Tensor Algebra and Composite Quantum Systems.

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Abstract

In this paper we develop a consistent formalism for constructing the tensor product of Hilbert spaces, by means of multilinear maps defined both on Hilbert spaces and their duals. We remark the difference between ordinary tensors and those which can be associated to quantum states of a many-particle physical system.

1 Introduction

Tensor product of infinite-dimension Hilbert spaces is widely used in quantum mechanics to describe complex systems in terms of its constituyents. In this manner, we can decompose states with many-degrees of freedom in terms of simpler ones, and in particular, this is useful to obtain many-particle states as combinations of products of single particle ones.

In spite of this fact, the rigorous mathematical construction of the tensor product of Hilbert spaces not only is not usually developed in standard texts of quantum mechanics, but also is difficult to find out in mathematical texts. Partial information is found in some textbooks, and with a formalism rather far away from the physical interpretations [1],[2].

For the students of Physics, the concept of tensor is usually restricted to the finite-dimensional ones, as it is done in standard texts of linear algebra, in spite of the fact that infinite dimensional spaces are of great importance in Physics, e.g. the spaces of squared-integrable functions over a subset of $\mathbb{R}^3$.

The main reason for this fact is that in Mathematics, finite-dimensional linear spaces are topics usually related to the field of linear algebra [3], whilst their extension to Hilbert spaces of infinite dimension has been historically related to Functional Analysis [1],[4] because their special application to the spaces of functions mentioned above.

In this work, our scope is to make this two ends meet and formulate, by extending the concept of tensor to arbitrary-dimension spaces, a consistent mathematical description of the tensor product of Hilbert spaces, focused towards their physical applications. This detailed formulation is not covered, to the best of the authors' knowledge, by standard texts at university level.

The description is planned for undergraduate students. The concepts which appear here are known, but our goal is to make a compendium of disperse information and make it understandable for physics students. It is worth to note that in Spanish universities, these students have to follow a semestral course on Mathematical Methods of Physics mainly devoted to Hilbert spaces and spectral theory, with emphasis in their applications in quantum mechanics. The formalism as described here is used succesfully in our class, which allows the students a better understanding of the mathematical framework of composite quantum systems.

The reader should be familiar with the main concepts and properties of the Hilbert spaces, and with standard quantum mechanics.

2 Tensor product of Hilbert spaces

It is useful in quantum mechanics, when describing a particular system with more than one degree of freedom which is represented by an element $|\psi\rangle$ of a Hilbert space $\mathcal{H}$ (we use the Dirac bra and ket notation), to decompose it in a product of elements of other Hilbert spaces which represent simpler systems. It is quite convenient to define the scalar product and obtain orthonormal bases in terms of properties of the simpler Hilbert spaces. We start now from the definitions of multilinear maps. Some concepts of the single linear maps, also called linear functionals, will be generalized. These concepts can be found in Ref.[5].

2.1 Multilinear maps

Let $\mathcal{H}_1$, $\mathcal{H}_2$, ... $\mathcal{H}_p$ be p complex Hilbert spaces. A p–linear map T on these spaces is defined in the usual way:

$$T : \mathcal{H}_1 \times \cdots \times \mathcal{H}_p \longrightarrow \mathbb{C}, \tag{1}$$

such as it is linear in each argument, i.e.

$$\begin{aligned} T(|x_1\rangle, \ldots, \alpha|x_i\rangle + \beta|y_i\rangle, \ldots, |x_p\rangle)x &= \alpha T(|x_1\rangle, \ldots, |x_i\rangle, \ldots, |x_p\rangle) \\ &\quad +\beta T(|x_1\rangle, \ldots, |y_i\rangle, \ldots, |x_p\rangle), \end{aligned} \tag{2}$$

for $i = 1, \ldots, p$, and for all $|x_j\rangle, |y_j\rangle \in \mathcal{H}_j$ and $\alpha, \beta \in \mathbb{C}$.

We say that the p–linear map T is bounded if there exist a constant $M > 0$, such that

$$|T(|x_1\rangle, \ldots, |x_p\rangle)| \leq M \| |x_1\rangle \| \cdots \| |x_p\rangle \|, \tag{3}$$

for all $|x_i\rangle \in \mathcal{H}_i$.

In this case, we can define a norm for T by means of

$$\|T\| = \sup_{|x_1\rangle, \ldots |x_p\rangle \neq 0} \frac{|T(|x_1\rangle, \ldots |x_p\rangle)|}{\| |x_1\rangle \| \cdots \| |x_p\rangle \|}. \tag{4}$$

This last property is equivalent to the continuity of T in each variable $|x_i\rangle$, as is shown in standard normed spaces theory [5].

Particular bounded p–linear maps can be defined in the following way: Let $\langle x_j|$ be the linear functional on $\mathcal{H}_j$ such that, for any $|y_j\rangle \in \mathcal{H}_j$, $\langle x_j|(|y_j\rangle) = \langle x_j|y_j\rangle$, being $\langle\ |\ \rangle$ the scalar product of $\mathcal{H}_j$. Then we can define the p–linear map

$$\langle x_1| \otimes \cdots \otimes \langle x_p|, \tag{5}$$

by acting on the cartesian product of $\mathcal{H}_1, \ldots \mathcal{H}_p$ as:

$$\langle x_1| \otimes \cdots \otimes \langle x_p|(|y_1\rangle, \ldots, |y_p\rangle) = \langle x_1|y_1\rangle \cdots \langle x_p|y_p\rangle. \quad (6)$$

By using the Schwarz inequality for every $\mathcal{H}_j$, we find that this p–linear map is bounded, with $M = \||x_1\rangle\| \cdots \||x_p\rangle\|$. The application of this map on $(|x_1\rangle, \ldots, |x_p\rangle)$ shows that the norm of this map is given by:

$$\| \langle x_1| \otimes \cdots \otimes \langle x_p| \| = \||x_1\rangle\| \cdots \||x_p\rangle\|. \quad (7)$$

Remember that the dual space of $\mathcal{H}_j$, denoted as $\mathcal{H}_j^*$ is the hilbert space of its linear bounded functionals. We can also define p–linear maps acting on these duals:

$$S : \mathcal{H}_1^* \times \cdots \times \mathcal{H}_p^* \longrightarrow \mathbb{C}, \quad (8)$$

with the same condition of linearity and boundedness as stated for T (Eqs.(2), (3).

Particular p–linear maps on $\mathcal{H}_1^* \times \cdots \times \mathcal{H}_p^*$ can be defined from the vectors $|x_1\rangle, \cdots, |x_p\rangle$, as

$$|x_1\rangle \otimes \cdots \otimes |x_p\rangle(\langle y_1|, \ldots, \langle y_p|) = \langle x_1|y_1\rangle \cdots \langle y_p|x_p\rangle, \quad (9)$$

which is bounded with the norm:

$$\||x_1\rangle \otimes \cdots \otimes |x_p\rangle\| = \||x_1\rangle\| \cdots \||x_p\rangle\|. \quad (10)$$

The multilinear bounded maps of T–type (Eq.1) are also frequently called *p–covariant tensors*, whilst those of S–type (Eq.8) are called *p–contravariant tensors*. Mixed tensors also can be defined:

$$R : \mathcal{H}_1 \times \cdots \times \mathcal{H}_p \times \mathcal{H}_{p+1}^* \times \cdots \mathcal{H}_{p+q}^* \longrightarrow \mathbb{C}. \quad (11)$$

This map is called a *p–covariant q–contravariant tensor* if it is linear and bounded. For example, $|y\rangle \otimes \langle x|$ represents a 1–covariant 1–contravariant tensor which acts on $\mathcal{H}_1^* \times \mathcal{H}_2$ as:

$$|y\rangle \otimes \langle x|(\langle u|, |v\rangle) = \langle u|y\rangle\langle x|v\rangle, \tag{12}$$

for any $\langle u| \in \mathcal{H}_1^*$ and $|v\rangle \in \mathcal{H}_2$.

2.2 Tensor components

Next we apply the above defined tensors to particular orthonormal bases of the Hilbert spaces and their duals.

Let us denote by $\left\{|e^j_{\alpha_j}\rangle\right\}_{\alpha_j \in A_j}$ an orthonormal basis of $\mathcal{H}_j$, such that any $|x_j\rangle \in \mathcal{H}_j$ can be written as

$$|x_j\rangle = \sum_{\alpha_j \in A_j} \lambda_j^{\alpha_j} |e^j_{\alpha_j}\rangle, \tag{13}$$

where $\lambda_j^{\alpha_j} = \langle e^j_{\alpha_j}|x_j\rangle$.

Then, the application of a multilinear map T on $\mathcal{H}_1 \times \cdots \times \mathcal{H}_p$ can be written as

$$T(|x_1\rangle, |x_2\rangle, \ldots, |x_p\rangle) = \sum_{\alpha_1 \in A_1} \cdots \sum_{\alpha_p \in A_p} \lambda_1^{\alpha_1} \cdots \lambda_p^{\alpha_p} T(|e^1_{\alpha_1}\rangle, \ldots, |e^p_{\alpha_p}\rangle). \tag{14}$$

The sums can be worked out from the arguments even if they are infinite because of the continuity of T.

We define the coordinates of T in the above basis by:

$$t_{\alpha_1,\ldots,\alpha_p} \equiv T(|e^1_{\alpha_1}\rangle, \ldots, |e^p_{\alpha_p}\rangle). \tag{15}$$

For a p-contravariant tensor, $S : \mathcal{H}_1^* \times \cdots \times \mathcal{H}_p^* \longrightarrow \mathbb{C}$, we obtain

$$S(\langle x_1|, \langle x_2|, \ldots, \langle x_p|) = \sum_{\alpha_1 \in A_1} \cdots \sum_{\alpha_p \in A_p} \lambda^1_{\alpha_1} \cdots \lambda^p_{\alpha_p} s^{\alpha_1,\ldots,\alpha_p}. \tag{16}$$

Where, as above, $\langle x_i| = \sum_{\alpha_i \in A_i} \lambda^i_{\alpha_i} \langle e^i_{\alpha_i}|$ and

$$s^{\alpha_1,\ldots,\alpha_p} = S(\langle e^1_{\alpha_1}|, \ldots, \langle e^p_{\alpha_p}|) \tag{17}$$

are the coordinates of S in the dual bases set $\langle e^i_{\alpha_i}|$, $i = 1, \ldots, p$.

2.3 Constructing the tensor product

Let us construct a Hilbert space of p contravariant tensors, including those of the kind $|x_1\rangle \otimes \cdots \otimes |x_p\rangle$.

First we will show how the application of any p-contravariant tensor S can be written in terms of the action of the different $|e^1_{\alpha_1}\rangle \otimes \cdots \otimes |e^p_{\alpha_p}\rangle$, defined from the orthonormal basis of $\mathcal{H}_1, \ldots \mathcal{H}_p$.

For any $(\langle x_1|, \ldots \langle x_p|) \in \mathcal{H}_1^* \times \cdots \times \mathcal{H}_p^*$, taking into account that $\langle x_i| = \langle x_i|e^i_{\alpha_i}\rangle \langle e^i_{\alpha_i}|$ we can write that

$$\begin{aligned} S(\langle x_1|, \ldots \langle x_p|) &= \sum_{\alpha_1,\ldots \alpha_p} s^{\alpha_1,\ldots,\alpha_p} \langle x_1|e^1_{\alpha_1}\rangle \cdots \langle x_p|e^p_{\alpha_p}\rangle \\ &= \left(\sum_{\alpha_1,\ldots \alpha_p} s^{\alpha_1,\ldots,\alpha_p} |e^1_{\alpha_1}\rangle \otimes \cdots \otimes |e^p_{\alpha_p}\rangle \right) (\langle x_1|, \ldots \langle x_p|) \end{aligned} \tag{18}$$

This expression suggests that

$$S = \sum_{\alpha_1,\ldots,\alpha_p} s^{\alpha_1,\ldots,\alpha_p} |e^1_{\alpha_1}\rangle \otimes \cdots \otimes |e^p_{\alpha_p}\rangle, \tag{19}$$

which, for infinite dimensional spaces, is only valid as an equality between two applications if the multiple series in the right hand side converges in the norm of the applications defined above, i.e. $\|S - S_n\| \longrightarrow 0$, where S_n is the partial sum of n terms.

Not every p–covariant tensor satisfies this statement, as we can see by the following example: Let S_0 be the tensor:

$$S_0 : \mathcal{H}^* \times \mathcal{H}^* \longrightarrow \mathbb{C}, \tag{20}$$

which, being $\mathcal{H}$ a infinte dimension separable Hilbert space and $|e_i\rangle$ the i–th vector of an orthonormal basis, acts as:

$$S_0(\langle x|, \langle y|) = \sum_{i=1}^{\infty} \langle x|e_i\rangle\langle y|e_i\rangle, \tag{21}$$

where the r.h.s. series is finite, because of

$$|\sum_{i=1}^{\infty} \langle x|e_i\rangle\langle y|e_i\rangle| \leq \sum_{i=1}^{\infty} |\langle x|e_i\rangle||\langle y|e_i\rangle| \leq \|\langle x|\|\|\langle y|\|, \tag{22}$$

which also proves the continuity of S_0. Hovewer, the expression

$$S_0 = \sum_{i=1}^{\infty} |e_i\rangle \otimes |e_i\rangle \tag{23}$$

is not valid because the r.h.s. is not convergent to S in the tensorial norm as we can prove easily

$$\|S_0 - (S_0)_n\| = \sup_{|x\rangle, |y\rangle \neq 0} \frac{|\sum_{i=n+1}^{\infty} \langle x|e_i\rangle\langle y|e_i\rangle|}{\|\langle x|\|\|\langle y|\|}, \tag{24}$$

which is greater or equal one, for any n, which is shown if we take $|x\rangle = |y\rangle = |e_{n+1}\rangle$.

Then, if we are interested in a Hilbert space of p–contravariant tensors described in terms of forms of the type $|e^1_{\alpha_1}\rangle \otimes \cdots \otimes |e^p_{\alpha_p}\rangle$, we must exclude those for which the expression (19) is not valid.

In addition, this space must include homogeneus tensors, that is, those of the form $|x_1\rangle \otimes \cdots \otimes |x_p\rangle$, for physical reasons. For example, if $|\psi\rangle$ and $|\varphi\rangle$ are describing two one-particle states, the tensor product $|\psi\rangle \otimes |\varphi\rangle$ actually represents the state of the compound system of the two particles without interacting between themselves.

These homogeneus tensors have a particular feature: if $|\psi\rangle$ and $|\varphi\rangle$ have components a^i and b^j, respectively in two ortonormal basis of the respective Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, then the tensor product has components (17) $s^{ij} = a^i b^j$ and the following sum is finite

$$\sum_{ij} |s^{ij}|^2 = \left(\sum_i |a^i|^2\right)\left(\sum_i |b^i|^2\right) = \||\psi\rangle\|^2 \||\varphi\rangle\|^2. \tag{25}$$

This fact suggest that we must impose the following condition to the components of a physical tensor:

$$\sum_{\alpha_1,\ldots,\alpha_p} |s^{\alpha_1,\ldots,\alpha_p}|^2 < \infty, \tag{26}$$

which inmediatelly guarantees that S is continuous, because

$$\begin{aligned} \|S\|_\infty &= \sup_{\langle x_1|,\ldots\langle x_p|\neq 0} \frac{|S(\langle x_1|,\ldots\langle x_p|)|}{\|\langle x_1|\|\cdots\|\langle x_p|\|} \\ &= \sup \frac{\left|\sum_{\alpha_1,\ldots\alpha_p} s^{\alpha_1,\ldots,\alpha_p}\langle x_1|e^1_{\alpha_1}\rangle\cdots\langle x_p|e^p_{\alpha_p}\rangle\right|}{\|\langle x_1|\|\cdots\|\langle x_p|\|} \\ &\leq \frac{\left(\sum_{\alpha_1,\ldots\alpha_p}|s^{\alpha_1,\ldots,\alpha_p}|^2\right)^{1/2}\left(\sum_{\alpha_1}|\langle x_1|e^1_{\alpha_1}\rangle|^2\right)^{1/2}\cdots\left(\sum_{\alpha_p}|\langle x_p|e^p_{\alpha_p}\rangle|^2\right)^{1/2}}{\|\langle x_1|\|\cdots\|\langle x_p|\|}, \end{aligned} \tag{27}$$

because of the recursive application of the Schwartz inequality in the multiple sum. The norms of $|x_i\rangle$ cancel, therefore:

$$\|S\|_\infty \leq \left(\sum_{\alpha_1,\ldots,\alpha_p} |s^{\alpha_1,\ldots,\alpha_p}|^2\right)^{1/2} < \infty. \tag{28}$$

Now we have restricted the set of tensors which may belong to the tensor product of $\mathcal{H}_1, \ldots \mathcal{H}_p$, we have to redefine the norm of these tensors. The reason is that we are interested in define a scalar product in this tensorial space which be compatible with the norm, and the norm $\|\cdot\|_\infty$ above defined does not follow from a scalar product, which can be easily proved just by seeing that it does not hold the parallelogram law with the following example: For $S_1 = |e_1\rangle \otimes |e_1\rangle$ and $S_2 = |e_2\rangle \otimes |e_2\rangle$ on $\mathcal{H}^* \times \mathcal{H}^*$, where $|e_1\rangle$ and $|e_2\rangle$ are two different elements of a orthonormal basis of $\mathcal{H}$, we can find that

$$|(S_1 + S_2)(\langle x|, \langle y|)| = |\langle x|[\langle y|e_1\rangle|e_1\rangle + \langle y|e_2\rangle|e_2\rangle]\rangle| \leq \||x\rangle\|\||y\rangle\|, \tag{29}$$

where the equality holds when, for example, $|x\rangle = |y\rangle = |e_1\rangle$. This means that

$$\|S_1 + S_2\| = 1 \tag{30}$$

as well as $\|S_1 - S_2\|$. Similarly, we can prove that $\|S_1\| = \|S_2\| = 1$, so

$$\|S_1 + S_2\|^2 + \|S_1 - S_2\|^2 \neq 2(\|S_1\|^2 + \|S_2\|^2). \tag{31}$$

Then, we define, for any p–contravariant tensor S

$$\|S\|_2 = \left(\sum_{\alpha_1,\ldots,\alpha_p} |s^{\alpha_1\ldots\alpha_p}|^2\right)^{1/2} \tag{32}$$

which evidently follows from the scalar product:

$$\langle S_1|S_2\rangle = \sum_{\alpha_1,\ldots,\alpha_p} \bar{s}_1^{\alpha_1\ldots\alpha_p} s_2^{\alpha_1\ldots\alpha_p}, \tag{33}$$

where s_1 and s_2 stand for the respective components of S_1 and S_2 and the bar stands for the complex conjugate. The reader can prove easily that this application is actually an scalar product and that it does not change when changing bases. Also, this is the scalar product which acts for homogeneous tensors as the product of the scalar products between vectors of the same space:

$$\langle |x\rangle \otimes |y\rangle || u\rangle \otimes |v\rangle\rangle = \langle x|u\rangle\langle y|v\rangle. \tag{34}$$

2.4 Definition of the tensor product of Hilbert spaces

We have completed our construction of the tensor product of the Hilbert spaces, which can be summarized by the following definition:

Let $\mathcal{H}_1, \ldots, \mathcal{H}_p$, *p Hilbert spaces, and* $\{|e_{\alpha_1}\rangle\}, \ldots, \{|e_{\alpha_p}\rangle\}$ *orthonormal basis of them, respectively. Then we define the tensor product* $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_p$ *as the set of all p–linear bounded maps S acting on* $\mathcal{H}_1^* \times \cdots \times \mathcal{H}_p^*$ *(p–contravariant tensors) such that*

$$\sum_{\alpha_1,\ldots,\alpha_p} |s^{\alpha_1,\ldots,\alpha_p}|^2 < \infty, \tag{35}$$

where

$$s^{\alpha_1,\ldots,\alpha_p} = S(\langle e^1_{\alpha_1}|, \ldots, \langle e^p_{\alpha_p}|), \tag{36}$$

with the scalar product defined in Eq.(33).

It can be seen how this space is isomorphic to the Hilbert space of sequences with p indices for which the sum of the square of their elements in every index is finite. The completeness of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_p$ can be checked by means of this isometric isomorphism.

In addition, it can be easily proved that the homogeneous tensors of the type $|e^1_{\alpha_1}\rangle \otimes \cdots \otimes |e^p_{\alpha_p}\rangle$ form an orthonormal basis of the tensor product space.

Finally, we want to remark that the elements of this tensor product of spaces are the only physical tensors, i.e. those which may represent the state of a quantum system with p–degrees of freedom. The other tensors, such as S_0 defined in Eq.(21), are non–physical tensors.

Example: Let us see, as an elementary application of the above definitions, that $L^2(\mathbb{R}^2)$ is isomorphic to the tensor product $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$.

Consider a orthonormal basis φ_n ($n = 0, \ldots, \infty$) of $L^2(\mathbb{R})$. The associated basis in $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ is $\varphi_n \otimes \varphi_m$. An element of this tensor space is of the kind

$$F = \sum_{nm} a_{nm} \varphi_n \otimes \varphi_m, \tag{37}$$

which univocally defines a function over $\mathbb{R}^2$

$$f(x, y) = \sum_{nm} a_{nm} \varphi_n(x) \varphi_m(y), \tag{38}$$

which norm is finite because

$$\int |f(x, y)|^2 dxdy = \sum_{nm} |a_{nm}|^2 < \infty, \tag{39}$$

and then it belongs to $L^2(\mathbb{R}^2)$.

Now comes the reciprocal argument: if f is a square–integrable function on $\mathbb{R}^2$, we can define a 2–contravariant tensor which acts on $L^2(\mathbb{R})^* \otimes L^2(\mathbb{R})^*$ as:

$$F(|g_1\rangle, |g_2\rangle) = \int f(x, y) \overline{g_1(x)}\, \overline{g_2(x)}\, dx\, dy, \tag{40}$$

its components on the orthonormal basis are found to be:

$$a_{nm} = \int f(x,y)\overline{\varphi_n(x)}\,\overline{\varphi_m(y)}dxdy, \tag{41}$$

and the sum of $|a_{nm}|^2$ is finite because Eq.(39) holds again.

It can be shown straightforward that the scalar product can be expressed as:

$$\langle F_1|F_2\rangle = \int \overline{f_1(x,y)}f_2(x,y)dxdy. \tag{42}$$

3 Comments about physical applications

It is a postulate of quantum mechanics that any physical system have an associate Hilbert space which elements represent all the possible states of the system. The description of a complex system in terms of simpler ones requieres the definition of the tensor product of Hilbert spaces in order to have the requiered mathematical structure.

This tensor product is often applied in quantum mechanics, although not always remarked explicitely. As as example, in the resolution of a three-dimensional Schrdinger equation for a particle under a central potential, solutions by means of a product of a radial function times an angular one, i.e. performing a factorization of the dependence upon the spherical coordinates, are found. This possibility comes from the fact that

$$L^2(\mathbb{R}^3) = L^2_{r^2}([0,\infty)) \otimes L^2(S_2), \tag{43}$$

where S_2 denotes the two-dimensional spherical surface of unit radius, and

r^2 is the weight in the scalar product of $L^2_{r^2}([0,\infty))$:

$$\langle f|g\rangle = \int_0^\infty r^2\overline{f(r)}g(r)\,dr. \tag{44}$$

The previous arguments then show that although not every element $L^2(\mathbb{R}^3)$ can be written as a single product of a radial and an angular function, it has a convergent expansion in a basis of this type, which can be constructed from orthonormal basis of $L^2([0,\infty))$ and $L^2(S_2)$. If we are interested in solutions with a certain symmetry, e.g. eigenstates of the angular momentum operators $\vec{L}^2$ and L_z, and we use the spherical harmonics basis set for $L^2(S_2)$, the expansion is then reduced to a single term in the spherical harmonics indexes, which allows us to write:

$$\Psi(\vec{r}) = R(r)Y_{lm}(\theta,\phi). \tag{45}$$

Another additional example can be found in the treatment of the spin of a particle in non-relativistic quantum mechanics. For a particle of spin s, it is introduced just by making the tensor product of the position space, $L^2(\mathbb{R}^3)$ and the $\mathbb{C}^{2s+1}$ space.

This tensor product of Hilbert spaces has further applications for describing a many-particle system in terms of elementary ones. The Hilbert space which describes the states of a N-particle system in three-dimensional space is

$$L^2(\mathbb{R}^3)\underbrace{\otimes\cdots\otimes}_{N\text{times}}L^2(\mathbb{R}^3) \tag{46}$$

and then the N-particle states can be expanded in a basis set of products of single-particle ones.

When dealing with the tensor product of N finite dimensional spaces, every p-contravariant tensor is an element of this product space and then define a physical state. However, we usually deal with infinite dimensional spaces such as $L^2(\mathbb{R}^3)$. Recalling the above example, not every N-contravariant tensor over $[L^2(\mathbb{R}^3)]^* \times \cdots (N\text{times}) \cdots \times [L^2(\mathbb{R}^3)]^*$ define a physical state, because the Hilbert structure requieres the additional condition written in Eq. (35), which select those here called *physical tensors*.

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