

PERIODIC SOLUTIONS OF TWIST TYPE OF AN EARTH SATELLITE EQUATION

DANIEL NÚÑEZ

Universidad de Granada,
Departamento de Matemática Aplicada, 18071 Granada, Spain.

PEDRO J. TORRES¹

International School for Advanced Studies
Via Beirut 2-4, 34013 Trieste, Italy

Abstract. We study Lyapunov stability for a given equation modelling the motion of an earth satellite. The proof combines bilateral bounds of the solution with the theory of twist solutions.

1. Introduction. In the recent paper [10], it is studied the following equation as a model of the motion of a satellite in orbit around the earth

$$x'' + \alpha \sqrt{1 + \sin^2 t} \sin x = 6 \sin 2t (1 + 3 \sin^2 t)^{-2}, \quad (1.1)$$

where α is a positive parameter related with magnetic intensity. Due to the invariance of the equation under transformation $(x, t) \rightarrow (-x, -t)$, it is natural to look for and study the properties of odd π -periodic solutions

$$x(t) = -x(-t) = x(t + \pi), \quad \forall t.$$

In particular, the question of stability of such a solution is of major importance because of the type of application. In this direction, in [10] it is proved that for $\alpha \leq 1/2$, there exists a stable solution in the linear sense (also called elliptic), that is, such that the linearized equation is stable. As it is known, this fact does not assure stability in the sense of Lyapunov of the mentioned equation, since it depends usually on the nonlinear term of the equation. The aim of this note is to prove Lyapunov stability under the more conservative bound $\alpha \leq 1/18$, by using a new result [6] on twist solutions and taking into account the bilateral bounds found in [10].

A periodic solution φ of a general periodic equation of second order is said to be 4-elementary if it is elliptic with Floquet's multipliers $\lambda, \bar{\lambda}$ verifying: $\lambda^p \neq 1$, $1 \leq p \leq 4$. The solution φ is called of twist type ([7, 8]) if it is 4-elementary and the first coefficient of Birkhoff of the associated Poincaré map is different to zero. From the Moser's invariant curve theorem it follows that a periodic solution of twist type is always Lyapunov stable ([4, 9]). Moreover, there is a rich dynamics around a twist periodic solution, namely: existence of sub-harmonics of order n

1991 *Mathematics Subject Classification.* 34C25, 34D20, 70F15.

Key words and phrases. odd periodic solution, Lyapunov stability, twist, earth satellite equation.

Research partially supported by C.I.C.T. PB98-1294, M.E.C., Spain.

¹Permanent address: Universidad de Granada, Departamento de Matemática Aplicada, 18071 Granada, Spain

with $n \rightarrow \infty$ as well as many quasi-periodic solutions. This is a consequence of the Poincaré-Birkhoff's fixed point theorem and the KAM theorem ([1, 9]).

In a forthcoming paper, the authors plan to extend the ideas of this note to a wider class of periodic differential equations combining the results in [6] with upper and lower solutions for the Dirichlet problem.

2. Stability by the third approximation. Let $x_0(t)$ be a periodic solution of a general π -periodic equation

$$x'' + f(t, x) = 0, \quad (2.1)$$

where f is continuous and π -periodic in the variable t and of class C^4 respect to x .

After a translation to the origin and a Taylor expansion, equation (2.1) can be written as

$$x'' + a(t)x + b(t)x^2 + c(t)x^3 + T(t, x) = 0, \quad (2.2)$$

where a, b, c are π -periodic and T denote the remaining terms.

The relation between Lyapunov stability and this expansion up to the third term is classical and at least goes back to Birkhoff. More recently, the mentioned relation is studied in the papers of Ortega [7, 8]. In these papers the author always assumes that the nonlinear coefficients b and c have constant sign because the technique employed to prove the stability relies on the maximum principle. Related results can be seen in [3, 5].

Also, the recent paper [6] study some cases which are not covered by the previous ones. Concretely, the main result of [6] is the following.

Theorem 2.1. *Assume that there exist σ, γ positive numbers such that*

$$\sigma^2 \leq a(t) \leq \gamma^2 \leq 1/9.$$

Then, the equilibrium of (2.2) is of twist type if the following condition holds

$$\sigma^6 \int_0^\pi c^-(t)dt - \sigma^2 \gamma^4 \int_0^\pi c^+(t)dt > 2\gamma^5 \int_0^\pi b^+(t)dt \int_0^\pi b^-(t)dt. \quad (2.3)$$

Let us note that the main advantage of this result is that the nonlinear coefficients are allowed to change sign.

3. Main result. Our main result is the following.

Theorem 3.1. *If $\alpha \leq 1/18$, equation (1.1) has an odd π -periodic solution of twist type.*

Proof: By using the theory of monotonic operators of Krasnoselskii, it is proved in [10] that for any α there is an odd π -periodic solution x_α such that

$$0 < x_\alpha(t) < x_\alpha^+(t), \quad \forall t \in (-\pi/2, 0),$$

being

$$x_\alpha^+(t) = (1 - \alpha\pi/2)t - \alpha t^2 - \arctan(2 \tan t).$$

This result can be proved directly by the theory of upper and lower solutions (see for instance [2]) if we note that $x_\alpha^+(t)$ is an upper solution and 0 is a constant lower solution of equation (1.1) with Dirichlet conditions

$$x(-\pi/2) = x(0) = 0.$$

By the equivalence between the odd-periodic problem and the Dirichlet problem, the result is done. Besides, by [10, Proposition 2.1], this solution is unique if $\alpha \leq 2$.

It is easy to verify that the function $A(\alpha) = \max \{x_\alpha^+(t) : t \in (-\pi/2, 0)\}$ is increasing with α , so in consequence for any $\alpha \leq 1/18$ we have the bound

$$\|x_\alpha\|_\infty < A(1/18) \simeq 0.37262$$

On the other hand, the coefficients of the expansion (2.2) are

$$\begin{aligned} a(t) &= \alpha \sqrt{1 + 3 \sin^2 t} \cos x_\alpha(t) \\ b(t) &= -\frac{\alpha}{2} \sqrt{1 + 3 \sin^2 t} \sin x_\alpha(t) \\ c(t) &= -\frac{\alpha}{6} \sqrt{1 + 3 \sin^2 t} \cos x_\alpha(t). \end{aligned}$$

In order to apply Theorem 2.1, let us fix $\gamma = \sqrt{2\alpha}$ and $\sigma = \sqrt{\alpha \cos A(1/18)}$. Then,

$$\sigma^2 \leq a(t) \leq \gamma^2 = 2\alpha \leq 1/9. \quad (3.1)$$

On the other hand, condition (2.3) reads

$$\sigma^6 \int_0^\pi c^-(t) dt > 2\gamma^5 \int_0^\pi b^+(t) dt \int_0^\pi b^-(t) dt.$$

Note that b is odd, so $b^+(t) = b^-(-t)$. Also, as we know that $x(t) > 0$ for all $t \in (-\pi/2, 0)$, the only change of sign is in 0. With these observations, we can write the previous condition as

$$\sigma^6 \int_0^\pi c^-(t) dt > 2\gamma^5 \left(\int_{-\pi/2}^0 b(t) dt \right)^2.$$

Calling $M = \int_0^\pi \sqrt{1 + 3 \sin^2 t} dt$, it is not hard to see that this inequality holds if

$$\frac{\alpha^4}{6} M \cos^4 A(1/18) > \frac{\sqrt{2}}{2} \alpha^{9/2} M^2 \sin^2 A(1/18),$$

that is,

$$\sqrt{\alpha} < \frac{\cos^4 A(1/18)}{3\sqrt{2} M \sin^2 A(1/18)}.$$

A numerical computation gives the approximate value $M \simeq 4.84422$, therefore condition (2.3) holds if

$$\alpha < \frac{\cos^8 A(1/18)}{18M^2 \sin^4 A(1/18)} \simeq 1.23,$$

which is evidently satisfied if $\alpha \leq 1/18$.

Remark: Numerical experiments show that one can expect stability for $\alpha < 1/2$ except possibly near to the strong resonances given by $\alpha = 1/8, 2/9$. However actually our theoretical approach do not allow to prove this. Only we can reach stability up to $\alpha = 1/18$. Note that the main assumption $\alpha \leq 1/18$ is needed essentially in (3.1). An improvement of our result would require a similar study as in [8] about the comparison of some coefficients of the 2-jet of the Poincaré map associated to the equation (2.2) (see [8, Lemma 4.7]). It seems to us a difficult and delicate question.

REFERENCES

- [1] Arnold, V.I.; Avez, A. (1968): *Ergodic Problems of Classical Mechanics*, Benjamin.
- [2] De Coster, C.; Habets, P. (1996): Upper and lower solutions in the theory of ODE boundary value problems: classical and recent results. *Nonlinear Analysis and Boundary Value Problems for Ordinary Differential Equations*, (F. Zanolin ed.), CISM-ICMS 371, Springer-Verlag.
- [3] Liu, B. (1996): The Stability of The Equilibrium of a Conservative System, *J. Math. Anal. Appl.* 202, 133-149.
- [4] Moser, J. (1962): On invariant curves of area preserving mappings of an annulus, *Nachr. Akad. Wiss. Göttingen Math. Phys. Kl. II*, 1-20.
- [5] Núñez, D.; Ortega, R.: Parabolic fixed points and stability criteria for nonlinear Hill's equation, to appear in *Z. Angew. Math. Phys.*
- [6] Núñez, D.: The Method of Lower and Upper Solutions and the Stability of Periodic Oscillations, preprint.
- [7] Ortega, R. (1992): The twist coefficient of periodic solutions of a time-dependent Newton's equation, *J. Dynam. Diff. Equ.*, 4, 651-665.
- [8] Ortega, R. (1996): Periodic solutions of a newtonian equation: stability by the third approximation, *J. Diff. Equ.*, 128, 491-518.
- [9] Siegel, C.L.; Moser, J.K. (1971): *Lectures on Celestial Mechanics*, Springer-Verlag.
- [10] Zevin, A.A.; Pinsky, M.A. (2000): Qualitative analysis of periodic oscillations of an earth satellite with magnetic attitude stabilization, *Discr. Cont. Dyn. Syst.*, 6, 293-297.

Revised version received November 2000.

E-mail address: dnunez@ugr.es, ptorres@ugr.es