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BOUNDED SOLUTIONS IN SINGULAR EQUATIONS OF REPULSIVE TYPE

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INTRODUCTION

In the 1960s, the work of Lazer about the periodic Duffing's equation

$$x''(t) + cx'(t) + g(x(t)) = p(t),$$

where p and g are continuous functions defined on $(-\infty, +\infty)$ and p is T-periodic, supposes a great advance in the comprehension of this equation. In his paper [1], it is proved the existence of at least one T-periodic solution under some conditions that include

$$\limsup_{x \to -\infty} g(x) < \bar{p} = \frac{1}{T} \int_0^T p(t)\, \mathrm{d}t < \liminf_{x \to +\infty} g(x),$$

usually referred as a Landesman–Lazer condition.

When p is not periodic but only bounded, periodic solutions cannot appear, but we can look for bounded solutions (together with its derivative). In this sense, Ahmad established in [2] some Landesman–Lazer-type conditions for existence of bounded solutions of (1) when c is positive and p is a continuous and bounded function having a generalized mean value. Moreover, properties for the totality of solutions defined on $[t_0, +\infty)$ (with a given $t_0 \in \mathbb{R}$) are shown: they are all bounded on this interval. Usually, it is said that such solutions are bounded in the future.

Recently, by using the method of guiding functions, Ahmad's results have been partially extended by Ortega in [3] to some cases where p does not need to be bounded, in concrete, if p can be decomposed as $p = p^* + p^{**}$, where p^* has a bounded primitive and p^{**} is bounded and continuous satisfying

$$\lim_{x \to -\infty} g(x) < \inf p^{**} \le \sup p^{**} < \lim_{x \to +\infty} g(x)$$

with at least one of these limits finite.

However, the previous papers do not consider the situation in which g is not defined on the whole real line. The aim of this paper is to extend the previous results to equations with a singular nonlinearity of repulsive type, that is,

$$x''(t) + cx'(t) - g(x(t)) = p(t), \tag{1}$$

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where c is positive, p is continuous and $g\colon (0, +\infty) \to (0, +\infty)$ is a continuous function such that

$$\lim_{x \to 0^+} g(x) = +\infty, \qquad \lim_{x \to +\infty} g(x) = 0 \tag{2}$$

and

$$\int_0^1 g(s)\, \mathrm{d}s = +\infty. \tag{3}$$

Equation (1) describes, for example, the motion of a piston submitted to an exterior force $p(t)$ in a cylinder closed at one extremity and filled with a perfect gas. The "model case" of such kind of singular nonlinearity is $g(x) = x^{-\alpha}$, and then condition (3) holds if and only if $\alpha \geq 1$. Besides, other classes of equations as Forbat's equation can be reduced to (1) by a suitable change of variables, as it is shown in [4].

Using the results in [2], we will prove that if p is a bounded function with negative generalized mean value, then the pair formed from any positive solution of (1) and its derivative remains in the future in a compact set of $\mathbb{R}^+ \times \mathbb{R}$. Moreover, it will be shown that every solution of (1) is defined in $(-\infty, +\infty)$. Also, results about existence of bounded solution of (1) and global stability when c is sufficiently large are stated. In particular, in the periodic case, the existence of a periodic solution is deduced.

Finally, we apply the results of [3] in order to give conditions for the existence of a bounded solution when p is only continuous and bounded below.

Note that since g is only assumed to be continuous, the initial value problem in (1) does not in general have a unique solution, but this is not a problem in the proof.

1. BOUNDED SOLUTIONS IN THE FUTURE

Let $t_0 \in \mathbb{R}$ be the initial time, fixed but arbitrary. From now on, we suppose that $g\colon (0, +\infty) \to (0, +\infty)$ is continuous and satisfies assumptions (2) and (3). In this section, p is a bounded and continuous function with a negative generalized mean value, that is,

$$\lim_{T \to +\infty} \frac{1}{T} \int_a^{a+T} p(t)\, \mathrm{d}t = p_0 < 0 \tag{4}$$

uniformly with respect to a. Moreover, c is supposed to be positive.

PROPOSITION 1. Consider equation (1) under the previous assumptions. Then, if we define the set

$$\mathcal{C} = \{x \in C^2([t_0, +\infty)) : \exists\, m, M, D > 0 \text{ such that } m < x(t) < M,\ |x'(t)| < D\ \forall t \geq t_0\},$$

any positive solution of (1) belongs to $\mathcal{C}$.

Proof. Let $x(t)$ be a positive solution of (1) with a maximal interval of existence (α, β). We claim that any $x(t)$ not belonging to $\mathcal{C}$ holds the property:

$$\text{there exists a sequence } \{r_n\} \to \beta \text{ such that } \{x'(r_n)\} \to -\infty \tag{5}$$

After proving this affirmation, we will show that it is not possible to be satisfied, so all the solutions will be in $\mathcal{C}$.

In order to prove the claim, we consider two cases:

(a) If $\beta < +\infty$, we prove that if (5) does not hold, then $x \in \mathfrak{A}$. In fact, if $x \notin \mathfrak{A}$ then, by the continuation theorem, two options are possible. The first of them is that $\lim_{t\to\beta} x(t) = \lim_{t\to\beta} x'(t) + \lim_{t\to\beta} x''(t) = +\infty$, but taking imits in (1) we have an absurdity. The second one is that $\lim_{t\to\beta} x(t) = 0$. In fact, if it were true, taking limits in the equation one of these alternatives would hold:

(i) $\lim_{t\to\beta} x'(t) = +\infty$, but then $\lim_{t\to\beta} x(t) = +\infty$,

or

(ii) $\lim_{t\to\beta} x''(t) = +\infty$, but then $\lim_{t\to\beta} x'(t) = +\infty$,

contradicting the fact that $x(t)$ tends to zero in both cases. Hence, when $\beta < +\infty$ the property (5) holds.

(b) Suppose now that $\beta = +\infty$. Then, if $x(t)$ does not belong to $\mathfrak{A}$, one of the following cases holds::

(i) There exists $\{t_n\}$ such that $\{t_n\} \to \beta$ and $x(T_b) \to 0$

or

(ii) There exists $\{\tau_n\}$ such that $\{\tau_n\} \to \beta$ and $x(\tau_n) \to +\infty$

or

(iii) There exists $\{\zeta_n\}$ such that $\{\zeta_n\} \to \beta$ and $|x'(\zeta_n)| \to +\infty$.

Suppose that (i) holds. By the same reasoning of the case $\beta < +\infty$, there is no limit of x as $t \to \beta$, so there exist two sequences $\{\bar{t}_n\}$, $\{\bar{\tau}_n\} \to \beta$ where $x(\bar{\tau}_n)$ are maximums and $x(\bar{t}_n)$ minumums intercalated, with

$$\{x(\bar{t}_n)\} \to 0, \qquad \{x(\bar{\tau}_n)\} \to K > 0 \tag{6}$$

as $n \to +\infty$. Now, if (5) is not true then

$$\text{there exists } B > 0 \text{ such that } x'(t) > -B,\ t \geq t_0. \tag{7}$$

Besides, $p(t)$ is bounded, so let $P > 0$ such that $|p(t)| < P$. Then

$$x''(t) + cx'(t) > g(x(t)) - P. \tag{8}$$

Using the assumption (2), there exists $R > 0$ such that $g(x) > P$ for all $x < R$. By (6), there exists n_0 such that $x(\bar{t}_n) < R$ whenever $n \geq n_0$. Moreover, any maximums of $x(t)$ are greater than R. Thus, it is possible to take a sequence $\{s_n\}_{n \geq n_0}$ such that $s_n < \bar{t}_n$, $x(s_n) = R$ and $x'(t) < 0$ for all $t \in]s_n, \bar{t}_n[$. Then $g(x(t)) > P$ for all $t \in]s_n, \bar{t}_n[$, and from (7) and (8) we have that

$$x''(t) + cx'(t) > [P - g(x(t))]\frac{x'(t)}{B}.$$

Integrating in $[s_n, \bar{t}_n]$,

$$-x'(s_n) + c[x(\bar{t}_n) - R] > \frac{1}{B}\int_{x(\bar{t}_n)}^{R} g(s)\,\mathrm{d}s - \frac{P}{B}[R - x(\bar{t}_n)],$$

but taking limits when $n \to +\infty$, the first member of the inequality is bounded, while the second tends to $+\infty$ by assumption (3), so we get a contadiction, that comes from (7).

Now, we are going to prove that (iii)$\Rightarrow$(i). In fact, if (i) is not true then there exists $\varepsilon_1 > 0$ such that $x(t) > \varepsilon_1$ whenever $t > t_0$. But now, for any $\varepsilon > 0$ smaller than ε_1 we define the following function

$$g_\varepsilon(x) = \begin{cases} g(x) & \text{if } x > \varepsilon \\ g(\varepsilon) & \text{if } x \leq \varepsilon. \end{cases}$$

and consider the equation

$$x''(t) + cx'(t) - g_\varepsilon(x(t)) = p(t)$$

From the results of [2], it follows that there is a uniform bound of $x'(t)$, a contradiction to (iii).

Similarly, it is proved that (ii)$\Rightarrow$(i).

Therefore, the claim has been proved. Now, for any $x(t)$ not belonging to $\mathcal{A}$, there exists a sequence $\{t_n\}$ such that

$$\{x'(r_n)\} \to -\infty, \qquad \text{as } t \to +\infty, \tag{9}$$

and a sequence $\{x(\bar{\tau}_n)\}$ of maximums such that $\times \tau V_n < r_n$.

Since g is positive,

$$x''(t) + cx'(t) = g(x(t)) + p(t) > -P.$$

But multiplying by ε^{ct} and integrating in $[\bar{\tau}_n, r_n]$ we have

$$x'(r_n)\,\mathrm{e}^{cr_n} > -\frac{P}{c}[\mathrm{e}^{cr_n} - \mathrm{e}^{c\bar{\tau}_n}] > -\frac{P}{c}\mathrm{e}^{cr_n},$$

so $x'(r_n) > -P/c$ for all n, a contradiction to (9). Thus, it is proved that any solution of (1) belongs to $\mathcal{A}$.

Proposition 2. With the previous assumptions, if we define the set

$$\mathcal{B} = \{x \in C^2(\mathbb{R}) : \exists\, m > 0 \text{ such that } x(t) > m\ \forall t \in \mathbb{R}\},$$

any positive solution of equation (1) belongs to $\mathcal{B}$.

Proof. If we invert the "time direction" doing the change $\tau = -t$, equation (1) becomes

$$x''(\tau) - cx'(\tau) - g(x(\tau)) = p(\tau) \tag{10}$$

Hence, we must prove that for any solution $x(\tau)$ of (10), there exists $m > 0$ such that $x(\tau) > m$ for all $\tau \geq t_0$. If such m does not exist, suppose that $\lim_{\tau\to\beta} x(\tau) = 0$. With similar arguments as in the proof of Proposition 1, we obtain $\lim_{\tau\to\beta} x'(\tau) = -\infty$, but then $\beta < +\infty$ and we can integrate (10) in the interval $[t_0, \beta - \varepsilon]$, resulting

$$x'(\beta - \varepsilon) - x'(t_0) - c[x(\beta - \varepsilon) - x(t_0)] - \int_{t_0}^{\beta-\varepsilon} g(x(\tau))\,\mathrm{d}\tau = \int_{t_0}^{\beta-\varepsilon} p(\tau)\,\mathrm{d}\tau$$

and taking limits when ε goes to zero, the first member of this equality tends to $-\infty$, unlike the second, so we have a contradiction. Thus, there exist two sequences $\{t_n\}, \{\tau_n\} \to \beta$ where $x(\tau_n)$ are maximums and $x(t_n)$ minimums intercalculated, with

$$\{x(t_n)\} \to 0, \qquad \{x(\tau_n)\} \to K > 0$$

and we can follow the same reasoning of the latter proof to get a contradiction. To finish the proof, we only have to verify that any solution $x(\tau)$ of (10) is defined in $(-\infty, +\infty)$. Suppose that the maximal interval of definition of $x(\tau)$ is (α, β). We know by Proposition 1 that $\alpha = -\infty$. Moreover, it is proved that there exists $m > 0$ such that $x(\tau) > m$ for all τ, so as a consequence there exists $G > 0$ such that $g(x(\tau)) < G$. Hence,

$$x''(\tau) - cx'(\tau) < P + G.$$

Multiplying by $e^{-c\tau}$ and integrating in $[t_0, \tau]$ we have

$$x'(\tau)\,e^{-c\tau} - x'(t_0)\,e^{-ct_0} < \frac{P+G}{c}\,e^{-ct_0}$$

and

$$x'(\tau) < \left\{\frac{P+G}{c} + x'(t_0)\right\} e^{c(\tau - t_0)}$$

so if β is finite, then $x'(\tau)$ is bounded above, but this is a contradiction.

Remark. Note that the proof of this result is still valid only supposing p to be bounded and continuous and $c \neq 0$, in virtue of the change $\tau\,3\,-t$.

THEOREM 1. Consider equation (1) such that $c \neq 0$, and (2) and (3) hold. Let $p(t)$ be a T-periodic continuous function with mean value $\bar{p} = (1/T)\int_0^T p(t)\,dt < 0$. Then, any positive solution of (1) belongs to $\mathfrak{A} \cap \mathfrak{B}$. Besides, (1) has at least one T-periodic solution.

Proof. For $c > 0$, this result is a direct consequence of Propositions 1 and 2 and the second theorem of Massera (see [5]). For $c < 0$, we only have to make the change $\tau = -t$.

Condition $\bar{p} < 0$ is necessary and sufficient for the existrence of periodic solution, as we can verify only integrating the equation in a period. Our last result of this Section shows that condition $p_0 < 0$ is also necessary and sufficient in Proposition 1.

PROPOSITION 3. Consider equation (1) with the assumptions of Theorem 1, and $p_0 \geq 0$. Then any solution $x(t)$ of (1) is defined in $(-\infty, +\infty)$ and is unbounded in $[t_0, +\infty)$. Moreover, if $p_0 > 0$ then $\lim_{t\to+\infty} x(t) = +\infty$.

Proof. By Proposition 2 and the remark at the end of its proof, every solution of (1) is defined in $(-\infty, +\infty)$. Integating (1) in $[t_0, t]$ we have

$$x'(t) + cx(t = x'(t_0) + cx(t_0) + \int_{t_0}^{t} g(x(s))\,ds + \int_{t_0}^{t} p(s)\,ds. \tag{11}$$

If $p_0 > 0$ then $\lim_{t\to+\infty} \int_{t_0}^t p(s)\,ds = +\infty$, so taking limits in (11) we have

$$\lim_{t\to+\infty} [x'(t) + cx(t)] = +\infty,$$

and the result holds.

If $p_0 = 0$ then from the definition, we get that the growth of $\int_{t_0}^t p(s)\,ds$ is sublinear, that is, for all $K > 0$, there exists $t_K > 0$ such that for $t > t_K$ one has

$$\left| \int_{t_0}^t p(s)\,ds \right| < K|t - t_0|. \tag{12}$$

Now, if $x(t)$ is bounded, then there exists some $\varepsilon > 0$ such that $g(x(t)) > \varepsilon$ for all t. Taking $K = \varepsilon/2$ in (12), from (11) we have that for t sufficiently large

$$x'(t) + cx(t) > x'(t_0) + cx(t_0) + |t - t_0|\varepsilon - |t - t_0|\frac{\varepsilon}{2} = |t - t_0|\frac{\varepsilon}{2} + x'(t_0) + cx(t_0),$$

and taking limits when $t \to +\infty$, we get a contadiction.

2. BOUNDED SOLUTIONS ON THE WHOLE REAL LINE

When p is not periodic, equation (1) cannot have periodic solutions, but bounded solutions still can appear. We understand that $x(t)$ is a bounded solution if $x(t)$ and $x'(t)$ are bounded.

THEOREM 2. Consider equation (1) with the assumptions of Proposition 1. Let ξ_0 be a positive number such that $g(\xi_0) = -p_0$ and assume that g is strictly decreasing in ξ_0. Then for any $\varepsilon > 0$ with $\varepsilon < \xi_0$, there exists $c_\varepsilon > 0$ such that for any $c \geq c_\varepsilon$, equation (1) has a solution satisfying

$$|x(t) - \lambda_0| < \xi_0 - \varepsilon, \ |x'(t)| < \xi_0 - \varepsilon, \qquad t \in \mathbb{R}.$$

Proof. The proof is obtained by using the function g_ε and applying Theorem 5.2 of [2] with $\alpha = \xi_0 - \varepsilon$. The solution obtained is greater than ε and hence is a solution of (1). ■

COROLLARY 1. Consider equation (1) with the assumptions of Proposition 1. If moreover g is of class C^1 and strictly decreasing, then there exists c^* such that for any $c \geq c^*$, there exists a unique bounded solution of (1) that is globally asymptotically stable.

Proof. Since g is strictly decreasing, trivially there exists an ξ_0 as in Theorem 2. Now, we only have to take $\varepsilon \to \xi_0$ and apply Theorem 1.2 in [6] for the corresponding equation ■

The condition of large friction can be removed, assuming a more restricting condition on g. From now on, let P be such that $|p(t)| \leq P$ for all $t \in \mathbb{R}$.

LEMMA 1. For every bounded solution of (1) on $(-\infty, +\infty)$ we have

$$x'(t) > -\frac{P}{c}, \qquad t \in \mathbb{R}.$$

Proof. Making the change $\tau = -t$, one obtains

$$x''(\tau) - cx'(\tau) - g(x(\tau)) = p(\tau), \qquad \tau \in \mathbb{R}.$$

We are going to prove that $x'(\tau) < P/c$ for all τ. In fact, if it is not true, there exists τ_0 such that $x'(\tau_0) > P/c > 0$, so

$$x''(\tau_0) = cx'(\tau_0) + g(x(\tau_0)) + p(\tau_0) > 0.$$

Thus, $x(\tau)$ is an increasing convex function, so it is unbounded, contradicting the hypothesis. Now, we only have to invert the change. ■

PROPOSITION 4. Let $g: (0, +\infty) \to (0, +\infty)$ be a continuous function satisfying assumptions (2) and (3). Suppose that

$$\lim_{\varepsilon \to 0^+} \varepsilon g(2\varepsilon) > \frac{3P2}{c^2} \tag{13}$$

and take $\varepsilon > 0$ such that

$$\varepsilon[g(2\varepsilon) - P] > \frac{P^2}{c^2}, \tag{14}$$

and

$$g(x) > P, \qquad 0 < x < 2\varepsilon. \tag{15}$$

Then, ε is a lower bound of every bounded solution of (1) on $(-\infty, +\infty)$.

Proof. It is easy to see that if the conclusion is not true, then there exists $t_0 \in \mathbb{R}$ such that $x(t_0) = \min x(t) < \varepsilon$; if not, we can take limits in the equation and get a contradiction. Let us take $t_1 < t_0$ such that $x(t_1) = 2\varepsilon$ and $x'(t) \le 0$ for all $t \in (t_1, t_0)$. If such t_1 does not exist, then we must have a maximum $x(t_M)$ less than 2ε, but from the equation and hypothesis (15) we would have

$$x''(t_M) = g(x(t_M)) + p(t) > g(2\varepsilon) - P > 0,$$

a contradiction.

Now, by the Mean Value Theorem, there exists $\xi \in (t_1, t_0)$ such that

$$\begin{aligned} x'(t_0) - x'(t_1) &= x''(\xi)(t_0 - t_1) \\ &= [-cx'(\xi) + g(x(\xi)) + p(\xi)](t_0 - t_1) \\ &> [g(2\varepsilon) - P](t_0 - t_1), \end{aligned}$$

and applying Lemma 1 we obtain

$$t_0 - t_1 < \frac{P}{c[g(2\varepsilon) - P]}.$$

A new application of the Mean Value Theorem gives

$$x(t_0) 1 x(t_1) = x'(\xi)(t_0 - t_1) > x'(\xi)\frac{P}{c[g(2\varepsilon) - P]} > \frac{-P^2}{c^2[g(2\varepsilon) - P]}.$$

Then,

$$\varepsilon > x(t_0) > \frac{-P^2}{c^2[g(2\varepsilon) - P]} + 2\varepsilon,$$

that is,

$$\frac{P^2}{c^2[g(2\varepsilon) - P]} > \varepsilon,$$

contradicting the assumption (14). ■

Remark. The hypothesis (13) holds for example if $g(x) = x^{-\alpha}$ with $\alpha > 1$ or if $\alpha = 1$ and $P < c$.

By using the same function as in Proposition 1 and the results of [2], the following theorem can be proved.

THEOREM 3. Consider equation (1) with the assumptions of Proposition 1 and (13). Then, there exists a bounded solution on the whole real line.

COROLLARY 2. Consider equation (1) with the assumptions of Proposition 1, and assume, moreover, that g is of class C^1 and strictly decreasing. Suppose that

$$\lim_{\varepsilon \to 0^+} \varepsilon g(2\varepsilon) = K > 0.$$

Let $\varepsilon^* = g_x^{-1}(-c^2/4)$ and P^* satisfying the conditions

$$P^* < \sqrt{K}c,$$

$$\varepsilon^* g(2\varepsilon^*) > P^*\left[\frac{P^*}{c^2} + \varepsilon^*\right]$$

and

$$g(x) > P^*, \qquad 0 < x < 2\varepsilon^*.$$

Then, for any $p(t)$ such that $|p(t)| \le P^*$, there exists a bounded solution on the whole real line that is globally asymptotically stable.

Proof. By using Proposition 4, there exists a bounded solution on the whole real line and ε^* is a lower bound. Now, the proof can be obtain by a linearization of the equation in order to apply the results of [6]. ■

Note that proofs of Lemma 1 and Proposition 4 are also true supposing that p is only continuous and bounded below. As in [8], denote by BC the set of bounded and continuous functions, and by C_0 the set of continuous functions with bounded primitive. From Theorem 2.1 of [3], we can state the following theorem.

THEOREM 4. Let p be a continous function bounded below by $-P$ and such that $p = p^* + p^{**}$ with $p^* \in C_0$ and $p^{**} \in BC$ and negative. Then, if g satisfies (2), (3) and (13), equation (1) has a bounded solution on the whole real line.

3. REMARKS AND GENERALIZATIONS

1. The existence result of a periodic solution was already obtained by Lazer–Solimini for the undamped case in [7] and by Habets–Sanchez in [8], by using a different method involving a priori bounds and topological degree.

2. When (3) is not true, it is easy to show that there are $p(t)$ with negative mean value such that there do not exist any bounded solutions.

3. For the undamped case ($c = 0$), Proposition 2 can be proved using similar techniques, but we do not know if Proposition 1 is true in general.

4. All the results of this paper are true if p is only bounded and we look for solutions in the weak sense, that is, in $W^{2,\infty}([t_0, +\infty))$ or $W^{2,\infty}(\mathbb{R})$.

5. In virtue of the remark of Theorem 3.1 in [2], Proposition 1 is still true if assumption (4) is changed by

$$\lim_{T \to +\infty} \frac{1}{T} \int_{t_0}^{t_0+T} p(t)\, dt = p_0 < 0.$$

However, (4) is needed in Theorem 2.

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