, Comm. Appl. Nonlinear Anal. 10 (2003) 1–20.
[26] F. Obersnel, P. Omari, Old and new results for first order periodic ODEs without uniqueness: a comprehensive study by lower and upper solutions, Adv. Nonlinear Stud. 4 (2004) 323.
[27] W. Magnus, S. Winkler, Hill's Equation, Dover Publ., 1979.
[28] M.A. Krasnosel'skii, V.Sh. Burd, Yu.S. Kolesov, Nonlinear Almost Periodic Oscillations, John Wiley & Sons, New York, 1973.
[29] V.E. Zakharov, V.S. Synakh, Nature of singularity in self-focusing of waves, Sov. Phys. JETP 41 (1975) 465.