Proyecto de Innovación docente

Teoría Cuántica de Campos aplicada a la Física de Partículas

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Introducción

En el curso de Teoría Cuántica de Campos se introducen conceptos físicos y matemáticos complejos, como Integrales Funcionales, Funciones de Green, Diagramas y Reglas de Feynman, el Grupo de Renormalización o Libertad Asintótica.

Por otro lado, estos mismos conceptos están en la base de la asignatura de Física de Partículas y su profunda comprensión es imprescindible para entender los fenómenos físicos que se estudian en el curso correspondiente.

Para los estudiantes de uno u otro curso resulta muy útil ver como el formalismo de la Teoría Cuántica de Campos tiene una aplicación inmediata en el marco de la Física de Partículas. En concreto, es útil que el alumnado aprenda a utilizar los conceptos complejos de la Teoría Cuántica de Campos en casos muy prácticos, es decir, a través del uso de herramientas pensadas para solucionar problemas concretos en Física de Partículas.

En este sentido, la Física de Partículas es un lugar *natural* para que los estudiantes utilicen lo que van aprendiendo en el curso de Teoría Cuántica de Campos.

Entonces, el primer objetivo de este Proyecto de Innovación docente es construir un puente entre los dos cursos, para que cada asignatura pueda sacar el máximo provecho de lo que se estudia en la otra, en el marco de una sinergia común.

Se pretende alcanzar este primer objetivo a través de una serie de problemas y de actividades prácticas que tienen como finalidad el aprendizaje de la utilización de los conceptos básicos y de los programas y herramientas informáticas por parte de los estudiantes de las dos asignaturas. Con ellas podrán efectuar *experimentos virtuales*, es decir simulaciones de procesos físicos que obedecen a las leyes estudiadas en el curso de Física de Partículas y al formalismo matemático de la Teoría Cuántica de Campos.

También se pretende que aprendan, en el mismo ciclo de prácticas, los fundamentos básicos de las técnicas de simulación numérica empleadas por los códigos que van utilizando.

Por otro lado, es útil que los estudiantes vean como todo lo que van aprendiendo en las dos asignaturas se relaciona directamente con temas de investigación de vanguardia en el experimento más grande y sofisticado construido por el hombre, es decir el Large Hadron Collider (LHC), que se acaba de inaugurar en el CERN en Ginebra, Suiza.

El segundo objetivo de este Proyecto es, además, sacar provecho del momento histórico particular que se vive en el campo de la Física de Partículas, para poner a los estudiantes en contacto directo con las actividades más avanzadas en el campo de la investigación teórica y experimental relacionadas con las dos asignaturas.

En efecto, estamos profundamente convencidos de que un estímulo tan grande como el seguimiento de los desarrollos a que dé lugar el LHC como pretendemos con este segundo objetivo del proyecto, pueda motivar y facilitar en gran medida el proceso de aprendizaje de conceptos complejos que, sin esta comparación con la realidad, tendrían tan sólo el mero valor de fórmulas escritas en los libros.

Este segundo objetivo se alcanzará a través de una serie de conferencias y ponencias de expertos que pongan la Teoría Cuántica de Campos y la Física de Partículas en el marco de la investigación básica contemporánea.

En resumen, el presente Proyecto de Innovación docente pretende alcanzar dos objetivos distintos:

1) Aplicación inmediata de los conceptos básicos de la Teoría Cuántica de Campos a la Física de Partículas, a través de problemas prácticos utilizando también herramientas y programas de simulación.

2) Ejemplificar lo aprendido en las asignaturas de Teoría Cuántica de Campos y de Física de Partículas en el contexto de un proyecto de investigación de vanguardia como el LHC y otros experimentos actuales de física de partículas.

Los dos Objetivos se han alcanzado a través de la preparación de los problemas prácticos que aquí presentamos. [1] En algunos de ellos se introduce el alumnado al uso de algunas de las herramientas más utilizadas en la simulación de problemas en física de partículas [1, 2], explicando también los fundamentos básicos de las técnicas empleadas por los programas.

[1] Los problemas con * tienen que ser solucionados por los alumnos, utilizando lo aprendido.

Además, cada año académico se organizará un ciclo de ponencias, a nivel básico, de expertos nacionales e internacionales en la física de LHC (John Ellis y Roger Bailey en 2009).

Para el desarrollo de todas las actividades previstas en este Proyecto, se propone que los estudiantes utilicen 1.5 de los créditos de prácticas del curso de Teoría Cuántica de Campos y 1 crédito de prácticas del curso de Física de Partículas, por un total de 25 horas (en su formulación actual, el curso de Teoría Cuántica de Campos tiene 5 créditos de Teoría y 2.5 de prácticas, mientras el de Física de Partículas 4 créditos de Teoría y 2 de Prácticas).

Las 25 horas serán así repartidas entre las varias actividades del proyecto:

a) Prácticas para familiarizar el alumnado con los programas y los algoritmos que tienen que utilizar: 6h

b) Trabajo individual o en grupo para solucionar problemas sencillos, utilizando los conceptos explicados: 16h

c) Asistencia a las ponencias de los expertos en de Física del LHC: 3h.

Finalmente, el material que aquí se presenta está en Inglés. En efecto, también el idioma se puede considerar, en el fondo, como una *herramienta* que el alumnado tiene que aprender y utilizar, especialmente en el ámbito cientifico.

Contents

Chapter 1

Classical fields

In point mechanics, Physics is described by dinamical variables

$$q_\alpha(t) \tag{1.1}$$

depending on the time t, whose equations of motions are fully determined once one knows the Lagrangian $\mathcal{L}(q_\alpha, \dot{q}_\alpha)$ of the system

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0. \tag{1.2}$$

In local field theory, at each point $x = (x_0, \vec{x})$ of the four-dimensional spacetime one associates one or more dynamical variables $\Phi_i(x)$ called *fields* obeying the equivalent of the Lagrange equations in (1.2)

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} - \frac{\partial \mathcal{L}}{\partial \Phi_i} = 0, \tag{1.3}$$

where $\mathcal{L}(\Phi_i, \partial_\mu \Phi_i)$ is the Lagrange density (often simply called Lagrangian) describing the field theory. Therefore, the formal transition between point mechanics and local field theory is

$$q_\alpha(t) \to \Phi_i(x). \tag{1.4}$$

The action S is defined as the integral of $\mathcal{L}$ over all the four-dimensional space

$$S = \int d^4x\, \mathcal{L}(\Phi_i, \partial_\mu \Phi_i). \tag{1.5}$$

1.1 Problem*: The principle of least action

Arrive at (1.3) by requiring $\delta S = 0$.

1.2 Problem: Adding a four-divergence to $\mathcal{L}$

Prove explicitly that

$$\mathcal{L}' = \mathcal{L} + \Delta\mathcal{L}, \tag{1.6}$$

where $\Delta\mathcal{L} = \partial_\beta G^\beta(\{\Phi_k\})$ is a four-divergence of an arbitrary function of the fields, is also a solution of (1.3).

Solution

Inserting $\mathcal{L}'$ in the l.h.s. of (1.3) gives

$$F \equiv \partial_\mu \frac{\partial \Delta\mathcal{L}}{\partial(\partial_\mu \Phi_i)} - \frac{\partial \Delta\mathcal{L}}{\partial \Phi_i}. \tag{1.7}$$

Hence, we have to show that $F = 0$. By rewriting

$$\Delta\mathcal{L} = \frac{\partial G^\beta}{\partial \Phi_j}(\partial_\beta \Phi_j), \tag{1.8}$$

one computes

$$\frac{\partial \Delta\mathcal{L}}{\partial(\partial_\mu \Phi_i)} = \frac{\partial G^\mu}{\partial \Phi_i}. \tag{1.9}$$

Inserting this in (1.7) and interchanging the order of the derivatives gives $F = 0$.

1.3 Problem: The Klein-Gordon and Dirac equations

Show that the Lagrangians

$$\begin{aligned}\mathcal{L}_1 &= \frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - \frac{1}{2}m^2\Phi^2,\\ \mathcal{L}_2 &= (\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi^*\phi,\\ \mathcal{L}_3 &= \bar{\Psi}(i\partial\!\!\!/ - m)\Psi \quad \text{with} \quad \bar{\Psi} = \Psi^\dagger\gamma_0,\end{aligned} \tag{1.10}$$

give the Klein-Gordon, the complex Klein-Gordon and the Dirac equation, respectively.

Solution

One computes

$$\frac{\partial\mathcal{L}_1}{\partial(\partial_\mu\Phi)} = \partial^\mu\Phi, \quad \frac{\partial\mathcal{L}_1}{\partial\Phi} = -m^2\Phi, \tag{1.11}$$

so that (1.3) gives $(\partial_\mu\partial^\mu + m^2)\Phi = 0$. Similarly

$$\frac{\partial\mathcal{L}_2}{\partial(\partial_\mu\phi)} = \partial^\mu\phi^*, \quad \frac{\partial\mathcal{L}_2}{\partial\phi} = -m^2\phi^*, \quad \frac{\partial\mathcal{L}_2}{\partial(\partial_\mu\phi^*)} = \partial^\mu\phi, \quad \frac{\partial\mathcal{L}_2}{\partial\phi^*} = -m^2\phi \tag{1.12}$$

give the two equations $(\partial_\mu\partial^\mu + m^2)\phi^* = (\partial_\mu\partial^\mu + m^2)\phi = 0$. Finally, by using the fact that four-divergences do not change the Physics content of the Lagrangian, one rewrites

$$\mathcal{L}_3 = -i\gamma_\mu(\partial^\mu\bar{\Psi})\Psi - m\bar{\Psi}\Psi, \tag{1.13}$$

so that

$$\frac{\partial\mathcal{L}_3}{\partial(\partial^\mu\bar{\Psi})} = -i\gamma_\mu\Psi, \quad \frac{\partial\mathcal{L}_3}{\partial\bar{\Psi}} = -m\Psi \tag{1.14}$$

which gives $(i\partial\!\!\!/ - m)\Psi = 0$.

1.4 Problem*: The conjugate Dirac equation

Show that taking the partial derivatives of $\mathcal{L}_3$ in (1.10) with respect to $\partial^\mu \Psi$ and Ψ one arrives at the conjugate transpose of the Dirac equation.

[Hint: use the result of problem 3.5.]

Chapter 2

Kinematics and special relativity

In Particle Physics and Quantum Field Theory, a fundamental role is played by special relativity, in the sense that it provides a common framework for both disciplines. In this chapter we recall the basic needed notions with the help of a few practical problems.

2.1 Problem: Momentum and speed of a particle

An electron has a total energy $E_{tot} = 5E_{quiet}$. Calculate its momentum and its speed.

Solution

One has

$$\begin{aligned} E_{tot} &= \sqrt{p^2 + m^2}, \\ E_{quiet} &= m = 0.5 \text{ MeV}, \\ \sqrt{p^2 + m^2} &= 5m \Rightarrow p^2 + m^2 = 25m^2, \\ p^2 &= 24m^2 \Rightarrow p = \sqrt{24}m \leftarrow \textit{momentum}. \end{aligned} \tag{2.1}$$

Furthermore

$$p = \beta E_{tot} \Rightarrow \beta \equiv \frac{v}{c} = \frac{p}{E_{tot}} = \frac{p}{5m} = \frac{\sqrt{24}m}{5m} = \frac{2\sqrt{6}}{5} \sim 0.98, \tag{2.2}$$

so that the speed of the electron is 98% of the speed of light.

2.2 Problem: Energy-momentum conservation

Why the process $e^- \to e^-\gamma$ doesn't occur?

Solution

We write the process as follows

$$e^-(p_i) \to e^-(p_f) + \gamma(k).$$

Then, in the system where the initial state electron has zero speed, one has the following kinematics

$$\begin{aligned} p_i &= (m, \vec{0}) \\ p_f &= (E_f, \vec{p}_f) \\ k &= (k_0, \vec{k}), \end{aligned} \tag{2.3}$$

together with the on-shell constraints:

$$\begin{aligned} m^2 &= E_f^2 - |\vec{p}_f|^2 \\ 0 &= k_0^2 - \vec{k}^2. \end{aligned} \tag{2.4}$$

From momentum conservation it should then happen

$$\vec{0} = \vec{p}_f + \vec{k} \Rightarrow |\vec{p}_f| = |\vec{k}| = k_0. \tag{2.5}$$

Now we can calculate the total energy in the final state by putting together all the previous results

$$E_f + k_0 = \sqrt{|\vec{p}_f|^2 + m^2} + |\vec{p}_f| > m. \tag{2.6}$$

On the other hand, by directly equating the energy components in (2.3), it should also be $E_f + k_0 = m$, in contradiction with (2.6). As a consequence, the process $e^- \to e^-\gamma$ cannot occur because one cannot simultaneously satisfy energy-momentum conservation and on-shell constraints.

2.3 Problem: Compton Scattering with e^- at rest

Show that in the Compton scattering, namely in the collision of γ against e^- at rest

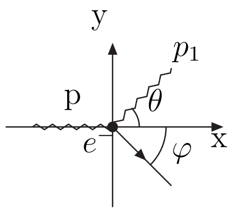
$$\Delta\lambda = \frac{2h}{m_e} sin^2\frac{\theta}{2}.$$

(Use a system with c=1)

Solution

The process can be written as follows

$$\gamma(p) + e^-(q) \to \gamma(p_1) + e^-(q_1)$$



The kinematics is given by $\begin{cases} p^\mu = (E_p, p, 0, 0) \\ q^\mu = (E_q, 0, 0, 0) \\ p_1^\mu = (E_p^{'}, p_1 cos\theta, p_1 sin\theta, 0) \\ q_1^\mu = (E_q^{'}, q_1 cos\varphi, q_1 sin\varphi, 0) \end{cases}$.

From energy-momentum conservation one obtains

$$\begin{cases} E_p + E_q = E'_p + E'_q \\ p = p_1 cos\theta + q_1 cos\varphi \\ p_1 sin\theta = -q_1 sin\varphi \end{cases} . \tag{2.7}$$

By eliminating φ from the last two equations one obtains

$$\begin{cases} q_1^2 cos\varphi^2 = (p - p_1 cos\theta)^2 \\ q_1^2 sin\varphi^2 = p_1^2 sin^2\theta \end{cases} \Rightarrow \tag{2.8}$$

$$q_1^2 = p^2 + p_1^2 cos^2\theta - 2pp_1 cos\theta + p_1^2 sin^2\theta \Rightarrow \tag{2.9}$$

$$q_1^2 = p^2 + p_1^2 - 2pp_1 cos\theta. \tag{2.10}$$

On the other hand, from the first of eqs. (2.7) one obtains ($m \equiv m_e$)

$$\begin{aligned} p + m &= p_1 + \sqrt{q_1^2 + m^2} \Rightarrow \\ p - p_1 + m &= \sqrt{q_1^2 + m^2} \Rightarrow \\ q_1^2 + m^2 &= p^2 + p_1^2 + m^2 - 2pp_1 + 2pm - 2mp_1, \end{aligned} \tag{2.11}$$

then

$$q_1^2 = p^2 + p_1^2 - 2pp_1 + 2pm - 2mp_1. \tag{2.12}$$

Equating eqs. (2.10) and (2.12) gives

$$\begin{aligned} -2pp_1 cos\theta &= -2pp_1 - 2m(p_1 - p) \Rightarrow \\ pp_1(1 - cos\theta) &= m(p - p_1) \Rightarrow \\ 2pp_1 sin^2\frac{\theta}{2} &= m(p - p_1). \end{aligned} \tag{2.13}$$

By remembering that

$$p = \frac{h}{\lambda} \quad \text{and} \quad p_1 = \frac{h}{\lambda'} \tag{2.14}$$

one obtains

$$2h^2\frac{1}{\lambda\lambda'}sin^2\frac{\theta}{2} = mh(\frac{1}{\lambda} - \frac{1}{\lambda'}) = mh\frac{\lambda' - \lambda}{\lambda\lambda'} = mh\frac{\Delta\lambda}{\lambda\lambda'}, \tag{2.15}$$

from which the desired result follows

$$\Delta\lambda = \frac{2h}{m}sin^2\frac{\theta}{2}. \tag{2.16}$$

2.4 Problem: Mandelstam variables

Given a $2 \to 2$ process

$$p_A + p_B \to p_C + p_D,$$

the Mandelstam variables are defined as

$$\begin{aligned} s &= (p_A + p_B)^2, \\ t &= (p_A - p_C)^2, \\ u &= (p_A - p_D)^2. \end{aligned} \tag{2.17}$$

a) Show that $s + t + u = \sum_i m_i^2$;

b) Express the total energy of the collision in the center-of-mass frame;

c) Compute the energy of particle A in the Laboratory system, where particle B is at rest;

d) Express the energy of particle A in the center-of-mass frame.

Solution

a) From the definition of the Mandelstam variables one computes

$$\begin{aligned} s = (p_A + p_B)^2 = m_A^2 + m_B^2 + 2p_A \cdot p_B, \\ t = (p_A - p_C)^2 = m_A^2 + m_C^2 - 2p_A \cdot p_C, \\ u = (p_A - p_D)^2 = m_A^2 + m_D^2 - 2p_A \cdot p_D. \end{aligned}$$

Therefore

$$s + t + u = 3m_A^2 + m_B^2 + m_C^2 + m_D^2 - 2p_A \cdot (p_D + p_C - p_B). \tag{2.18}$$

By using in the previous equation energy-momentum conservation, namely $p_D + p_C - p_B = p_A$, one immediately obtains the desired results

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2. \tag{2.19}$$

b) By definition of center-of-mass frame

$$\vec{p}_A + \vec{p}_B = 0 \Rightarrow s = (E_A + E_B, \vec{0})^2 = (E_T, \vec{0})^2.$$

Thus, $s = E_T^2 \Rightarrow E_T = \sqrt{s}$.

c) If B is at rest

$$p_B^\mu = (m_B, \vec{0}) \Rightarrow p_A \cdot p_B = E_A m_B \Rightarrow s = m_A^2 + m_B^2 + 2E_A m_B.$$

Therefore, $E_A = \dfrac{s - m_A^2 - m_B^2}{2m_B}$.

d) In the center-of-mass frame $\vec{p}_A + \vec{p}_B = 0 \Rightarrow \vec{p}_A = -\vec{p}_B \Rightarrow |\vec{p}_A| = |\vec{p}_B| = |\vec{p}| = p$. Thus

$$\begin{array}{l} m_A^2 = E_A^2 - p^2 \\ m_B^2 = E_B^2 - p^2 \end{array} \Rightarrow \left\{ \begin{array}{l} E_A = \sqrt{m_A^2 + p^2} \\ E_B = \sqrt{m_B^2 + p^2}. \end{array} \right.$$

On the other hand

$$s = (E_A + E_B, \vec{0})^2 = (E_A + E_B)^2 = m_A^2 + m_B^2 + 2p^2 + 2\sqrt{(m_A^2 + p^2)(m_B^2 + p^2)} \Rightarrow$$

$$\frac{s - m_A^2 - m_B^2 - 2p^2}{2} = \sqrt{(m_A^2 + p^2)(m_B^2 + p^2)}.$$

Therefore

$$\frac{1}{4}\left[s^2 + m_A^4 + m_B^4 + 4p^4 - 2sm_A^2 - 2sm_B^2 - 4sp^2 + 2m_A^2 m_B^2 + 4m_A^2 p^2 + 4m_B^2 p^2\right]$$
$$= m_A^2 m_B^2 + p^4 + m_A^2 p^2 + m_B^2 p^2$$

$$\Rightarrow s^2 + m_A^4 + m_B^4 - 2sm_A^2 - 2sm_B^2 - 4sp^2 - 2m_A^2 m_B^2 = 0$$
$$\Rightarrow p = \frac{1}{2\sqrt{s}} \lambda^{\frac{1}{2}}(s, m_A^2, m_B^2).$$

This gives

$$E_A = \left\{ m_A^2 + \frac{1}{4s}\left[s^2 + m_A^4 + m_B^4 - 2sm_A^2 - 2sm_B^2 - 2m_A^2 m_B^2\right] \right\}^{\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{s}} \left\{4sm_A^2 + s^2 + m_A^4 + m_B^4 - 2sm_A^2 - 2sm_B^2 - 2m_A^2 m_B^2\right\}^{\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{s}}(s + m_A^2 - m_B^2).$$

2.5 Problem: Compton Scattering with e^- not at rest

Show that, in the case where initial electron is moving with momentum q along x, the formula reads

$$\Delta\lambda = \frac{2\lambda(p+q)}{E_q - q} sin^2\frac{\theta}{2}.$$

Solution

$$\gamma(p) + e^-(q) \to \gamma(p_1) + e^-(q_1)$$

$$\begin{cases} E_p + E_q = E'_p + E'_q \\ p + q = p_1 cos\theta + q_1 cos\varphi \\ p_1 sin\theta = -q_1 sin\varphi \end{cases} \Rightarrow$$

$$\begin{cases} q_1 cos\varphi = p + q - p_1 cos\theta \\ q_1 sin\varphi = -p_1 sin\theta \end{cases} \Rightarrow$$

$$q_1^2 = p^2 + q^2 + p_1^2 + 2pq - 2pp_1cos\theta - 2qp_1cos\theta. \tag{2.20}$$

On the other hand, from energy conservation one obtains

$$\begin{aligned} p + \sqrt{m^2 + q^2} &= p_1 + \sqrt{m^2 + q_1} \Rightarrow \\ (p - p_1) + E_q &= \sqrt{m^2 + q^2} \Rightarrow \\ p^2 + p_1^2 - 2pp_1 + q^2 + m^2 + 2E_q(p - p_1) &= m^2 + q_1^2. \end{aligned} \tag{2.21}$$

Therefore

$$q_1^2 = p^2 + p_1^2 - 2pp_1 + 2E_q(p - p_1) + q^2. \tag{2.22}$$

Equating eqs. (2.20) and (2.22) gives

$$\begin{aligned} p^2 + q^2 + p_1^2 + 2pq - 2pp_1cos\theta - 2qp_1cos\theta &= p^2 + q^2 + p_1^2 - 2pp_1 + 2E_q(p - p_1) \Rightarrow \\ pq - (p+q)p_1cos\theta &= E_q(p - p_1) - pp_1 \Rightarrow \\ pp_1 + pq - (p+q)p_1cos\theta &= E_q(p - p_1). \end{aligned} \tag{2.23}$$

By adding and subtracting the quantity qp_1 in the l.h.s. of the previous equation, one obtains the desired result

$$\begin{aligned}
&p_1(p+q)(1-cos\theta)+q(p-p_1)-E_q(p-p_1)=0 \Rightarrow \\
&2p_1(p+q)sin^2\frac{\theta}{2}=(p-p_1)(E_q-q) \Rightarrow \\
&(p-p_1)=\frac{2p_1(p+q)sin^2\frac{\theta}{2}}{E_q-q} \Rightarrow \\
&h(\frac{\lambda'-\lambda}{\lambda\lambda'})=\frac{2h}{\lambda'}\frac{(p+q)}{E_q-q}sin^2\frac{\theta}{2} \Rightarrow \\
&\Delta\lambda=2\lambda\frac{p+q}{E_q-q}sin^2\frac{\theta}{2}.
\end{aligned} \quad (2.24)$$

2.6 Problem∗: The Lorentz transformations

Write down the Lorentz transformations between two inertial frames moving with relative speed v along the z axis. How a generic 4-vectors p_μ in one frame is transformed in the other frame? If $p^2 \equiv s > 0$, show that is it always possible to find a frame in which

$$p=(\sqrt{s},0,0,0). \quad (2.25)$$

Write down explicitly the corresponding Lorentz transformation.

Chapter 3

Trace theorems and γ matrices

When dealing with fermions some properties of the gamma matrices are necessary. We review them here and present a few problems on the subject.

3.1 Traces of γ matrices in 4 dimensions

1. $Tr\{\mathbf{I}\} = 4$.

2. $Tr\{\gamma_5\} = 0$, where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

3. A trace of an odd number of $\gamma's$ vanishes.

4. $Tr\{\not{a}\not{b}\} = 4(a \cdot b)$.

5. $Tr\{\gamma_5\not{a}\not{b}\} = 0$.

6. $Tr\{\not{a}\not{b}\not{c}\not{d}\} = 4\left[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)\right]$.

7. $Tr\{\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\sigma\gamma_5\} = 4i\epsilon_{\mu\nu\lambda\sigma} = -4i\epsilon^{\mu\nu\lambda\sigma}$,

   where $\epsilon_{\mu\nu\lambda\sigma} = 1$ when $(\mu, \nu, \lambda, \sigma)$ is an even permutation of $(0, 1, 2, 3)$, -1 for an odd permutation, 0 otherwise.

8. $\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu}_{\alpha\beta} = -2\left(g_{\rho\alpha}g_{\sigma\beta} - g_{\rho\beta}g_{\sigma\alpha}\right)$.

3.2 Identities in 4 dimensions and other useful relations

1. $\gamma_\mu \gamma^\mu = 4$.
2. $\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$.
3. $\gamma_\mu \not{a}\not{b} \gamma^\mu = 4a \cdot b$.
4. $\gamma_\mu \not{a}\not{b}\not{c} \gamma^\mu = -2\not{c}\not{b}\not{a}$.
5. $\gamma_\mu \not{a}_1 \not{a}_2 \ldots \not{a}_{2n-1} \gamma^\mu = -2(\not{a}_{2n-1} \ldots \not{a}_2 \not{a}_1)$.

3.3 Problem*: Gamma Matrices

With the help of the fundamental anticommutation relation

$$\{\gamma^\mu \gamma^\nu\} = 2\, g^{\mu\nu} \tag{3.1}$$

prove all the identities in the previous sections.

3.4 Problem*: An explicit representation for the γ matrices

By using the properties of the Pauli matrices, prove that a possible explicit representation of the γ matrices is given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{3.2}$$

where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.3}$$

3.5 Problem*: Complex conjugation of the γ matrices

By using the explicit representation in (3.2) show that

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu. \tag{3.4}$$

Chapter 4

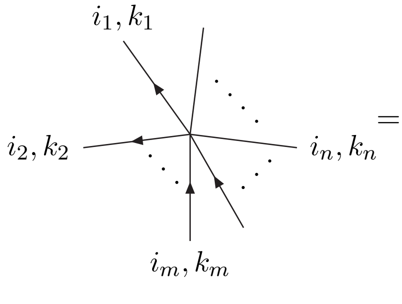
Feynman rules

In this Chapter we derive the Feynman rules needed to perform the calculations presented in the following sections.

A Lagrangian $\mathcal{L}$ involving complex fields $\{\phi^*_{i_1}, \phi^*_{i_2}, \ldots\}$, $\{\phi_{i_m}, \phi_{i_{m+1}}, \ldots\}$ and real fields $\{\Phi_{i_n}, \ldots\}$

$$\mathcal{L} = \alpha_{i_1 i_2 \ldots}(\{\partial^\mu_{i_\ell}\})\phi^*_{i_1}(x)\cdots\phi_{i_m}(x)\cdots\Phi_{i_n}(x)\cdots \tag{4.1}$$

produces a vertex in the momentum-space defined as [3]



$$= i \sum_{\{1\ldots m-1\}} \sum_{\{m\ldots n-1\}} \sum_{\{n\ldots\}} (-1)^P \tilde{\alpha}_{i_1 i_2 \ldots}(\{-ik^\mu_{i_\ell}\}).$$

The indices of the fields stand for any kind of index, such as Lorentz, spin and isospin, and $\tilde{\alpha}_{i_1 i_2 \ldots}(\{-ik^\mu_{i_\ell}\})$ is the Fourier transform of $\alpha_{i_1 i_2 \ldots}(\{\partial^\mu_{i_\ell}\})$. The momenta k_j are incoming and each of the derivatives in the set $\{\partial^\mu_{i_\ell}\}$, acting on the i^{th}_ℓ field, is replaced by $-i$ times the momentum of the field. The sums are over the permutations of the indicated indices and $(-1)^P$ is only relevant if several fermion fields occur: each fermion(antifermion) is taken to anticommute with any other fermion(antifermion).

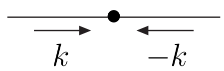
The quadratic part of $\mathcal{L}$ defines a 2-point vertex. The propagator is defined to be minus the inverse of such a 2-point vertex.

4.1 Problem: The scalar and fermion propagators

Derive the propagators P_1, P_2 and P_3 from the three Lagrangians in (1.10).

Solution

The 2-point vertex one reads from $\mathcal{L}_1$ is

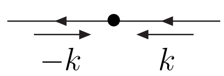


$$= \frac{i}{2}\big[(-ik_\mu)(ik^\mu) + (ik^\mu)(-ik_\mu) - m^2 - m^2\big] = i(k^2 - m^2) = V_1.$$

Therefore

$$P_1 = -\frac{1}{V_1} = \frac{i}{k^2 - m^2}. \tag{4.2}$$

The 2-point vertex produced by $\mathcal{L}_2$ is



$$= i\big[-i(-k_\mu)(-ik^\mu) - m^2\big] = i(k^2 - m^2) = V_2.$$

Thus

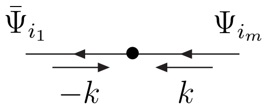
$$P_2 = -\frac{1}{V_2} = \frac{i}{k^2 - m^2}. \tag{4.3}$$

Finally, rewriting $\mathcal{L}_3$ in components gives

$$\mathcal{L}_3 = \bar{\Psi}_{i_1}(i\partial\!\!\!/ - m)_{i_1 i_m}\Psi_{i_m}, \tag{4.4}$$

so that [1]

[1] Note that $\bar{\Psi}_{i_1}$ is used instead of $\Psi^\dagger_{i_1}$. However, this still produces the correct result if $\Psi^\dagger$ is replaced by $\bar{\Psi}$ also in the interaction vertices [3].



$$= i(\not{k} - m)_{i_1 i_m} = (V_3)_{i_1 i_m}.$$

and

$$P_3 = -(V_3)^{-1} = \frac{i}{\not{k} - m}. \tag{4.5}$$

Note that k is the momentum in the direction of the fermion arrow.

4.2 Problem*: Interactions involving scalars

Show that the interaction Lagrangian

$$\mathcal{L}_{\text{INT}} = -\frac{g}{n_1! n_1! n_2!} (\phi^*(x)\phi(x))^{n_1} \Phi(x)^{n_2} \tag{4.6}$$

gives the interaction vertex $V_{\text{INT}} = -ig$.

4.3 Tree-level electroweak interactions between two massless fermions

The part of the Standard Model Lagrangian needed to study electroweak interactions between two massless fermions at the tree-level is as follows

$$\begin{aligned} \tilde{\mathcal{L}}^{\text{SM}} = {} & \mathcal{L}^{\text{QED}}_{\text{INT}} + \mathcal{L}^{\text{EW}}_{\text{INT}} + \mathcal{L}_{\text{YM,A}} + \mathcal{L}^{(2)}_{\text{YM,Z}} + \mathcal{L}^{(2)}_{\text{YM,W}} + \mathcal{L}_{\text{GF,A}} + \mathcal{L}^{(2)}_{\text{GF,Z}} + \mathcal{L}^{(2)}_{\text{GF,W}} \\ & + \sum_f \bar{f}_j (i\not{\partial}) f_j + M_W^2 W^{+\alpha} W^-_\alpha + \frac{M_Z^2}{2} Z^\alpha Z_\alpha. \end{aligned} \tag{4.7}$$

As a matter of notation, the tilde on $\mathcal{L}^{\text{SM}}$ indicates that only the relevant terms are included. Furthermore, the superscript $^{(2)}$ means that only the terms quadratic in the massive gauge boson fields are considered. The last three mass terms in the second line are assumed to be generated by the Higgs mechanism. The interaction terms read

$$\mathcal{L}_{\text{INT}}^{\text{QED}} = -eA_\alpha \sum_f Q_f \bar{f}_j \gamma^\alpha f_j, \qquad e \equiv g s_\theta,$$

$$\mathcal{L}_{\text{INT}}^{\text{EW}} = -\frac{g}{2c_\theta} Z_\alpha \sum_f \bar{f}_j \gamma^\alpha (v_f + a_f \gamma_5) f_j - \frac{g}{2\sqrt{2}} W_\alpha^+ \sum_f \frac{2I_{3f}+1}{2} \bar{f}_j \gamma^\alpha (1-\gamma_5) f'_j$$
$$- \frac{g}{2\sqrt{2}} W_\alpha^- \sum_f \frac{1-2I_{3f}}{2} \bar{f}_j \gamma^\alpha (1-\gamma_5) f'_j. \tag{4.8}$$

The photon and the massive gauge boson fields are denoted by A_α, Z_α and $W_\alpha^\pm$, respectively. The spinor associated with a fermion f with color j is denoted by f_j, with the convention that $j = 1 \div 3$ for quarks and $j = 1$ for leptons. The sum runs over all fermions and f' is the isospin partner of f in the limit of diagonal CKM quark-mixing matrix. The vector and axial couplings are

$$v_f = I_{3f} - 2s_\theta^2 Q_f, \quad a_f = -I_{3f}, \tag{4.9}$$

where I_{3f} is the third isospin component, Q_f the electric charge and s_θ (c_θ) is the sine (cosine) of the weak mixing angle defined by the relation

$$M_Z^2 = \frac{M_W^2}{c_\theta^2}. \tag{4.10}$$

The Yang-Mills parts are

$$\mathcal{L}_{\text{YM,A}} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu),$$
$$\mathcal{L}_{\text{YM,Z}}^{(2)} = -\frac{1}{4}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)(\partial^\mu Z^\nu - \partial^\nu Z^\mu),$$
$$\mathcal{L}_{\text{YM,W}}^{(2)} = -\frac{1}{2}(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)(\partial^\mu W^{-\nu} - \partial^\nu W^{-\mu}). \tag{4.11}$$

Finally, the gauge fixing terms read

$$\mathcal{L}_{\text{GF,A}} = -\frac{1}{2}(\partial^\mu A_\mu)^2,$$
$$\mathcal{L}_{\text{GF,Z}}^{(2)} = -\frac{1}{2}(\partial^\mu Z_\mu)^2,$$
$$\mathcal{L}_{\text{GF,W}}^{(2)} = -(\partial^\mu W_\mu^+)(\partial^\nu W_\nu^-). \tag{4.12}$$

4.4 Problem: The gauge boson propagators

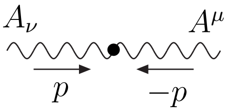
Derive from $\tilde{\mathcal{L}}^{\text{SM}}$ the propagators of the A, W and Z bosons.

Solution

We start with the photon. $\mathcal{L}_{\text{YM,A}}$ is the gauge invariant kinetic term of the A field. It can be rewritten as

$$\mathcal{L}_{\text{YM,A}} = -\frac{1}{2}\left[(\partial_\alpha A_\nu)(\partial^\alpha A^\mu)g^\nu{}_\mu - (\partial_\mu A_\nu)(\partial^\nu A^\mu)\right], \tag{4.13}$$

which gives the 2-point vertex



$$= -i(p^2 g^\nu{}_\mu - p^\nu p_\mu) = (V_{\text{YM,A}})^\nu{}_\mu .$$

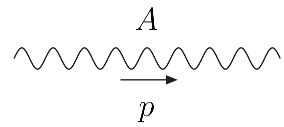
The matrix $(V_{\text{YM,A}})^\nu{}_\mu$ has 0 as an eigenvalue, $p^\mu (V_{\text{YM,A}})^\nu{}_\mu = p_\nu (V_{\text{YM,A}})^\nu{}_\mu = 0$. Hence, it does not have an inverse. [2] This is why one needs to introduce in $\tilde{\mathcal{L}}^{\text{SM}}$ the gauge fixing term $\mathcal{L}_{\text{GF,A}}$, which gives an additional 2-point vertex

$$(V_{\text{GF,A}})^\nu{}_\mu = -ip^\nu p_\mu . \tag{4.14}$$

Adding the two contributions gives the matrix

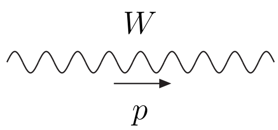
$$(V_{\text{A}})^\nu{}_\mu = (V_{\text{YM,A}})^\nu{}_\mu + (V_{\text{GF,A}})^\nu{}_\mu = -ip^2 g^\nu{}_\mu , \tag{4.15}$$

whose inverse is $ig^{\mu\rho}/p^2$. Thus, the photon propagator reads

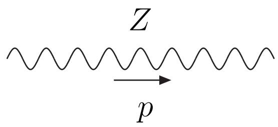


$$\nu = -ig_{\mu\nu}\frac{1}{p^2}.$$

In an analogous way, adding also the mass terms in the second line of (4.7), one derives the W and Z propagators [3]



$$\nu = -ig_{\mu\nu}\frac{1}{p^2 - M_W^2}, \tag{4.16}$$



$$\nu = -ig_{\mu\nu}\frac{1}{p^2 - M_Z^2}. \tag{4.17}$$

[2] Prove explicitly that one cannot find two constants $C_{1,2}$ such that $(V_{\text{YM,A}})^\nu{}_\mu (C_1 g^{\mu\rho} + C_2 p^\mu p^\rho) = g^{\nu\rho}$.

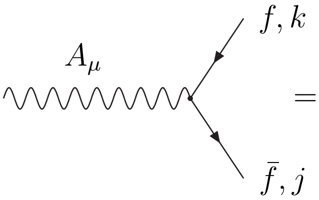
[3] In this case a ghost contribution has to be included as well, that is omitted in (4.7) because it does not contribute at the tree-level.

4.5 Problem*: The fermion-fermion-boson vertices

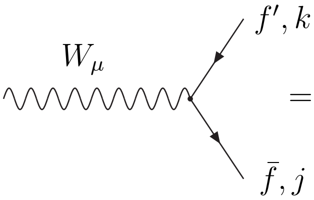
Derive from $\tilde{\mathcal{L}}^{\text{SM}}$ all possible interaction vertices between gauge bosons and fermions.

Solution

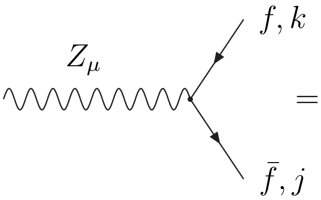
As stated in footnote 1, one can replace $\Psi^\dagger$ by $\bar{\Psi}$ in the vertices. This gives



$$= -igs_\theta Q_f \gamma_\mu \delta_{jk},$$



$$= -ig\frac{1}{2\sqrt{2}}\gamma_\mu(1-\gamma_5)\delta_{jk},$$



$$= \frac{-ig}{2c_\theta}\gamma_\mu(v_f + a_f\gamma_5)\delta_{jk},$$

where j and k are color indices.

4.6 Problem: The QCD Feynman rules

Derive the full set of QCD Feynman rules from the QCD Lagrangian

$$\mathcal{L}^{\rm QCD} = \mathcal{L}^{\rm INV} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm Ghost}, \tag{4.18}$$

where the various pieces read as follows

$$\mathcal{L}^{\rm INV} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \bar{\Psi}_j \big(i \not{D}_{jk} - m \delta_{jk} \big) \Psi_k \tag{4.19}$$

with

$$F^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g c^{abc} G^b_\nu G^c_\mu \qquad \text{and} \qquad D_\mu = \partial_\mu + i g t^a G^a_\mu, \tag{4.20}$$

$$\mathcal{L}_{\rm GF} = -\frac{1}{2} (\partial_\mu G^{\mu a})(\partial_\nu G^{\nu a}), \tag{4.21}$$

$$\mathcal{L}_{\rm Ghost} = -\bar{\eta}^a \partial^2 \eta^a - g c^{abc}\, \bar{\eta}^a \partial_\mu \big(G^{\mu c} \eta^b \big). \tag{4.22}$$

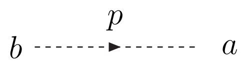
In the previous formulas our conventions on the color indices are as follows

$$a, b, c, d, e = 1, 2, \ldots, 8 \quad \text{and} \quad j, k = 1, 2, 3. \tag{4.23}$$

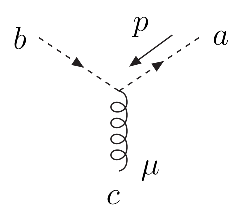
The matrices t^a_{jk} are defined in section 13.2 and $c^{abc} = f^{abc}$ are the SU(3) structure constants.

Solution

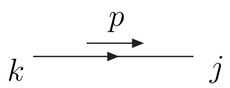
Using the definition of vertices and propagators gives



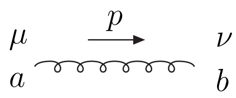
$= i\dfrac{\delta_{ab}}{p^2}$,



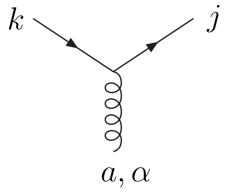
$= g c^{abc} p_\mu$,



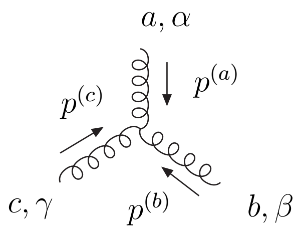
$$= \quad i\frac{1}{\not{p} - m}\delta_{kj} \ ,$$



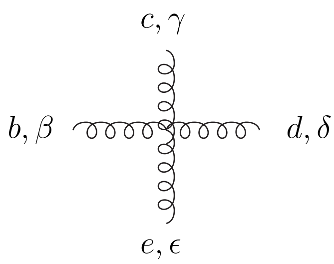
$$= \quad -\, i\frac{g^{\mu\nu}}{p^2}\delta^{ab} \ ,$$



$$= \quad -\, ig\gamma_\alpha t^a_{jk} \ ,$$



$$= \quad gc^{abc}\left\{g_{\alpha\beta}(p^{(b)} - p^{(a)})_\gamma \quad + \quad g_{\beta\gamma}(p^{(c)} - p^{(b)})_\alpha \right.$$
$$\left. + \quad g_{\gamma\alpha}(p^{(a)} - p^{(c)})_\beta \right\} \ ,$$



$$= \quad -\, ig^2\left\{c^{abc}c_{ade}(g_{\beta\delta}g_{\gamma\epsilon} - g_{\beta\epsilon}g_{\gamma\delta}) \right.$$
$$+ c^{abd}c_{aec}(g_{\beta\epsilon}g_{\delta\gamma} - g_{\beta\gamma}g_{\delta\epsilon})$$
$$\left. + c^{abe}c_{acd}(g_{\beta\gamma}g_{\epsilon\delta} - g_{\beta\delta}g_{\epsilon\gamma})\right\} \ .$$

Chapter 5

Conservation laws and symmetries

Whenever it exists a *global* symmetry, namely an invariance under a transformation that does not depend on the space time, it exists a current and a conserved quantity, that can be determined by using the Nöther's theorem. *Local* symmetries, i.e. invariance under transformations that depend on the coordinates, determine, instead, the dynamic of the interactions [4], as we will discuss in chapter 11.
One can look for symmetries by looking at the Lagrangian $\mathcal{L}$ of the Theory at hand. The Lagrangian $\mathcal{L}$ is therefore the fundamental quantity one has to know to study Particle Physics, meaning that any symmetry and conservation law

is completely determined by the form of the Lagrangian.

In this chapter, we demonstrate the Nöther's theorem and propose a few practical problems on this subject.

5.1 The Nöther's theorem

Consider a transformation

$$\mathrm{T}\colon \begin{cases} x \stackrel{\mathrm{T}}{\to} \bar{x} = x + \delta x \\ \Phi_i(x) \stackrel{\mathrm{T}}{\to} \bar{\Phi}_i(\bar{x}) = \Phi_i(x) + \delta\Phi_i \end{cases} \tag{5.1}$$

and a Lagrangian $\mathcal{L}$ invariant in form under T

$$\mathcal{L}(\Phi_i, \partial_\mu \Phi_i) \stackrel{\mathrm{T}}{\to} \mathcal{L}'(\bar{\Phi}_i, \partial_\mu \bar{\Phi}_i) = \mathcal{L}(\bar{\Phi}_i, \partial_\mu \bar{\Phi}_i). \tag{5.2}$$

The Nöther's theorem states that if the action is unchanged before and after applying the transformatio T, namely if

$$\int_{\bar{R}} d^4\bar{x}\,\mathcal{L}(\bar{\Phi}_i, \partial_\mu \bar{\Phi}_i) - \int_R d^4x\,\mathcal{L}(\Phi_i, \partial_\mu \Phi_i) = 0, \tag{5.3}$$

where $R \xrightarrow{\mathrm{T}} \bar{R}$, there is a conserved corrent.

We work at the first order in all δs, hence we can replace in the first term of (5.3)

$$\begin{aligned} d^4\bar{x} &= d^4x\,(1 + \partial_\mu(\delta x^\mu)) \\ \mathcal{L}(\bar{\Phi}_i, \partial_\mu \bar{\Phi}_i) &= \mathcal{L}(\Phi_i, \partial_\mu \Phi_i) + \frac{\partial \mathcal{L}}{\partial \Phi_i}\delta\Phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)}\delta(\partial_\mu \Phi_i), \end{aligned} \tag{5.4}$$

giving

$$\int_R d^4x\,\left\{\mathcal{L}\partial_\mu(\delta x^\mu) + \frac{\partial \mathcal{L}}{\partial \Phi_i}\delta\Phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)}\delta(\partial_\mu \Phi_i)\right\} = 0. \tag{5.5}$$

Next we introduce the variation of Φ_i at a fixed point x

$$\delta_* \Phi_i = \bar{\Phi}_i(x) - \Phi_i(x). \tag{5.6}$$

Therefore

$$\delta\Phi_i = \bar{\Phi}_i(\bar{x}) - \Phi_i(x) = \bar{\Phi}_i(\bar{x}) - \bar{\Phi}_i(x) + \delta_*\Phi_i. \tag{5.7}$$

This gives at the first order

$$\begin{aligned} \delta\Phi_i &= (\partial_\mu \Phi_i)\delta x^\mu + \delta_*\Phi_i \\ \delta(\partial_\nu \Phi_i) &= \partial_\mu(\partial_\nu \Phi_i)\delta x^\mu + \partial_\nu(\delta_*\Phi_i). \end{aligned} \tag{5.8}$$

Putting (5.8) in (5.5) results in

$$\begin{aligned} \int_R d^4x \Big\{ \mathcal{L}\partial_\mu(\delta x^\mu) &+ \frac{\partial \mathcal{L}}{\partial \Phi_i}\Big((\partial_\mu \Phi_i)\delta x^\mu + \delta_*\Phi_i\Big) \\ &+ \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi_i)}\Big(\partial_\mu(\partial_\nu \Phi_i)\delta x^\mu + \partial_\nu(\delta_*\Phi_i)\Big)\Big\} = 0. \end{aligned} \tag{5.9}$$

But

$$\frac{\partial \mathcal{L}}{\partial \Phi_i}\partial_\mu \Phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi_i)}\partial_\mu(\partial_\nu \Phi_i) = \partial_\mu \mathcal{L}, \tag{5.10}$$

and the Lagrange equations applied to the third term in (5.9) give

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} = \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi_i)}. \tag{5.11}$$

This allows one to rewrite (5.9) as follows

$$\int_R d^4x\, \partial_\mu \left\{ \mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \delta_* \Phi_i \right\} = 0, \tag{5.12}$$

and since R is generic, it exists a conserved current

$$\partial_\mu(\delta J^\mu) = 0, \qquad \delta J^\mu = -\mathcal{L}\delta x^\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \delta_* \Phi_i. \tag{5.13}$$

5.2 Exact symmetries

Exact symmetries are fundamental symmetries of the theory at hand, that are never broken at all orders in perturbation theory, such as Charge conservation and Lepton number conservation in Quantum Electrodynamics.

5.3 Problem: Charge conservation

Given the complex Klein-Gordon Lagrangian, describing a self-interacting scalar particle

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi - \frac{\lambda}{4}(\phi^* \phi)^2 \tag{5.14}$$

a) Show that $\mathcal{L}$ is invariant under the global transformation

$$\phi(x) \to e^{i\theta} \phi(x), \qquad \theta \in \mathbb{R} \text{ constant } \forall\, x. \tag{5.15}$$

b) Show that this symmetry gives rise to the following conserved current

$$J^\mu = i\phi^*(\partial^\mu \phi) - i(\partial^\mu \phi^*)\phi \tag{5.16}$$

by explicitly checking that

$$\partial_\mu J^\mu = 0. \tag{5.17}$$

c) Show that the quantity

$$Q_0 \equiv \int d^3x J^0 \tag{5.18}$$

does not depend on the time.

Solution

a) We know that under a phase transformation T, the field ϕ and ϕ^* transform as

$$\phi(x) \xrightarrow{\text{T}} e^{i\theta}\phi(x) \;\Rightarrow\; \phi^*(x) \xrightarrow{\text{T}} e^{-i\theta}\phi(x)^*. \tag{5.19}$$

One then obtains, for the product of the two

$$\phi^*\phi \xrightarrow{\text{T}} \phi^* \underbrace{e^{i\theta}e^{-i\theta}}_{1}\phi = \phi^*\phi.$$

Analogously, given the fact that θ does not depend on x, the derivatives of the fields transform as follows

$$\partial^\mu\phi \xrightarrow{\text{T}} e^{i\theta}\partial^\mu\phi \quad \text{and} \quad \partial_\mu\phi^* \xrightarrow{\text{T}} e^{-i\theta}\partial_\mu\phi^*, \tag{5.20}$$

so that, for the product of two derivatives one has

$$(\partial_\mu\phi^*)(\partial^\mu\phi) \xrightarrow{\text{T}} (\partial_\mu\phi^*)(\partial^\mu\phi).$$

Therefore, all three terms in (5.14) remains unchanged under the transformation of (5.15). Therefore the full $\mathcal{L}$ is invariant.

b) We use the Nöther's theorem, that states that the explicit form of the infinitesimal current δJ^μ is given by

$$\delta J^\mu = -\mathcal{L}\delta x^\mu - \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\partial_*\varphi_i$$

under the following infinitesimal transformation laws

$$x^\mu \to \bar{x^\mu} = x^\mu + \delta x^\mu$$
$$\phi_i(x) \to \bar{\phi_i}(\bar{x}) = \phi_i(x) + \delta\varphi_i$$

where

$$\delta_*\varphi_i \equiv \bar{\phi_i}(x) - \varphi_i(x)$$

is the change in form of the field.

In our case $\delta x^\mu = 0$, because we are dealing with an internal symmetry, and

$$\phi(x) \to \bar{\phi}(\bar{x}) = \bar{\phi}(x) = (1 + i\delta\theta)\varphi(x)$$

$$\Longrightarrow \delta_*\varphi = i\delta\theta\varphi \qquad \text{and} \qquad \delta_*\varphi^* = -i\delta\theta\varphi^*.$$

By using the previous theorem we find

$$\delta J^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_* \varphi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\delta_* \varphi^*,$$

but

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^* \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \phi,$$

so that the infinitesimal current reads

$$\begin{aligned} \delta J^\mu &= -\partial^\mu \phi^*(i\delta\theta\varphi) + \partial^\mu \phi(i\delta\theta\varphi^*) \\ &= \delta\theta\{i(\partial^\mu\phi)\varphi^* - i(\partial^\mu\phi^*)\varphi\} \end{aligned}$$

and, since the only infinitesimal parameter in the previous equation is $\delta\theta$, the finite conserved current is

$$J^\mu = i\phi^*(\partial^\mu\phi) - i\phi(\partial^\mu\phi^*).$$

We now explicitly check that J^μ is a conserved current, namely $\partial_\mu J^\mu = 0$. Consider

$$\partial_\mu J^\mu = i\{(\partial_\mu\phi^*)(\partial^\mu\phi) + \phi^*\partial^2\phi - (\partial_\mu\phi)(\partial^\mu\phi^*) - \phi\partial^2\phi^*\} = i\{\phi^*\partial^2\phi - \phi\partial^2\phi^*\}.$$

To show that this is zero one must use the equations of the motion

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \\[2ex] \dfrac{\partial \mathcal{L}}{\partial \varphi^*} - \partial_\mu \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = 0 \end{cases} \Rightarrow$$

$$\begin{cases} -m^2\phi^* - \dfrac{\lambda}{2}(\phi^*)^2\phi - \partial_\mu\partial^\mu\phi^* = 0 \\[2ex] -m^2\varphi - \dfrac{\lambda}{2}(\phi)^2\phi^* - \partial_\mu\partial^\mu\varphi = 0 \end{cases}$$

Inserting this equation in $\partial_\mu J^\mu$ gives the desired result

$$\partial_\mu J^\mu = i\left\{\phi^*\left[-m^2\varphi - \frac{\lambda}{2}(\phi)^2\varphi^*\right] - \phi\left[-m^2\phi^* - \frac{\lambda}{2}(\phi^*)^2\phi\right]\right\} = 0.$$

c) From the previous equation, one obtains the desired result by applying the Gauss theorem

$$\partial_\mu J^\mu = 0 \quad \Rightarrow \quad \partial_0 J^0 + \bar\nabla\cdot\bar J = 0 \quad \Rightarrow \quad \textstyle\int_V \partial_0 J^0 d^3x = -\int_V d^3x \bar\nabla\cdot\bar J \quad \Rightarrow$$

$$\partial_0 \textstyle\int_V J^0 d^3x = -\int_\Sigma \bar J\cdot\bar n\, d\Sigma \xrightarrow{\Sigma\to\infty} 0$$

where Σ is the surface of the volume V.

5.4 Problem: Charge and Lepton number conservation in QED

Show that the QED interactions

$$\begin{aligned}\mathcal{L}_{\text{INT}}^{\text{QED}} = &\bar{\Psi}_e(i\not{\partial} - m_e)\Psi_e + \bar{\Psi}_\mu(i\not{\partial} - m_\mu)\Psi_\mu + \bar{\Psi}_\tau(i\not{\partial} - m_\tau)\Psi_\tau \\ &-eA_\alpha\bar{\Psi}_e\gamma^\alpha\Psi_e - eA_\alpha\bar{\Psi}_\mu\gamma^\alpha\Psi_\mu - eA_\alpha\bar{\Psi}_\tau\gamma^\alpha\Psi_\tau\end{aligned} \tag{5.21}$$

conserve charge and the lepton numbers.

Solution

- The charge is conserved because $\mathcal{L}_{\text{INT}}^{\text{QED}}$ is invariant under the global $U_c(1)$ transformation:

$$\Psi_j \longrightarrow e^{i\theta_c}\Psi_j \qquad j = e, \mu, \tau \qquad \theta_c \in \mathbb{R}.$$

- In addition, each family is separately invariant under another global $U_{L_j}(1)$ transformation, with $j = e, \mu, \tau$.

$$\Psi_j \longrightarrow e^{i\theta_{L_j}}\Psi_j.$$

  Therefore there exist three conserved quantities L_j, that can be identified with the three lepton numbers.

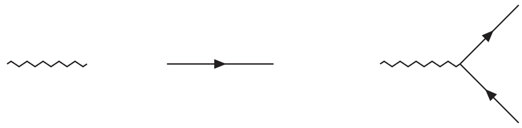
5.5 Problem: Conservation laws and Feynman Diagrams

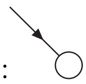
By using the QED Feynman rules, show, diagrammatically, that the charge is conserved by the Lagrangian

$$\mathcal{L} = \bar{\Psi}_e(i\not{\partial} - m)\Psi_e - eA_\mu\bar{\Psi}_e\gamma^\mu\Psi_e - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Solution

The Vertices and Propagators of the Theory, namely the Feynman rules, are as follows



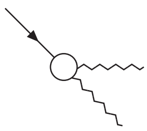
Start with an incoming electron: 

Whatever happens in the blob, the above Feynman rules tells us that the arrow must exit either in the initial or in the final state. Namely one of the following 2 situation should be verified

Arrow exiting in initial state: Arrow exiting in final state:

In both cases $\Delta Q_e \equiv Q_{final} - Q_{initial} = 0$, namely the charge must be conserved.

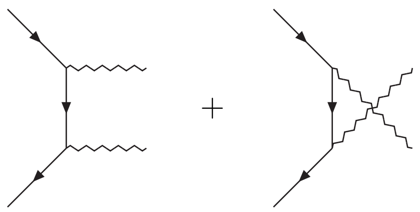
An explicit example of blob is



Then the, initial state electron must either exit in the initial or final state as follows

(a) (b)

The 2 Feynman diagrams contributing, at the tree-level, to case (a) are

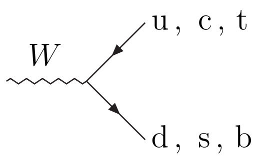


Note, however, that our reasoning based of Feynman diagrams hold at all orders (namely at all loops). For this reason, some authors think that

there is more truth in Feynman diagrams than in Quantum Field Theory.

5.6 Approximated symmetries

Approximated symmetries are symmetries that are broken by weaker interactions like for example "strangeness" in QCD, broken by the weak interactions through the vertex

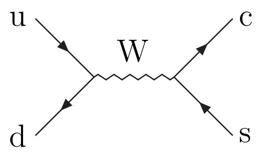


5.7 Problem: a process with $\Delta s = 1$

Write, at the tree-level, a process with variation of strangeness $\Delta s = 1$, by assuming a diagonal CKM matrix.

Solution

A possible process is

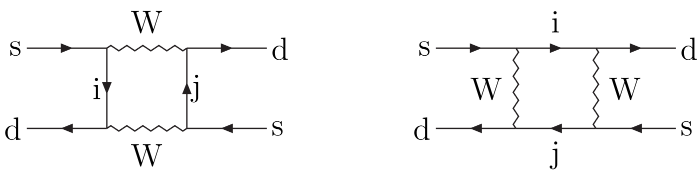


5.8 Problem: a process with $\Delta s = 2$

By assuming a non diagonal CKM matrix, write the diagrams contributing to the process with $\Delta s = 2$ $s\bar{d} \to d\bar{s}$.

Solution

There are two contributing Feynman diagrams



where $i, j = u, c, t$.

As a last remark, note that, once again, is the form of the $\mathcal{L}$, namely the graphical Feynman rules, that determine everything.

5.9 Problem*: Coupled electrons and muons

Given a theory described by a Lagrangian containing electronic (Ψ_e) and muonic (Ψ_μ) fields coupled as follows

$$\mathcal{L} = \bar{\Psi}_e(i\partial\!\!\!/ - m_e)\Psi_e + \bar{\Psi}_\mu(i\partial\!\!\!/ - m_\mu)\Psi_\mu - eA_\alpha\bar{\Psi}_e\gamma^\alpha\Psi_\mu - eA_\alpha\bar{\Psi}_\mu\gamma^\alpha\Psi_e$$

- Is the charge conserved is such a theory?
- Are the lepton numbers L_μ and L_e conserved?

Chapter 6

Green's functions and S matrix

In this chapter we define Green's functions in terms of Feynman rules. The scattering matrix S is also introduced and its connection with the Green's functions discussed.

6.1 Green's functions

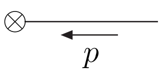
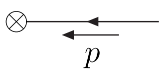
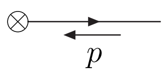
For the sake of definiteness, we consider a Lagrangian containing neutral and charged fields, which may also carry additional indices $1 \leq a \leq N_a$ and $1 \leq b \leq N_b$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi_a)(\partial^\mu \Phi_a) - \frac{M^2}{2}\Phi_a\Phi_a + (\partial_\mu \phi_b^*)(\partial^\mu \phi_b) - m^2 \phi_b^* \phi_b + \mathcal{L}_{\text{INT}}(\Phi, \phi, \phi^*). \tag{6.1}$$

The interaction Lagrangian $\mathcal{L}_{\text{INT}}$ is assumed to be polynomial. [1] To define the Green's functions we introduce a source for each field

$$\mathcal{L} \to \mathcal{L} - K_a(x)\Phi_a(x) - J_b^*(x)\phi_b(x) - J_b(x)\phi_b^*(x). \tag{6.2}$$

This generates the following extra Feynman rules

 $= -iK_a(p)$,  $= -iJ_b^*(p)$,  $= -iJ_b(p)$,

where sources are denoted by the symbol $\otimes$, and $K_a(p)$, $J_b^*(p)$ and $J_b(p)$ are the Fourier transforms of $K_a(x)$, $J_b^*(x)$ and $J_b(x)$, respectively.

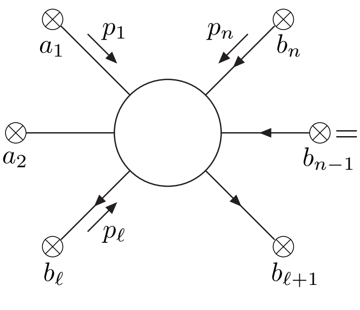
[1]An explicit example with $N_a = N_b = 1$ is given in (4.6).

Diagrams are constructed by connecting vertices and sources by means of propagators. In addition

- there is an integral $\int \frac{d^4 q_\ell}{(2\pi)^4}$ over the unbounded four-momentum q_ℓ in each loop ℓ of the diagram;
- there is a minus sign for each closed fermion loop;
- diagrams related by the exchange of two fermion lines have a relative minus sign;
- energy-momentum conservation is assumed at each interaction vertex;
- any diagram is provided with a combinatorial factor.

As for the latter rule, the combinatorial factor is always 1 for tree-level diagrams. In the one-loop case one has to multiply by 1/2 diagrams in which a particle starts and ends at the same vertex. Diagrams where two *identical* particles connect two vertices need to be multiplied by 1/2 as well. For two loops and more see e.g. [4].

The sum of all possible diagrams connecting n sources is of the form



$$= i^n K_{a_1}(-p_1)..J^*_{b_\ell}(-p_\ell)..J_{b_n}(-p_n) G_{a_1 \ldots b_n}(p_1, .., p_\ell, .., p_n), \tag{6.3}$$

where all momenta flow from the sources into the diagrams. The function $G_{a_1 \ldots b_n}$ is the n-point connected Green's function for the given configuration of the external lines.

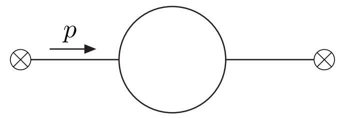
Green's functions which cannot be separated into two pieces by cutting an *internal* propagator are dubbed *one-particle irreducible* (1PI).

6.2 Problem: Perturbative Green's functions I

Assuming the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi(x)\partial^\mu \Phi(x) - \frac{M^2}{2}\Phi^2(x) - \frac{g}{4!}\Phi^4(x) - K(x)\Phi(x), \tag{6.4}$$

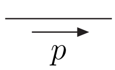
write down, up to the second perturbative order in g, the connected two-point Green's function for the following configuration



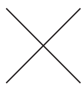
.

Solution

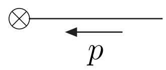
The Feynman rules are as follows



$= \frac{i}{p^2 - M^2}$,

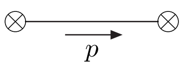


$= -ig$,



$= -iK(p)$.

At the lowest order one has



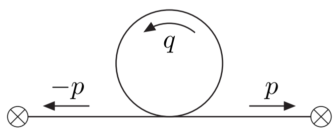
$$= K(p)K(-p)\frac{-i}{p^2 - M^2} = i^2 K(p)K(-p)\frac{i}{p^2 - M^2},$$

so that the perturbative expansion of the Green's function at the 0^{th} order in g reads

$$G^{(0)}(p, -p) = \frac{i}{p^2 - M^2}, \tag{6.5}$$

which coincides with the propagator.

The only diagram contributing at the next perturbative order is



$$= i^2 K(p)K(-p)\frac{-g}{2(p^2 - M^2)^2}\int_R \frac{d^4q}{(2\pi)^4}\frac{1}{q^2 - M^2 + i\epsilon},$$

which gives

$$G^{(1)}(p,-p) = -\frac{g}{2(p^2-M^2)^2}\int_R \frac{d^4q}{(2\pi)^4}\frac{1}{q^2-M^2+i\epsilon}. \tag{6.6}$$

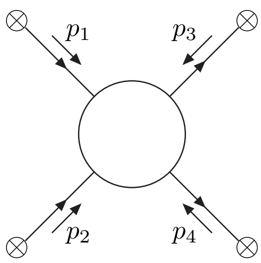
A few remarks are in order. The 1/2 is a combinatorial factor. The notation $\int_R$ means that a UV regulator must be used to compute the integral. Finally, the $+i\epsilon$ prescription defines the causal Feynman propagator.

6.3 Problem: Perturbative Green's functions II

Assuming the Lagrangian

$$\mathcal{L} = \partial_\mu\phi^*(x)\partial^\mu\phi(x) - m^2\phi^*(x)\phi(x) - \frac{g}{2!2!}(\phi^*(x)\phi(x))^2 - J^*(x)\phi(x) - J(x)\phi^*(x), \tag{6.7}$$

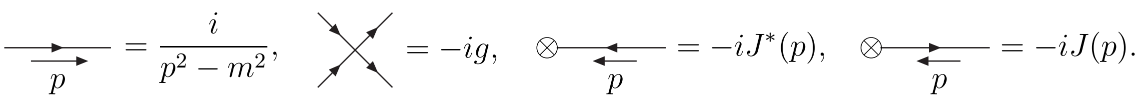
write down, up to the second perturbative order in g, the connected 1PI four-point Green's function for the following configuration



.

Solution

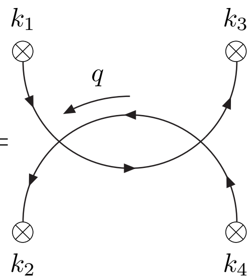
The Feynman rules one reads from (6.7) are

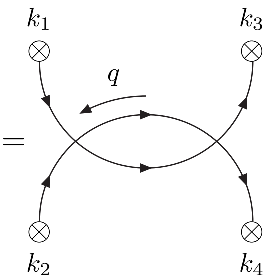


At the first order in g, one links the four-particle vertex directly to the sources. This gives

$$G^{(0)}(p_1, p_2, p_3, p_4) = -ig \frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)(p_3^2 - m^2)(p_4^2 - m^2)}. \tag{6.8}$$

To obtain the 1PI $G^{(2)}(p_1, p_2, p_3, p_4)$, we first write down all possible four-point 1PI one-loop diagrams. They are

$D_1(k_1, k_2, k_3, k_4) =$ 

, $D_2(k_1, k_2, k_3, k_4) =$ 

,

where the momenta k_j flow out from the sources. One computes

$$D_1(k_1, k_2, k_3, k_4) = i^4 J(-k_1) J^*(-k_2) J^*(-k_3) J(-k_4) \frac{g^2 F(k_1, k_2)}{\prod_j (k_j^2 - m^2)},$$
$$D_2(k_1, k_2, k_3, k_4) = i^4 J(-k_1) J(-k_2) J^*(-k_3) J^*(-k_4) \frac{g^2 F(k_1, k_2)}{\prod_j (k_j^2 - m^2)} \frac{1}{2},$$
$$F(k_1, k_2) = \int_R \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2 - m^2 + i\epsilon)\big((q + k_1 + k_2)^2 - m^2 + i\epsilon\big)}, \tag{6.9}$$

where the 1/2 in D_2 is a combinatorial factor. Therefore

$$G^{(2)}(p_1, p_2, p_3, p_4) = \frac{g^2}{\prod_j (p_j^2 - m^2)} \left(\frac{1}{2} F(p_1, p_2) + F(p_1, p_3) + F(p_1, p_4) \right). \tag{6.10}$$

6.4 The S matrix

We work in the interaction picture, where operators evolve according to the free theory and the evolution of the states is dictated by the interaction.

Consider an initial-state particle configuration

$$|\Phi_i> \equiv |\Phi_i(t = -\infty)> . \tag{6.11}$$

We assume that the interaction affects the states during a finite amount of time, so that $|\Phi_i>$, being evaluated at $t = -\infty$, is an *asymptotically free state* made of non-interacting particles. The time evolution of $|\Phi_i>$ to a state $|\Psi(t)>$ is controlled by the time-evolution operator U

$$|\Psi(t)> = \mathrm{U}(-\infty, t)|\Phi_i> . \tag{6.12}$$

The S matrix elements are defined as the $t \to \infty$ limit of projections of $|\Psi(t)>$ on non-interacting, asymptotically free final states

$$|\Phi_f> \equiv |\Phi_f(t = +\infty)> . \tag{6.13}$$

Thus

$$\mathrm{S}_{fi} \equiv \lim_{t\to+\infty} <\Phi_f|\Psi(t)> = <\Phi_f|\mathrm{U}(-\infty, \infty)|\Phi_i>, \tag{6.14}$$

which gives

$$\mathrm{S} = \mathrm{U}(-\infty, \infty). \tag{6.15}$$

Therefore, the probability of an asymptotically free initial state $|\Phi_i>$ to evolve to an asymptotically free final states $|\Phi_f>$ is

$$\mathrm{P}_{fi} = |\mathrm{S}_{fi}|^2 = | <\Phi_f|\mathrm{S}|\Phi_i> |^2. \tag{6.16}$$

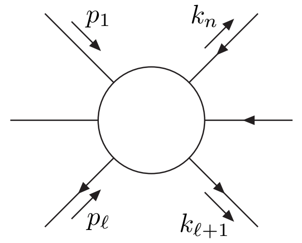
The S matrix elements are obtained from the connected Green's functions in two steps

- The momenta of the external lines are put on-shell;
- The sources are normalized in such a way that they emit or absorb one particle.

For example, if in (6.3) particles $1 \div \ell$ are incoming and particles $(\ell+1) \div n$ outgoing, one has

$$\begin{aligned} &< b_{\ell+1}k_{\ell+1},.., b_n k_n|S|a_1 p_1,.., b_\ell p_\ell > = \lim_{p_1^2=M^2} K_{a_1}^N(-p_1)(p_1^2 - M^2)\dots \\ &\dots \lim_{p_\ell^2=m^2} J_{b_\ell}^{N*}(-p_\ell)(p_\ell^2 - m^2) \lim_{k_{\ell+1}^2=m^2} J_{b_{\ell+1}}^{N*}(k_{\ell+1})(k_{\ell+1}^2 - m^2)\cdots \\ &\dots \lim_{k_n^2=m^2} J_{b_n}^N(k_n)(k_n^2 - m^2) G_{a_1\dots b_n}(p_1, ..p_\ell, -k_{\ell+1}, .., -k_n), \end{aligned} \tag{6.17}$$

with $k_j = -p_j$ for $j = (\ell + 1) \div n$, so that the momenta of the outgoing particles are taken to be flowing out. Diagrams contributing to the S matrix are denoted without drawing external sources, putting incoming particles to the left and outgoing particles to the right. For instance, in the case at hand,



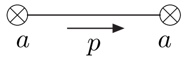
Equation (6.17) is called the LSZ reduction formula. The propagator of each external line is amputated by multiplying by its inverse, and the normalized sources K^N, J^N and J^{N*} are defined below.

In the general case of the transition from N_i initial state particles with momenta and indices $\{p_i, a_i\}$, to N_f final state particles $\{k_f, b_f\}$ one has

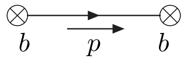
$$
\begin{aligned}
< \{k_f, b_f\}|S|\{p_i, a_i\} > \;=\; & \prod_{f=1}^{N_f} \lim_{k_f = m_f^2} (k_f^2 - m_f^2) S_{b_f}^N(k_f) \prod_{i=1}^{N_i} \lim_{p_i = m_i^2} (p_i^2 - m_i^2) S_{a_i}^N(-p_i) \\
& \times G_{\{b_f\}\{a_i\}}(\{-k_f\}, \{p_i\}), \qquad (6.18)
\end{aligned}
$$

where the normalizes sources S^N correspond to any type of field and the energy flows from left to right, namely $p_i^0 > 0$ and $k_f^0 > 0$.

As for the correct normalization of the sources, it is obtained by considering diagrams connecting two sources. The tree-level two-point contributions corresponding to the interchange of real and complex fields are



$$= \frac{i}{p^2 - M^2} K_a^N(p) K_a^N(-p)\, i^2 \qquad \forall\, a = 1 \div N_a,$$



$$= \frac{i}{p^2 - m^2} J_b^{N*}(p) J_b^N(-p)\, i^2 \qquad \forall\, b = 1 \div N_b.$$

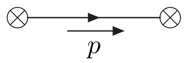
They represent the probability density of emission and absorption of one particle if

$$K_a^N(p) K_a^N(-p) = J_b^{N*}(p) J_b^N(-p) = 1 \qquad (a \text{ and } b \textit{ not} \text{ summed}), \qquad (6.19)$$

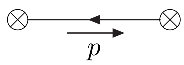
which gives

$$K_a^N(p) = K_a^N(-p) = J_b^{N*}(p) = J_b^N(-p) = 1 \quad \forall\, a, b. \tag{6.20}$$

In the case of fermions f and antifermions $\bar{f}$ in a state of spin s one computes, at the tree-level,



$$= \frac{i}{p^2 - m_f^2}\, \bar{\Psi}_f^N(p,s)(\not{p} + m_f)\Psi_f^N(-p,s)\, i^2,$$



$$= \frac{i}{p^2 - m_f^2}\, \bar{\Psi}_{\bar{f}}^N(-p,s)(-\not{p} + m_f)\Psi_{\bar{f}}^N(p,s)\, i^2.$$

Hence, the normalization conditions are

$$\bar{\Psi}_f^N(p,s)(\not{p} + m_f)\Psi_f^N(-p,s) = \bar{\Psi}_{\bar{f}}^N(-p,s)(-\not{p} + m_f)\Psi_{\bar{f}}^N(p,s) = 1. \tag{6.21}$$

In terms of the solutions of the Dirac equation

$$\begin{aligned}
&(\not{p} - m_f)u^s(p) = 0, && (\not{p} + m_f)v^s(p) = 0,\\
&\bar{u}^s(p)u^r(p) = 2m_f\delta^{rs}, && \bar{v}^s(p)v^r(p) = -2m_f\delta^{rs},\\
&\textstyle\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m_f, && \textstyle\sum_s v^s(p)\bar{v}^s(p) = \not{p} - m_f,
\end{aligned} \tag{6.22}$$

one finds that (6.21) is fulfilled by taking

$$\bar{\Psi}_f^N(p,s) = \frac{1}{2m_f}\bar{u}^s(p), \quad \Psi_f^N(-p,s) = \frac{1}{2m_f}u^s(p), \tag{6.23}$$

and [2]

$$\bar{\Psi}_{\bar{f}}^N(-p,s) = -\frac{1}{2m_f}\bar{v}^s(p), \quad \Psi_{\bar{f}}^N(p,s) = \frac{1}{2m_f}v^s(p). \tag{6.24}$$

For (real or complex) vector fields with mass M_V, properly normalized tree-level sources are

$$\epsilon^\mu(p,s) \quad \text{with} \quad s = 1,2,3, \quad p_\mu\epsilon^\mu(p,s) = 0, \quad \text{and} \quad \epsilon^{*\mu}(p,r)\epsilon_\mu(p,s) = -\delta_{rs}. \tag{6.25}$$

[2]The minus sign associated to the incoming anti-particle in the first of (6.24) is a phase common to all diagrams contributing to a given process. Nevertheless, it is deeply connected to the minus sign to be given to fermion loops, and is relevant in the proof of the unitarity of the S matrix [3].

They obey the completeness relation

$$\sum_{s=1}^{3} \epsilon^{*\mu}(p,s)\epsilon^{\nu}(p,s) = -g^{\mu\nu} + \frac{p^\mu p^\nu}{M_V^2}. \tag{6.26}$$

Massless vectors only have 2 spin components. Their sources are as in (6.25), but with $s = 1, 2$. Furthermore, the completeness relation reads

$$\sum_{s=1}^{2} \epsilon^{*\mu}(p,s)\epsilon^{\nu}(p,s) = -g^{\mu\nu} + \frac{p^\mu \bar{p}^\nu + \bar{p}^\mu p^\nu}{p \cdot \bar{p}}, \tag{6.27}$$

where $p^\mu = (E, \vec{p})$ and $\bar{p}^\mu = (E, -\vec{p})$.

In summary, the S matrix is constructed from the Green's functions by amputating the external propagators and by multiplying by properly normalized sources. The sources to be used at the tree-level in the case of scalars, spinors and vectors are listed in Equations (6.20), (6.22), (6.25)-(6.27).

6.5 Problem: Unitarity of the S matrix

Show that requiring the sum of the transition probabilities from $|\Phi_i>$ to any possible $|\Phi_f>$ to be 1 implies that the S matrix is unitary.

Solution

Equation (6.16) gives

$$1 = \sum_f \mathrm{P}_{fi} = \sum_f <\Phi_i|\mathrm{S}^\dagger|\Phi_f><\Phi_f|\mathrm{S}|\Phi_i> = <\Phi_i|\mathrm{S}^\dagger\mathrm{S}|\Phi_i>, \tag{6.28}$$

where we have used the completeness of the final states $|\Phi_f>$. Therefore $\mathrm{S}^\dagger\mathrm{S} = 1$ if the state $|\Phi_i>$ is normalized to 1.

6.6 Problem: Wave function renormalization

Take a scalar particle and suppose that higher order corrections modify its propagator as follows

$$\frac{1}{p^2 - M^2} \to \frac{1}{Z^2}\frac{1}{p^2 - M^2}. \tag{6.29}$$

Derive the properly normalized sources to be used in the LSZ reduction formula.

Solution

The new normalization condition is

$$\frac{1}{Z^2} K^N(p) K^N(-p) = 1, \tag{6.30}$$

which gives

$$K^N(p) = K^N(-p) = Z. \tag{6.31}$$

Chapter 7

Green's functions and path integrals

According to the Feynman's path-integral formulation, n-point Green's functions in the position-space can be defined as products of n fields averaged over all possible field configurations weighted by the exponential of i times the action. Here we illustrate the perturbative approach to Quantum Field Theory from this point of view, taking the Lagrangian in (6.4) as a case study.

7.1 The path-integral definition of the Green's functions

Imagine a discretized world made of only N space-time points x_a^μ, with $a = 1, \ldots, N$. The action corresponding to (6.4) is now

$$S = S_0 + S_{\text{INT}}, \tag{7.1}$$

with

$$S_0 = \frac{1}{2}\sum_{a,b=1}^{N} \Phi(x_a) W_{ab} \Phi(x_b) - \sum_{a=1}^{N} K(x_a)\Phi(x_a), \quad S_{\text{INT}} = -\frac{g}{4!}\sum_{a=1}^{N} \Phi^4(x_a), \tag{7.2}$$

and where $W_{ab} = W_{ba}$ is the discretized variant of the quadratic part of the Lagrangian, where derivatives are replaced by differences. Green's functions involving

n space-time points are defined as follows

$$G(x_1, \ldots, x_n) = \left. \frac{\int \mathcal{D}\Phi \, \Phi(x_1) \ldots \Phi(x_n) \, e^{iS}}{\int \mathcal{D}\Phi \, e^{iS}} \right|_{K=0}, \tag{7.3}$$

where we use the notation $\mathcal{D}\Phi \equiv \prod_{i=1}^{N} d\Phi(x_i)$. [1]

Dubbing $\tilde{G}(p_1, \ldots, p_n)$ the Fourier transform of (7.3)

$$\tilde{G}(p_1, \ldots, p_n) = \int \left\{ \prod_{j=1}^{n} d^4 x_j e^{-ix_j \cdot p_j} \right\} G(x_1, \ldots, x_n), \tag{7.4}$$

the Green's functions $G(p_1, \ldots, p_n)$ normalized as in (6.3) are given by

$$\tilde{G}(p_1, \ldots, p_n) = G(p_1, \ldots, p_n) \, (2\pi)^4 \delta^4 \left(\sum_{i=1}^{n} p_i \right). \tag{7.5}$$

7.2 Free fields

First we solve the free case, in which $S_{\text{INT}} = 0$. For ease of notation, we define $\Phi_a \equiv \Phi(x_a)$, $K_a \equiv K(x_a)$ and understand summation over repeated indices. That yields

$$S_0 = \frac{1}{2} \Phi_a W_{ab} \Phi_b - K_a \Phi_a. \tag{7.6}$$

The product of fields in (7.3) can be replaced by derivatives over sources $\partial_j \equiv \frac{\partial}{\partial K_j}$, giving

$$G_0(x_1, \ldots, x_n) = \frac{\left(\prod_{j=1}^{n} i\partial_j \right) \int \mathcal{D}\Phi \, e^{iS_0} \Big|_{K=0}}{\int \mathcal{D}\Phi \, e^{iS_0} \Big|_{K=0}}. \tag{7.7}$$

Thus, one needs to single out the source dependence of

$$\mathcal{Z}_0(K) \equiv \int \mathcal{D}\Phi \, e^{iS_0}. \tag{7.8}$$

[1] The continuum limits of the numerator and denominator in (7.3) (if they exist) are examples of path integrals.

This is achieved by introducing the inverse of W_{ab}, namely a Δ_{ab} such that

$$\Delta_{ab}W_{bc} = W_{ab}\Delta_{bc} = \delta_{ac}. \tag{7.9}$$

Changing variables in (7.8)

$$\Phi_a = \Phi'_a + \Delta_{ab}K_b, \quad \mathcal{D}\Phi = \mathcal{D}\Phi', \tag{7.10}$$

gives

$$\mathcal{Z}_0(K) = \mathcal{Z}_0(0)\exp\left\{-\frac{i}{2}K_a\Delta_{ab}K_b\right\}, \tag{7.11}$$

so that

$$G_0(x_1,\ldots,x_n) = \left(\prod_{j=1}^{n} i\partial_j\right)\exp\left\{-\frac{i}{2}K_a\Delta_{ab}K_b\right\}\Bigg|_{K=0}. \tag{7.12}$$

Equation (7.12) allows one to express any possible Green's functions of the free theory in terms of the Δ_{ab}.

7.3 The free propagator

Using $\partial_j K_\ell = \delta_{j\ell}$ in (7.12) one easily derives the free 2-point Green's function

$$G_0(x_1,x_2) = i\Delta_{12}. \tag{7.13}$$

Now we go back to the physical continuum space-time. In this case

$$\begin{aligned} W_{ab} &\to W = -\partial^2 - M^2, \\ \Delta_{12} &\to \Delta(x-y), \end{aligned} \tag{7.14}$$

where $\Delta(x-y)$ is the inverse of W, as in (7.9)

$$-(\partial^2 + M^2)\Delta(x-y) = \delta^4(x-y). \tag{7.15}$$

To solve (7.15), we introduce the Fourier transform $\tilde{\Delta}(p^2)$ of $\Delta(x-y)$

$$\Delta(x-y) = \frac{1}{(2\pi)^4}\int d^4p\,\tilde{\Delta}(p^2)e^{-ip\cdot(x-y)}. \tag{7.16}$$

Inserting this in (7.15) gives

$$\tilde{\Delta}(p^2) = \frac{1}{p^2 - M^2}, \tag{7.17}$$

so that the Fourier transform of $i\Delta(x-y)$ is nothing but the propagator in the momentum-space. To fix the prescription to go around the two poles $p_0 = \pm\sqrt{\vec{p}^2 + M^2}$ in the p_0 integral of (7.16), we go back to (7.7) and replace

$$iS_0 \to iS_0 - \frac{1}{2}\epsilon\,\Phi_a\Phi_a, \quad \text{with} \quad \epsilon \to 0^+. \tag{7.18}$$

This change makes the integrals in (7.7) well defined also when $\Phi_a \to \infty$, and can be achieved by changing $M^2 \to M^2 - i\epsilon$. In summary, the causal Feynman propagator reads

$$i\Delta(x-y) = \frac{1}{(2\pi)^4}\int d^4p\,\frac{i}{p^2 - M^2 + i\epsilon}e^{-ip\cdot(x-y)}. \tag{7.19}$$

Very often, the change $M^2 \to M^2 - i\epsilon$ is understood.

Finally, we compute the Fourier transform of (7.13):

$$\begin{aligned}
\tilde{G}_0(p_1,p_2) &= \int d^4x_1 d^4x_2 e^{-ip_1\cdot x_1}e^{-ip_2\cdot x_2}\,G_0(x_1,x_2)\\
&= \frac{1}{(2\pi)^4}\int d^4x_1 d^4x_2 d^4p\, e^{-ip_1\cdot x_1}e^{-ip_2\cdot x_2}e^{-ip\cdot x_1}e^{ip\cdot x_2}\frac{i}{p^2 - M^2 + i\epsilon}\\
&= \frac{1}{(2\pi)^4}\int d^4p\,\frac{i}{p^2 - M^2 + i\epsilon}\int d^4x_1 e^{-i(p_1+p)\cdot x_1}\int d^4x_2 e^{-i(p_2-p)\cdot x_2}\\
&= (2\pi)^4\int d^4p\,\frac{i}{p^2 - M^2}\,\delta^4(p_1+p)\delta^4(p_2-p)\\
&= (2\pi)^4\frac{i}{p_1^2 - M^2}\delta^4(p_1+p_2).
\end{aligned} \tag{7.20}$$

Thus,

$$G_0(p_1,-p_1) = \frac{i}{p_1^2 - M^2 + i\epsilon}, \tag{7.21}$$

in agreement with (6.5).

7.4 Interacting fields

We rewrite in (7.3)

$$e^{iS} = e^{iS_0} \exp\left\{-i\frac{g}{4!}\sum_v \Phi_v^4\right\}, \tag{7.22}$$

and replace fields by derivatives over sources. This gives

$$\begin{aligned} G(x_1,\dots,x_n) &= \frac{\left(\prod_{j=1}^n i\partial_j\right)\exp\left\{-i\frac{g}{4!}\sum_v (i\partial_v)^4\right\}\mathcal{Z}_0(K)\Big|_{K=0}}{\exp\left\{-i\frac{g}{4!}\sum_v (i\partial_v)^4\right\}\mathcal{Z}_0(K)\Big|_{K=0}} \\ &= \frac{\left(\prod_{j=1}^n i\partial_j\right)\exp\left\{-i\frac{g}{4!}\sum_v (i\partial_v)^4\right\}\exp\left\{-\frac{i}{2}K_a\Delta_{ab}K_b\right\}\Big|_{K=0}}{\exp\left\{-i\frac{g}{4!}\sum_v (i\partial_v)^4\right\}\exp\left\{-\frac{i}{2}K_a\Delta_{ab}K_b\right\}\Big|_{K=0}}. \end{aligned} \tag{7.23}$$

Expanding (7.23) in powers of the coupling constant g generates all the perturbative Green's functions of the interacting theory.

The generalization of the described technique to Lagrangians depending on many fields with arbitrary polynomial interactions is straightforward.

7.5 Problem: Perturbative Green's functions III

Use the path integral approach to rederive the two-point Green's function in (6.6).

Solution

We rewrite $G(x_1, x_2) = N(x_1, x_2)/D$, where $N(x_1, x_2)$ is the numerator of (7.23) with $n = 2$. Expanding $N(x_1, x_2)$ to the first order in g produces six derivatives. Therefore, a result different from zero is generated only by the fourth term in the expansion of $\exp\left\{-\frac{i}{2}K_a\Delta_{ab}K_b\right\}$

$$\begin{aligned} N(x_1,x_2) &= (i\partial_2)(i\partial_1)\left(-i\frac{g}{4!}\sum_v (i\partial_v)^4\right)\frac{1}{3!}\left(\frac{-i}{2}\right)^3 \Delta_{a_1a_2}\Delta_{a_3a_4}\Delta_{a_5a_6}\prod_{k=1}^6 K_{a_k} \\ &= -\frac{g}{1152}\sum_v \partial_v^4\partial_2\partial_1\prod_{k=1}^6 K_{a_k}\,\Delta_{a_1a_2}\Delta_{a_3a_4}\Delta_{a_5a_6}. \end{aligned} \tag{7.24}$$

Acting with the derivatives on the sources produces 6! terms. However, due to the summations over indices, only two possible contributions may arise, proportional to $\Delta_{1v}\Delta_{v2}\Delta_{vv}$ or $\Delta_{12}\Delta_{vv}\Delta_{vv}$, respectively. Thus

$$N(x_1,x_2) = -\frac{g}{1152}\sum_v \left(N_1\Delta_{1v}\Delta_{v2}\Delta_{vv} + N_2\Delta_{12}\Delta_{vv}\Delta_{vv}\right), \tag{7.25}$$

with $N_1+N_2=6!$. To determine $N_{1,2}$, we first compute

$$\begin{aligned}\partial_2\partial_1\prod_{k=1}^{6}K_{a_k}\,\Delta_{a_1a_2}\Delta_{a_3a_4}\Delta_{a_5a_6} &= 6\,\partial_2\prod_{k=1}^{5}K_{a_k}\,\Delta_{1a_1}\Delta_{a_2a_3}\Delta_{a_4a_5}\\ &= 6\prod_{k=1}^{4}K_{a_k}\left(\Delta_{12}\Delta_{a_1a_2}\Delta_{a_3a_4}+4\Delta_{1a_1}\Delta_{a_22}\Delta_{a_3a_4}\right).\end{aligned} \tag{7.26}$$

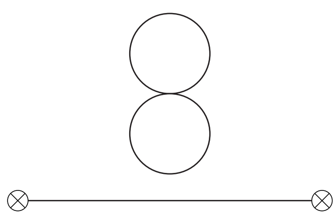
Acting now with ∂_v^4 on the r.h.s. of (7.26) generates 4! contributions from each of the two terms, with all summation indices replaced by v

$$\begin{aligned}&\partial_v^4\,6\prod_{k=1}^{4}K_{a_k}\left(\Delta_{12}\Delta_{a_1a_2}\Delta_{a_3a_4}+4\Delta_{1a_1}\Delta_{a_22}\Delta_{a_3a_4}\right)=\\ &\qquad(6\cdot 4!)\,\Delta_{12}\Delta_{vv}\Delta_{vv}+(6\cdot 4\cdot 4!)\,\Delta_{1v}\Delta_{v2}\Delta_{vv}.\end{aligned} \tag{7.27}$$

Therefore $N_1 = 576$ and $N_2 = 144$, which gives

$$N(x_1,x_2) = -g\sum_v\left(\frac{1}{2}\Delta_{1v}\Delta_{v2}\Delta_{vv}+\frac{1}{8}\Delta_{12}\Delta_{vv}\Delta_{vv}\right). \tag{7.28}$$

The first term is the connected Green's function of (6.6) in the (discretized) position-space. As for the second term, it corresponds to the *vacuum bubble* contribution



and it is canceled by the denominator D of (7.23) expanded at order g. [2]

[2]This is a general feature. At each perturbative order, *vacuum bubbles* generated by the numerator are canceled by the denominator. We leave to the reader to verify this explicitly for the case at hand.

Finally, we go back to the continuum and rewrite the first term of (7.28) as

$$N_c(x_1, x_2) = -\frac{g}{2} \int d^4y \Delta(x_1 - y)\Delta(y - x_2)\Delta(y - y), \tag{7.29}$$

with $\Delta(x - y)$ given in (7.19).

The Fourier transform of (7.29) reads

$$\tilde{G}_c(p_1, p_2) = G^{(1)}(p_1, p_2)\,(2\pi)^4\delta^4\,(p_1 + p_2)\,, \tag{7.30}$$

with $G^{(1)}(p_1, -p_1)$ in (6.6).

7.6 Problem*: Perturbative Green's functions IV

Use the path integral approach to rederive the two-point Green's functions in (6.8) and (6.10).

Chapter 8

Cross sections and decay rates

Cross sections (σ) and decay rates (Γ) are fundamental measurable quantities that provide the link between the underlying Quantum Field Theory and the experimental data measurable in Particle Physics experiments. In this chapter, we recall the basic formulas and give examples on how to compute, analytically, the phase space integrals appearing in the definition of σ and Γ. Since, in practical cases, it is not always possible to perform the phase space integration analytically, one has to rely, in general, on Monte Carlo methods. For this reason, at the end of the chapter, we also propose a few practical problems on the latter subject.

8.1 The definition of phase space

For a generic process with n particle in the final state, the total n-body phase space integrals is defined by

$$\Phi_n = \int d\phi_n = \int d^4p_1 \cdots d^4p_n \, \delta_+(p_1^2 - m_1^2) \cdots \delta_+(p_n^2 - m_n^2) \, \delta^4(Q_{init} - \sum_i p_i), \quad (8.1)$$

with

$$\delta_+(p^2 - m^2) \equiv \delta(p^2 - m^2)\theta(p_0). \quad (8.2)$$

8.2 The definition of Cross section

Given a generic $2 \to n$ process

$$q_1 + q_2 \to p_1 + p_2 + \cdots + p_n, \quad \text{with} \quad q_i^2 = M_i^2 \quad \text{and} \quad p_i^2 = m_i^2, \tag{8.3}$$

the total cross section is defined as

$$\sigma = \frac{(2\pi)^{4-3n}}{4\left[(q_1 \cdot q_2)^2 - M_1^2 M_2^2\right]^{\frac{1}{2}}} \int d\phi_n \, |\bar{\mathcal{M}}|^2, \tag{8.4}$$

where $|\bar{\mathcal{M}}|^2$ is the S matrix element squared summed over the final state polarizations and averaged over the initial state ones.

8.3 The definition of Decay Rate

Given a generic $1 \to n$ decay

$$Q \to p_1 + p_2 + \cdots + p_n, \quad \text{with} \quad Q^2 = M^2 \quad \text{and} \quad p_i^2 = m_i^2, \tag{8.5}$$

the total decay rate is defined as

$$\Gamma = \frac{(2\pi)^{4-3n}}{2M} \int d\phi_n \, |\bar{\mathcal{M}}|^2, \tag{8.6}$$

and it is linked to the mean lifetime τ of the decaying particle by the relation

$$\tau = \frac{1}{\Gamma}. \tag{8.7}$$

8.4 Problem: The massless 2-body phase space

Given a process

$$p_1 + p_2 \to p_3 + p_4, \quad \text{with} \quad p_3^2 = p_4^2 = 0,$$

show that

$$\Phi_2 = \int d\phi_2 = \frac{1}{8} \int d\Omega_3 = \frac{1}{8} 4\pi = \frac{\pi}{2}, \tag{8.8}$$

where $d\Omega_3$ is the solid angle of particle 3.

Solution

In the center-of-mass frame one has

$$\vec{p_1} + \vec{p_2} = 0, \quad \text{so that} \quad P \equiv (p_1 + p_2) = (\sqrt{s}, 0, 0, 0). \tag{8.9}$$

Therefore

$$\begin{aligned} \int d\phi_2 &= \int d^4p_3 d^4p_4 \delta_+(p_3^2)\delta_+(p_4^2)\delta^4(P - p_3 - p_4) \\ &= \int d^4p_3 \delta(p_3^2)\theta(E_3)\delta((P - p_3)^2)\theta(\sqrt{s} - E_3). \end{aligned} \tag{8.10}$$

Now, by calling $q_3 \equiv \mid \vec{p_3} \mid$ we have

$$\int d\phi_2 = \int dE_3 \int d\Omega_3 \int dq_3 q_3^2 \delta(E_3^2 - q_3^2)\theta(E_3)\delta(s - 2\sqrt{s}E_3)\theta(\sqrt{s} - E_3). \tag{8.11}$$

But

$$\theta(E_3)\delta(E_3^2 - q_3^2) = \frac{1}{2q_3}\delta(E_3 - q_3), \tag{8.12}$$

because $\theta(E_3)$ selects the positive solution. Then

$$\begin{aligned} \int d\phi_2 &= \int d\Omega_3 \int dq_3 \frac{q_3^2}{2q_3} \underbrace{\delta(s - 2\sqrt{s}q_3)}_{\frac{1}{2\sqrt{s}}\delta(q_3 - \frac{\sqrt{s}}{2})} \theta(\sqrt{s} - q_3) \\ &= \int d\Omega_3 \left. \frac{q_3^2}{2q_3} \frac{1}{2\sqrt{s}} \right|_{q_3 = \frac{\sqrt{s}}{2}} = \frac{1}{8} \int d\Omega_3 = \frac{\pi}{2}. \end{aligned} \tag{8.13}$$

From the previous result, one can immediately write down the following explicit parametrization for the momenta:

$$p_1 = \left(\frac{\sqrt{s}}{2}, \frac{\sqrt{s}}{2}, 0, 0 \right)$$

$$p_2 = \left(\frac{\sqrt{s}}{2}, -\frac{\sqrt{s}}{2}, 0, 0 \right)$$

$$p_3 = \left(\frac{\sqrt{s}}{2}, \frac{\sqrt{s}}{2} cos\theta_3, \frac{\sqrt{s}}{2} sin\theta_3 cos\varphi_3, \frac{\sqrt{s}}{2} sin\theta_3 sin\varphi_3 \right)$$

$$p_3 = \left(\frac{\sqrt{s}}{2}, -\frac{\sqrt{s}}{2} cos\theta_3, -\frac{\sqrt{s}}{2} sin\theta_3 cos\varphi_3, -\frac{\sqrt{s}}{2} sin\theta_3 sin\varphi_3 \right).$$

The massless n-body phase space

The massless case is simple enough that a general formula can be derived

$$\Phi_n = \int d\phi_n = \frac{\left(\frac{\pi}{2}\right)^{(n-1)} s^{(n-2)}}{(n-1)!(n-2)!}. \tag{8.14}$$

8.5 Problem: The massive 2-body phase space

Compute $\Phi_2 = \int d\phi_2$ when $p_3^2 = m_3^2$ and $p_4^2 = m_4^2$.

Solution

Let $P \equiv (p_1 + p_2) = (\sqrt{s}, 0, 0, 0)$ and $q_3 \equiv |\vec{p}_3|$. Then

$$\begin{aligned} \int d\phi_2 &= \int d^4 p_3 \delta\left(p_3^2 - m_3^2\right) \theta(E_3) \delta((p-p_3)^2 - m_4^2)\, \theta(\sqrt{s} - E_3) \\ &= \int dE_3 \int d\Omega_3 \int dq_3 q_3^2 \delta(E_3^2 - q_3^2 - m_3^2)\theta(E_3)\delta(s + m_3^2 - 2\sqrt{s}E_3 - m_4^2) \\ &\quad \times\theta(\sqrt{s} - E_3). \end{aligned} \tag{8.15}$$

Since

$$\theta(E_3)\delta(E_3^2 - (q_3^2 + m_3^2)) = \frac{1}{2\sqrt{q_3^2 + m_3^2}}\delta(E_3 - \sqrt{q_3^2 + m_3^2}) \tag{8.16}$$

one obtains

$$\int d\phi_2 = \int d\Omega_3 \int dq_3 \frac{q_3^2}{2\sqrt{q_3^2 + m_3^2}} \delta(s + m_3^2 - 2\sqrt{s}\sqrt{q_3^2 + m_3^2} - m_4^2)\theta(\sqrt{s} - E_3). \tag{8.17}$$

But

$$\delta(s + m_3^2 - 2\sqrt{s}\sqrt{q_3^2 + m_3^2} - m_4^2) = \frac{\sqrt{q_3^2 + m_3^2}}{2\sqrt{s}q_3}\delta(q_3 - q_3^0), \tag{8.18}$$

where q_3^0 is the value of q_3 that nullifies the argument of the delta function. Therefore

$$\begin{aligned}\int d\phi_2 &= \int d\Omega_3 \int dq_3 \frac{q_3^2}{2\sqrt{q_3^2+m_3^2}} \frac{1}{2\sqrt{s}q_3} \sqrt{q_3^2+m_3^2}\,\delta(q_3-q_3^0)\\ &= \int d\Omega_3 \int dq_3 \frac{q_3^0}{4\sqrt{s}}\delta(q_3-q_3^0) = \frac{q_3^0}{4\sqrt{s}} \int d\Omega_3. \end{aligned} \tag{8.19}$$

Now, one has just to compute q_3^0,

$$\begin{aligned}&\frac{1}{4s}(s+m_3^2-m_4^2)^2 = (q_3^0)^2 + m_3^2 \Rightarrow\\ &\frac{1}{4s}\left[-4sm_3^2 + s^2 + m_3^4 + m_4^4 + 2sm_3^2 - 2sm_4^2 - 2m_3^2m_4^2\right] = (q_3^0)^2. \end{aligned} \tag{8.20}$$

Thus

$$\frac{1}{4s}\lambda(s,m_3^2,m_4^2) = (q_3^0)^2, \tag{8.21}$$

where $\lambda(x,y,z) \equiv x^2+y^2+z^2-2xy-2xz-2yz$ is the Källén function. Then

$$q_3^0 = \frac{1}{2\sqrt{s}}\lambda^{\frac{1}{2}}(s,m_3^2,m_4^2), \tag{8.22}$$

so that

$$\int d\phi_2 = \frac{\lambda^{\frac{1}{2}}(s,m_3^2,m_4^2)}{8s} \int d\Omega_3. \tag{8.23}$$

8.6 Problem: The massless 3-body phase space

For a process

$$p_1 \to p_2 + p_3 + p_4, \quad \text{with} \;\; p_1^2 = m^2 \;\; \text{and} \;\; p_2^2 = p_3^2 = p_4^2 = 0, \tag{8.24}$$

show that

$$\int d\phi_3 = \pi^2 \int_0^{\frac{m}{2}} dE_2 \int_{\frac{m}{2}-E_2}^{\frac{m}{2}} dE_3. \tag{8.25}$$

Solution

$$\begin{aligned}
\int d\phi_3 &= \int d^4p_2 d^4p_3 d^4p_4\, \delta_+(p_2^2)\delta_+(p_3^2)\delta_+(p_4^2)\, \delta^4(p_1 - p_2 - p_3 - p_4) \\
&= \int d^3p_2 d^3p_3 \frac{1}{4E_2E_3} \delta\left[(p_1 - p_2 - p_3)^2\right] \\
&= \int d\Omega_2 d\Omega_3 dE_2 dE_3 \frac{E_2E_3}{4} \delta(m^2 - 2(p_1 \cdot p_2) - 2(p_1 \cdot p_3) + 2(p_2 \cdot p_3)),
\end{aligned}$$

where we understand $E_4 > 0$. Now we choose a convenient reference frame where the spatial components of the momenta p_2, p_3, p_4 lie in the (x,y) plane with p_2 along x,

$$\begin{aligned}
p_1 &= (m, 0, 0, 0) \\
p_2 &= E_2(1, 1, 0, 0) \\
p_3 &= E_3(1, c_3, s_3, 0) \\
p_4 &= E_4(1, c_4, s_4, 0).
\end{aligned} \tag{8.26}$$

Therefore we have

$$\begin{aligned}
\delta(m^2 - 2(p_1 \cdot p_2) - 2(p_1 \cdot p_3) + 2(p_2 \cdot p_3)) &= \delta(m^2 - 2mE_2 - 2mE_3 + 2E_2E_3(1 - c_3)) \\
&= \frac{1}{2E_2E_3} \delta\left(c_3 - \frac{2E_2E_3 - 2m(E_2 + E_3) + m^2}{2E_2E_3}\right),
\end{aligned} \tag{8.27}$$

so that

$$\int d\phi_3 = (4\pi)(2\pi) \int_{-1}^{1} dc_3 dE_2 dE_3 \frac{E_2E_3}{4} \frac{1}{2E_2E_3} \delta(c_3 - \dots) = \pi^2 \int dE_2 dE_3. \tag{8.28}$$

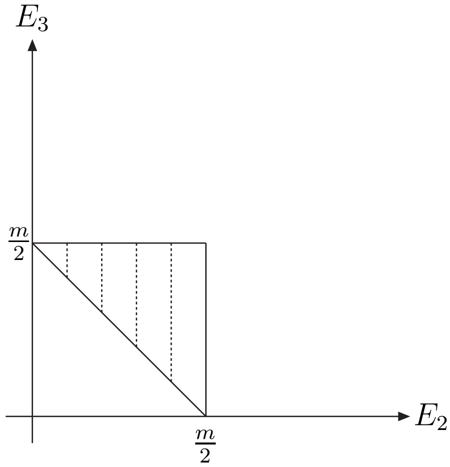
The integration boundaries for E_2 and E_3 can be determined by observing that $-1 < c_3 < 1$

$$\begin{aligned}
c_3 < 1 \quad &\Rightarrow \quad E_2 + E_3 > \frac{m}{2}, \\
-1 < c_3 \quad &\Rightarrow \quad (2E_2 - m)(2E_3 - m) > 0.
\end{aligned} \tag{8.29}$$

Thus one has

$$E_2 + E_3 > \frac{m}{2}, \quad E_2 < \frac{m}{2}, \quad E_3 < \frac{m}{2}. \tag{8.30}$$

Note that the second solution of (8.29), namely $E_{2,3} > \frac{m}{2}$, is discarded because the condition $E_4 > 0$ implies $m - E_2 - E_3 > 0 \;\Rightarrow\; E_2 + E_3 < m$. Therefore, we have to remain inside the dashed part of the following figure



from which the desired result follows

$$\int d\phi_3 = \pi^2 \int_0^{\frac{m}{2}} dE_2 \int_{\frac{m}{2}-E_2}^{\frac{m}{2}} dE_3. \tag{8.31}$$

8.7 Monte Carlo Numerical Integration

In this section we recall the basic principles of the Monte Carlo numerical integration. Given a one dimensional integral over a function $f(x)$,

$$I = \int_a^b dx\, f(x), \tag{8.32}$$

one can always change variables and put the integration domain in the interval $[0, 1]$,

$$I = \int_0^1 d\rho\, g(\rho). \tag{8.33}$$

Then, I can be estimated by taking N values of ρ (which we dub $\rho^{(i)}$, with $i = 1 \div N$) *randomly* in $[0, 1]$,

$$I \simeq \frac{1}{N} \sum_{i=1}^{N} g(\rho^{(i)}) \equiv < g >, \tag{8.34}$$

where the symbol $\simeq$ means that equality is reached in the $N \to \infty$ limit. The error ΔI of this estimate is given by the formula

$$\Delta I = \sqrt{\frac{< g^2 > - < g >^2}{N}}. \tag{8.35}$$

Therefore, the Monte Carlo estimate of I is

$$I \simeq < g > \pm \Delta I. \tag{8.36}$$

A nice property of the Monte Carlo method is that *it does not depend on the dimensionality of the function g*, in the sense that it can be immediately translated to functions of n variables $g(\vec{\rho}) := g(\rho_1, \cdots, \rho_n)$. Given

$$J = \int_0^1 d\rho_1 \cdots \int_0^1 d\rho_n \, g(\vec{\rho}), \tag{8.37}$$

a Monte Carlo estimate is given by the formula

$$J \simeq \frac{1}{N} \sum_{i=1}^{N} g(\vec{\rho}^{(i)}) \equiv < g > \tag{8.38}$$

where $\vec{\rho}^{(i)}$ are randomly taken values in the hypercube $[0,1]^n$. The error is again given by

$$\Delta J = \sqrt{\frac{< g^2 > - < g >^2}{N}}. \tag{8.39}$$

8.8 Problem*: Numerical integration of a 5-body phase space

Compute numerically with RAMBO the massless phase space integral $\int_{cut} d\Phi_5$ using the following input values in the center-of-mass frame:

1. $\sqrt{s} = 200$ GeV,
2. $E_i > 10$ GeV ($i = 1 \div 5$),
3. $|cos\theta_i| \leq 0.9$ ($i = 1 \div 5$).

A version of RAMBO and an example of FORTRAN program implementing it can be found in

www.ugr.es/local/pittau/particulas1.f.

Chapter 9

Problems at the tree-level

In this chapter we compute a few processes at the lowest order in the perturbation theory, namely at the *tree-level*. The steps needed to produce physical predictions can be summarized as follows:

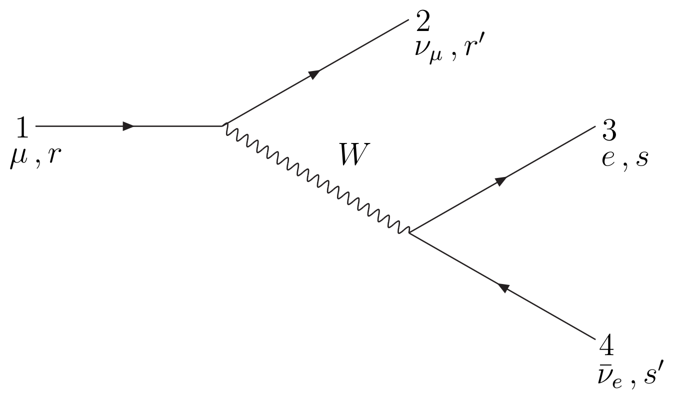
1. Calculation of the amplitude squared
   (a) draw the Feynman diagrams for the process;
   (b) apply the Feynman rules for propagators and vertices;
   (c) calculate the amplitude squared by using trace theorems and γ matrix properties.
2. Calculation of the phase space
   (a) identify the number and the masses of the particles in the process;
   (b) fix the reference frame and the momenta of the particles;
   (c) calculate the integrals using the properties of the δs, as we have seen in chapter 8.
3. Calculation of cross sections or decay rates
   (a) by using (8.4) or (8.6), respectively.

9.1 Problem: The μ decay in the large M_W limit

Compute the muon lifetime.

Solution

The only contributing Feynman diagram is



where we take $m_\mu = m$ and $m_e = m_{\bar{\nu}_e} = m_{\nu_\mu} = 0$.

Using Feynman rules for propagators and vertices gives the following expression for the *invariant amplitude*

$$\mathcal{M} = \left(-\frac{ig}{2\sqrt{2}} \right)^2 (-i) \, \bar{u}_{(2)} \gamma_\mu (1-\gamma_5) u_{(1)} \bar{u}_{(3)} \gamma^\mu (1-\gamma_5) v_{(4)} \frac{1}{(-M_W{}^2)} . \tag{9.1}$$

Note that the exact propagator $\frac{1}{p^2 - M_W^2}$ has been replaced by $\frac{1}{(-M_W^2)}$. This is so because we assume to work in the large M_W limit, namely at low energy. We need an expression for $|\mathcal{M}|^2$, so that we must find the complex conjugate of $\mathcal{M}$. A bi-spinor product such as $\bar{v}\gamma^\mu u$ can be complex-conjugated as follows

$$\left(\bar{v} \gamma^\mu u \right)^* = u^\dagger \left(\gamma^\mu \right)^\dagger \left(\gamma^0 \right)^\dagger v = u^\dagger \left(\gamma^\mu \right)^\dagger \left(\gamma^0 \right) v = u^\dagger \gamma^0 \gamma^\mu v = \bar{u} \gamma^\mu v .$$

Thus, the squared matrix element reads

$$\begin{aligned} |\mathcal{M}|^2 = & \frac{g^4}{64 M_W{}^4} \left\{ \bar{u}_{(1)} \gamma^\nu (1-\gamma_5) u_{(2)} \bar{v}_{(4)} \gamma_\nu (1-\gamma_5) u_{(3)} \right\} \\ & \times \left\{ \bar{u}_{(2)} \gamma_\mu (1-\gamma_5) u_{(1)} \bar{u}_{(3)} \gamma^\mu (1-\gamma_5) v_{(4)} \right\} . \end{aligned} \tag{9.2}$$

We are still free to specify particular polarizations r, r', s, s' for the fermions. However, in actual experiments it is difficult to retain control over spin states. In most experiments the initial state is unpolarized, so the measured cross section, or decay rate, is an *average* over the spin of the initial particles and a *sum* over the final state polarizations. Besides, the expression for $|\mathcal{M}|^2$ simplifies considerably when we throw away the spin information. In summary, the quantity we want to compute is

$$|\bar{\mathcal{M}}|^2 \equiv \frac{1}{2}\sum_r \sum_s \sum_{r'} \sum_{s'} |\mathcal{M}\left(r \to r', s, s'\right)|^2, \tag{9.3}$$

namely, the average over the μ spin (r), and the sums over the spins of the e (s), ν_μ (r') and $\bar{\nu}_e$ (s').

The spins sums can be performed by using the completeness relation for spinors:

$$\sum_{spin} u_{(j)}\bar{u}_{(j)} = \not{p}_j + m_j, \qquad \sum_{spin} v_{(j)}\bar{v}_{(j)} = \not{p}_j - m_j. \tag{9.4}$$

By working explicitly with spinor indices and taking the trace, one can freely move all $\bar{u}$ next to the u and all $\bar{v}$ next to the v, so that (9.3) can be written as follows

$$\begin{aligned}
|\bar{\mathcal{M}}|^2 &= \frac{g^4}{128{M_W}^4} Tr\left\{\sum_{r'} u_{(2)}\bar{u}_{(2)}\gamma_\mu(1-\gamma_5)\sum_r u_{(1)}\bar{u}_{(1)}\gamma^\nu(1-\gamma_5)\right\} \\
&\quad \times Tr\left\{\sum_s u_{(3)}\bar{u}_{(3)}\gamma^\mu(1-\gamma_5)\sum_{s'} v_{(4)}\bar{v}_{(4)}\gamma_\nu(1-\gamma_5)\right\} \\
&= \frac{g^4}{128{M_W}^4} Tr\left\{\not{p}_2\gamma_\mu(1-\gamma_5)\left(\not{p}_1+m\right)\gamma^\nu(1-\gamma_5)\right\} Tr\left\{\not{p}_3\gamma^\mu(1-\gamma_5)\not{p}_4\gamma_\nu(1-\gamma_5)\right\} \\
&= \frac{g^4}{128{M_W}^4} 4\, Tr\left\{\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu(1-\gamma_5)\right\} Tr\left\{\not{p}_3\gamma^\mu\not{p}_4\gamma^\nu(1-\gamma_5)\right\}.
\end{aligned} \tag{9.5}$$

By computing the traces one obtains

$$\begin{aligned}
|\bar{\mathcal{M}}|^2 = \frac{g^4}{128{M_W}^4}\, 64\, &\left\{(p_{2\mu}p_{1\nu}+p_{2\nu}p_{1\mu}) - (p_1\cdot p_2)g_{\mu\nu} - i\epsilon_{2\mu1\nu}\right\} \\
\times &\left\{(p_3^\mu p_4^\nu + p_3^\nu p_4^\mu) - (p_4\cdot p_3)g^{\mu\nu} - i\epsilon^{3\mu4\nu}\right\}.
\end{aligned} \tag{9.6}$$

The crossed terms in the previous equation vanish because symmetric tensors are contracted with antisymmetric ones, therefore

$$\begin{aligned}
|\bar{\mathcal{M}}|^2 = \frac{g^4}{2M_W^4} &\{2(p_2\cdot p_3)(p_1\cdot p_4) + 2(p_1\cdot p_3)(p_2\cdot p_4) + 4(p_1\cdot p_2)(p_3\cdot p_4) \\
&-2(p_3\cdot p_4)(p_1\cdot p_2) - 2(p_1\cdot p_2)(p_3\cdot p_4) - \epsilon_{2\mu1\nu}\epsilon^{3\mu4\nu}\}.
\end{aligned} \tag{9.7}$$

In addition

$$\epsilon_{2\mu 1\nu}\epsilon^{3\mu 4\nu} = -2\left[(p_2\cdot p_3)(p_1\cdot p_4) - (p_2\cdot p_4)(p_1\cdot p_3)\right],$$

so the final result reads

$$|\bar{\mathcal{M}}|^2 = \frac{g^4}{2M_W^4}\left\{4(p_2\cdot p_3)(p_1\cdot p_4)\right\} = \frac{2g^4}{M_W^4}(p_1\cdot p_4)(p_2\cdot p_3). \tag{9.8}$$

The needed 3-body phase space has already been computed in Problem 8.6

$$\int d\phi_3 = \pi^2 \int_0^{\frac{m}{2}} dE_2 \int_{\frac{m}{2}-E_2}^{\frac{m}{2}} dE_3, \tag{9.9}$$

with momenta given by

$$\begin{aligned} p_1 &= (m,0,0,0) \\ p_2 &= E_2(1,1,0,0) \\ p_3 &= E_3(1,c_3,s_3,0) \\ p_4 &= E_4(1,c_4,s_4,0), \end{aligned} \tag{9.10}$$

where

$$c_3 = \frac{2E_2E_3 - 2m(E_2+E_3) + m^2}{2E_2E_3}. \tag{9.11}$$

We are now ready to compute the decay rate for the process with the help of the formula

$$\Gamma = \frac{(2\pi)^{-5}}{2m}\int d\phi_3\,|\bar{\mathcal{M}}|^2.$$

With our explicit choice of momenta we have

$$\begin{aligned} (p_1\cdot p_4) &= mE_4 = m(m-E_2-E_3) \\ (p_2\cdot p_3) &= E_2E_3(1-c_3). \end{aligned} \tag{9.12}$$

Therefore

$$|\bar{\mathcal{M}}|^2 = \frac{2g^4}{M_W^4} m E_2 E_3 (m-E_2-E_3)(1-c_3). \tag{9.13}$$

By using (9.11) one obtains

$$c_3 = 1 + \frac{m}{2E_2E_3}(m - 2E_2 - 2E_3), \tag{9.14}$$

so that

$$|\bar{\mathcal{M}}|^2 = \frac{g^4}{M_W^4}(m - E_2 - E_3)m^2(2E_2 + 2E_3 - m). \tag{9.15}$$

Finally, the decay rate is given by

$$\begin{aligned}\Gamma &= \frac{(2\pi)^{-5}}{2m}\pi^2 \frac{g^4}{m_W^4} m^2 \int_0^{\frac{m}{2}} dE_2 \int_{\frac{m}{2}-E_2}^{\frac{m}{2}} dE_3 (m - E_2 - E_3)(2E_2 + 2E_3 - m) \\ &\equiv \frac{g^4 m}{64\pi^3 M_W^4}\mathcal{I}. \end{aligned} \tag{9.16}$$

To compute $\mathcal{I}$ we change variables as follows

$$t_{2,3} = \frac{2}{m}E_{2,3}.$$

Thus,

$$\begin{aligned}\mathcal{I} &= \int_0^1 dt_2 \int_{1-t_2}^1 dt_3 \left(\frac{m}{2}\right)^2 \left\{m - \frac{m}{2}t_2 - \frac{m}{2}t_3\right\} \{mt_2 + mt_3 - m\} \\ &= \left(\frac{m}{2}\right)^3 m \int_0^1 dt_2 \int_{1-t_2}^1 dt_3 (2 - t_2 - t_3)(t_3 + t_2 - 1). \end{aligned} \tag{9.17}$$

A further change $t_2 \to 1 - t_2$ gives

$$\begin{aligned}\mathcal{I} &= \frac{m^4}{8}\int_0^1 dt_2 \int_{t_2}^1 dt_3 (1 + t_2 - t_3)(t_3 - t_2) \\ &= \frac{m^4}{8}\int_0^1 dt_2 \int_{t_2}^1 dt_3 \left\{(t_3 - t_2) - (t_3 - t_2)^2\right\}. \end{aligned} \tag{9.18}$$

Finally, redefining

$$\begin{aligned} &x = t_2; \quad y = \frac{t_3 - x}{1 - x} \quad \Rightarrow \quad t_3 = x + y(1 - x) \\ &dt_2 = dx; \quad dt_3 = (1 - x)dy \end{aligned}$$

gives

$$\mathcal{I} = \frac{m^4}{8} \int_0^1 dx \int_0^1 dy (1-x) \left\{ y(1-x) - y^2(1-x)^2 \right\}, \tag{9.19}$$

and by further shifting $x \to (1-x)$ one obtains

$$\mathcal{I} = \frac{m^4}{8} \int_0^1 dx \int_0^1 dy \left\{ yx^2 - y^2x^3 \right\} = \frac{m^4}{8} \left\{ \frac{1}{2}\frac{1}{3} - \frac{1}{3}\frac{1}{4} \right\} = \frac{m^4}{8 \times 12}, \tag{9.20}$$

which gives

$$\Gamma = \frac{g^4 m^5}{\pi^3 M_W^4} \cdot \frac{1}{6144} = \frac{m^5 G_F^2}{192\pi^3}, \tag{9.21}$$

where we have defined

$$g^2 = \frac{G_F}{\sqrt{2}} 8 M_W^2.$$

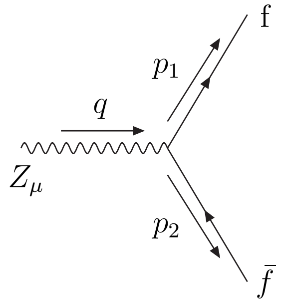
Finally, the muon lifetime is $\tau = \frac{1}{\Gamma}$.

9.2 Problem: The Z decay width

Compute the total decay width Γ_Z of the Z boson to massless fermions in terms of G_F, M_Z and M_W.

Solution

The only contributing Feynman diagram is



from which we compute the amplitude

$$\mathcal{M} = \epsilon^{\mu}_{(q)} \left(\frac{-ig}{2c_\theta} \right) \bar{u}_{(1)} \gamma_\mu (v_f + a_f \gamma_5) v_{(2)}, \tag{9.22}$$

with $v_f = I_{3f} - 2s_\theta^2 Q_f$ and $a_f = -I_{3f}$ and where, by momentum conservation, $q = p_1 + p_2$. The squared amplitude for the process, summed over the final state polarizations and averaged over the initial state ones, can be calculated as in the previous problem

$$|\bar{\mathcal{M}}|^2 = \frac{g^2}{4c_\theta^2}\frac{1}{3}\sum_{spin}\epsilon^{\mu}_{(q)}\epsilon^{*\nu}_{(q)}\sum_{spin}\left\{\bar{u}_{(1)}\gamma_\mu(v_f + a_f\gamma_5)v_{(2)}\right\}\times\left\{\bar{v}_{(2)}\gamma_\nu(v_f + a_f\gamma_5)u_{(1)}\right\}, \tag{9.23}$$

where the factor $\frac{1}{3}$ comes from the average over the initial spin. Using the trace technique gives

$$\begin{aligned} |\bar{\mathcal{M}}|^2 &= \frac{g^2}{4c_\theta^2}\frac{1}{3}\left(\sum_{spin}\epsilon^{\mu}_{(q)}\epsilon^{*\nu}_{(q)}\right) \\ &\quad \times Tr\left\{\sum_{spin} u_{(1)}\bar{u}_{(1)}\gamma_\mu(v_f + a_f\gamma_5)\times\sum_{spin} v_{(2)}\bar{v}_{(2)}\gamma_\nu(v_f + a_f\gamma_5)\right\} \\ &= \frac{g^2}{4c_\theta^2}\frac{1}{3}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M_Z^2}\right) Tr\left\{\not{p}_1\gamma_\mu(v_f + a_f\gamma_5)\not{p}_2\gamma_\nu(v_f + a_f\gamma_5)\right\}. \end{aligned} \tag{9.24}$$

Now we are going to work in terms of $\omega^\pm = 1/2(1 \pm \gamma_5)$, so that $v_f + a_f\gamma_5 = v_f^+\omega^+ + v_f^-\omega^-$ with $v_f^\pm = v_f \pm a_f$. The projectors properties $\omega^+\omega^- = 0$ and $(\omega^\pm)^2 = \omega^\pm$ allow us to rewrite

$$\begin{aligned} |\bar{\mathcal{M}}|^2 &= \frac{g^2}{4c_\theta^2}\frac{1}{3}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M_Z^2}\right) Tr\left\{\not{p}_1\gamma_\mu(v_f^+\omega_+ + v_f^-\omega_-)\not{p}_2\gamma_\nu(v_f^+\omega_+ + v_f^-\omega_-)\right\} \\ &= \frac{g^2}{4c_\theta^2}\frac{1}{3}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M_Z^2}\right)\left[(v_f^+)^2\, Tr\left\{\not{p}_1\gamma_\mu\not{p}_2\gamma_\nu\omega_+\right\} + (v_f^-)^2\, Tr\left\{\not{p}_1\gamma_\mu\not{p}_2\gamma_\nu\omega_-\right\}\right]. \end{aligned}$$

The traces containing γ_5 do not contribute upon contraction with the symmetric tensor $\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M_Z^2}\right)$. Therefore

$$|\bar{\mathcal{M}}|^2 = \frac{g^2}{8c_\theta^2}\frac{1}{3}\left[(v_f^+)^2 + (v_f^-)^2\right]\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M_Z^2}\right) Tr\left\{\not{p}_1\gamma_\mu\not{p}_2\gamma_\nu\right\}. \tag{9.25}$$

Due to *gauge invariance* the $q^\mu q^\nu$ piece does not contribute, as can be explicitly checked,

$$Tr\left\{\not{p}_1\not{q}\not{p}_2\not{q}\right\} = Tr\left\{\not{p}_1(\not{p}_1 + \not{p}_2)\not{p}_2(\not{p}_1 + \not{p}_2)\right\} = 0.$$

The $g_{\mu\nu}$ piece gives, instead,

$$-Tr\left\{\not{p}_1\gamma_\mu\not{p}_2\gamma^\mu\right\} = 2\,Tr\left\{\not{p}_1\not{p}_2\right\} = 8(p_1\cdot p_2) = 4M_Z^2. \tag{9.26}$$

In conclusion

$$|\bar{\mathcal{M}}|^2 = \frac{g^2}{2c_\theta^2}\frac{1}{3}\left[(v_f^+)^2 + (v_f^-)^2\right]M_Z^2. \tag{9.27}$$

To compute the width we use the formula

$$\Gamma_f = \frac{(2\pi)^{4-3n}}{2M}\int d\phi_n\,|\bar{\mathcal{M}}|^2, \tag{9.28}$$

which gives, for $n = 2$,

$$\Gamma_f = \frac{(2\pi)^{-2}}{2M_Z}\int d\phi_2\,|\bar{\mathcal{M}}|^2. \tag{9.29}$$

The 2-body massless phase space has been already calculated in chapter 8,

$$\int d\phi_2 = \frac{1}{8}\int d\Omega = \frac{\pi}{2}. \tag{9.30}$$

Finally the partial width decay $\Gamma_f(Z\to f\bar{f})$ for one family of fermions reads

$$\begin{aligned}\Gamma_f &= \frac{(2\pi)^{-2}}{2M_Z}\frac{\pi}{2}|\bar{\mathcal{M}}|^2 \\ &= \frac{1}{3}\frac{(2\pi)^{-2}}{2M_Z}\frac{\pi}{2}\frac{g^2}{2c_\theta^2}M_Z^2\left[(v_f^+)^2 + (v_f^-)^2\right] = \frac{G_F M_Z^3}{12\sqrt{2}\pi}\left[(v_f^+)^2 + (v_f^-)^2\right] \\ &= \frac{G_F M_Z^3}{6\sqrt{2}\pi}\left[v_f^2 + a_f^2\right].\end{aligned} \tag{9.31}$$

To compute the total width one has to sum over all possibilities

$$\Gamma_Z = \sum_{f\neq \mathrm{top}} N_{cf}\Gamma_f, \tag{9.32}$$

where N_{cf} is the colour factor, namely $N_{cf} = 1$ for leptons and $N_{cf} = 3$ for quarks. We do not consider the top quark in the sum because, due to its large mass, the decay into it is not kinematically allowed. We can then compute Γ_Z by using as input parameters

$$\begin{aligned}G_F &= 1.16637\times 10^{-5}\ \mathrm{GeV}^{-2}, \\ M_Z &= 91.1867\ \mathrm{GeV}, \\ M_W &= 80.450\ \mathrm{GeV},\end{aligned} \tag{9.33}$$

which give the numerical value

$$\Gamma_Z = 2.447 \text{ GeV}, \tag{9.34}$$

to be compared with the experimental value $\Gamma_Z^{\text{EXP}} = (2.495 \pm 0.002)$ GeV.

As a last remark, one could use, instead of the previous one, the following set of parameters

$$\begin{aligned} \alpha(0) &= 1/137.0359895, \\ M_Z &= 91.1807 \text{ GeV}, \\ M_W &= 80.450 \text{ GeV}. \end{aligned} \tag{9.35}$$

To achieve this, the following relations should be used connecting the two sets

$$\begin{aligned} g^2 &= \frac{G_F}{\sqrt{2}} 8 M_W^2, \\ c_\theta^2 &= \frac{M_W^2}{M_Z^2}, \\ 4\pi\alpha &= g^2 s_\theta^2, \\ G_F &= \frac{\pi\alpha}{\sqrt{2} M_W^2 s_\theta^2}. \end{aligned} \tag{9.36}$$

The result in this case would be

$$\Gamma'_Z = 2.371 \text{ GeV}. \tag{9.37}$$

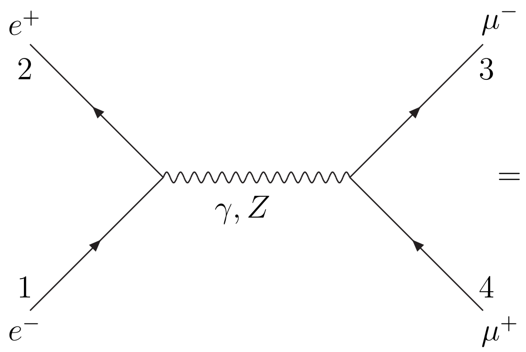
The numerical difference between Γ_Z and Γ'_Z is due to the neglected higher order corrections, being our calculation at the tree-level only.

9.3 Problem: Cross section and FB asymmetry for $e^+e^- \to \mu^+\mu^-$

Compute, in the limit of massless fermions, the electroweak cross section and the forward-backward asymmetry for the process $e^+e^- \to \mu^+\mu^-$.

Solution

There are two contributing Feynman diagrams, where a photon and a Z boson are exchanged, respectively



$$= \mathcal{M}_{\gamma,Z}.$$

By dubbing $\mathcal{M}_{\gamma,Z}$ the corresponding amplitudes, one obtains

$$\begin{aligned}
\mathcal{M}_\gamma &= (ie)^2 \frac{(-i)}{s} \bar{v}_{(2)}\gamma_\mu u_{(1)} \bar{u}_{(3)} \gamma^\mu v_{(4)}, \\
\mathcal{M}_Z &= \left(\frac{-ig}{2c_\theta}\right)^2 \frac{(-i)}{s - M_0^2} \bar{v}_{(2)}\gamma_\mu (v + a\gamma_5) u_{(1)} \bar{u}_{(3)} \gamma^\mu (v + a\gamma_5) v_{(4)},
\end{aligned} \tag{9.38}$$

where we have used

$$s = (p_1 + p_2)^2, \quad v = -\frac{1}{2} + 2s_\theta^2, \quad a = \frac{1}{2}. \tag{9.39}$$

Introducing the projectors $\omega_\pm = \frac{1}{2}(1 \pm \gamma_5)$ gives $v + a\gamma_5 = v_+\omega_+ + v_-\omega_-$, with $v_\pm = v \pm a$, in terms of which the amplitudes read

$$\begin{aligned}
\mathcal{M}_\gamma &= \frac{ie^2}{s} \left[\bar{v}_{(2)}\gamma_\mu \omega_+ u_{(1)} + \bar{v}_{(2)}\gamma_\mu \omega_- u_{(1)}\right] \left[\bar{u}_{(3)}\gamma^\mu \omega_+ v_{(4)} + \bar{u}_{(3)}\gamma^\mu \omega_- v_{(4)}\right], \\
\mathcal{M}_Z &= \frac{ig^2}{4c_\theta^2} \frac{1}{s - M_0^2} \left\{v_+ \bar{v}_{(2)}\gamma_\mu \omega_+ u_{(1)} + v_- \bar{v}_{(2)}\gamma_\mu \omega_- u_{(1)}\right\} \\
&\quad \times \left\{v_+ \bar{u}_{(3)}\gamma^\mu \omega_+ v_{(4)} + v_- \bar{u}_{(3)}\gamma^\mu \omega_- v_{(4)}\right\}.
\end{aligned} \tag{9.40}$$

The full amplitude is the sum of the two

$$\begin{aligned}
\mathcal{M}_\gamma + \mathcal{M}_Z &= i \sum_{\lambda,\sigma=\pm 1} \left(\frac{e^2}{s} + \frac{g^2}{4c_\theta^2(s - M_0^2)} v_\lambda v_\sigma\right) \times \left[\bar{v}_{(2)}\gamma_\mu \omega_\lambda u_{(1)} \bar{u}_{(3)}\gamma^\mu \omega_\sigma v_{(4)}\right] \\
&:= i \sum_{\lambda,\sigma=\pm 1} \left(\frac{e^2}{s} + \frac{g^2}{4c_\theta^2(s - M_0^2)} v_\lambda v_\sigma\right) \times A_{\lambda\sigma}.
\end{aligned} \tag{9.41}$$

When computing $|\bar{\mathcal{M}}|^2$ each term in $\sum_{\lambda,\sigma=\pm 1}$ does not interfere with the others. In fact

$$\sum_{spin} A_{\lambda\sigma} A^*{}_{\lambda'\sigma'} = \sum_{spin} (\bar{v}_{(2)}\gamma_\mu \omega_\lambda u_{(1)})(\bar{u}_{(3)}\gamma^\mu \omega_\sigma v_{(4)}) \times (\bar{v}_{(4)}\gamma^\alpha \omega_{\sigma'} u_{(3)})(\bar{u}_{(1)}\gamma_\alpha \omega_{\lambda'} v_{(2)})$$
$$= Tr[\not{p}_2 \gamma_\mu \omega_\lambda \not{p}_1 \gamma_\alpha \omega_{\lambda'}]\, Tr[\not{p}_3 \gamma^\mu \omega_\sigma \not{p}_4 \gamma^\alpha \omega_{\sigma'}]$$
$$= Tr[\not{p}_2 \gamma_\mu \not{p}_1 \gamma_\alpha \omega_\lambda \omega_{\lambda'}]\, Tr[\not{p}_3 \gamma^\mu \not{p}_4 \gamma^\alpha \omega_\sigma \omega_{\sigma'}] \propto \delta_{\lambda\lambda'}\delta_{\sigma\sigma'}. \tag{9.42}$$

Therefore

$$|\bar{\mathcal{M}}|^2 = \frac{1}{4} \sum_{\lambda,\sigma=\pm 1} \left| \frac{e^2}{s} + \frac{g^2}{4c_\theta^2(s - M_0^2)} v_\lambda v_\sigma \right|^2 \left(\sum_{spin} A_{\lambda\sigma} A^*{}_{\lambda\sigma} \right). \tag{9.43}$$

When $\lambda = \sigma = 1$ or $\lambda = \sigma = -1$ the product of traces is the same that appeared in the computation of the μ decay (see Problem 9.1)

$$\sum_{spin} A_{++} A^*{}_{++} = \sum_{spin} A_{--} A^*{}_{--} = 16(p_1 \cdot p_4)(p_2 \cdot p_3) = 16(p_2 \cdot p_3)^2. \tag{9.44}$$

On the contrary, when $\lambda \neq \sigma$ the sign in front of $\epsilon_{2\mu 1\nu}\epsilon^{3\mu 4\nu}$ changes, giving

$$\sum_{spin} A_{-+} A^*{}_{-+} = \sum_{spin} A_{+-} A^*{}_{+-} = 16(p_1 \cdot p_3)(p_2 \cdot p_4) = 16(p_1 \cdot p_3)^2. \tag{9.45}$$

Hence

$$|\bar{\mathcal{M}}|^2 = 4 \left\{ \left| \frac{e^2}{s} + \frac{g^2}{4c_\theta^2(s - M_0^2)} v_+ v_+ \right|^2 (p_2 \cdot p_3)^2 \right.$$
$$+ \left| \frac{e^2}{s} + \frac{g^2}{4c_\theta^2(s - M_0^2)} v_- v_- \right|^2 (p_2 \cdot p_3)^2$$
$$\left. +2 \left| \frac{e^2}{s} + \frac{g^2}{4c_\theta^2(s - M_0^2)} v_+ v_- \right|^2 (p_1 \cdot p_3)^2 \right\}. \tag{9.46}$$

In terms of Mandelstam variables

$$t = (p_1 - p_3)^2 = -2(p_1 \cdot p_3) = -2(p_2 \cdot p_4),$$
$$u = (p_1 - p_4)^2 = -2(p_1 \cdot p_4) = -2(p_2 \cdot p_3), \tag{9.47}$$

the amplitude becomes

$$|\bar{\mathcal{M}}|^2 = \frac{e^4}{s^2} \left\{ \left| 1 + \frac{s}{4s_\theta^2 c_\theta^2 (s - M_0^2)} v_+^2 \right|^2 u^2 + \left| 1 + \frac{s}{4s_\theta^2 c_\theta^2 (s - M_0^2)} v_-^2 \right|^2 u^2 \right.$$
$$\left. + 2 \left| 1 + \frac{s}{4s_\theta^2 c_\theta^2 (s - M_0^2)} v_+ v_- \right|^2 t^2 \right\}. \tag{9.48}$$

To describe the peak $s \sim M_0^2$ one introduces the Z width as follows

$$M_0^2 \to M_0^2 - i\Gamma_Z M_0.$$

Therefore, defining

$$\chi_Z = \frac{s}{4s_\theta^2 c_\theta^2 (s - M_0^2 + i\Gamma_Z M_0)}$$

gives

$$|\bar{\mathcal{M}}|^2 = \frac{e^4}{s^2}\Big\{2\big[u^2+t^2\big] + |\chi_Z|^2\big[u^2(v_+^4+v_-^4) + 2t^2 v_+^2 v_-^2\big] \\ +2\,\Re e\,\chi_Z\big[u^2(v_+^2+v_-^2)+2t^2 v_+ v_-\big]\Big\}. \tag{9.49}$$

In the center-of-mass frame

$$p_1 = \frac{\sqrt{s}}{2}(1,1,0,0), \quad p_2 = \frac{\sqrt{s}}{2}(1,-1,0,0), \quad p_3 = \frac{\sqrt{s}}{2}(1, c_{\theta'}, s_{\theta'}s_\varphi, s_{\theta'}c_\varphi), \tag{9.50}$$

one computes

$$t = -\frac{s}{2}(1-c_{\theta'}), \quad u = -\frac{s}{2}(1+c_{\theta'}).$$

The 2-body phase space is

$$\int d\phi_2 = \frac{\pi}{4}\int_{-1}^{1} dc_{\theta'}, \tag{9.51}$$

so that the differential cross section reads

$$\frac{d\sigma}{dc_{\theta'}} = \frac{1}{32\pi s}|\bar{\mathcal{M}}|^2. \tag{9.52}$$

Let us take the pure QED limit to begin with. That means $M_0 \to \infty$, namely $\chi_Z \to 0$. Then

$$|\bar{\mathcal{M}}|^2 = 16\pi^2\alpha^2(1+c_{\theta'}^2). \tag{9.53}$$

Thus the differential QED cross section is

$$\frac{d\sigma}{dc_{\theta'}} = \frac{\pi\alpha^2}{2s}(1+c_{\theta'}^2). \tag{9.54}$$

Note that $d\sigma/dc_{\theta'}$ is symmetric when $c_{\theta'} \to -c_{\theta'}$. The total QED cross section is easily computed by integrating the previous equation

$$\sigma = \frac{4}{3}\pi\frac{\alpha^2}{s}. \tag{9.55}$$

Now we take the full result parametrized as

$$
\begin{aligned}
|\bar{\mathcal{M}}|^2 &= 16\pi^2\alpha^2 \left\{A(1+c_{\theta'}^2) + Bc_{\theta'}\right\}, \quad \text{with} \\
A &= 1 + \frac{|\chi_Z|^2}{4}(v_+^2+v_-^2)^2 + \frac{\Re e\,\chi_Z}{2}(v_+ + v_-)^2, \\
B &= \frac{|\chi_Z|^2}{2}(v_+^2-v_-^2)^2 + \Re e\,\chi_Z(v_+ - v_-)^2.
\end{aligned} \tag{9.56}
$$

The differential cross section then reads

$$
\frac{d\sigma}{dc_{\theta'}} = \frac{\pi\alpha^2}{2s}\left\{A(1+c_{\theta'}^2) + Bc_{\theta'}\right\}. \tag{9.57}
$$

Now we have an asymmetry when $c_{\theta'} \to -c_{\theta'}$, so we define a forward-backward asymmetry as follows

$$
\Delta_{FB} = \frac{1}{\sigma}\left\{\int_0^1 dc_{\theta'}\frac{d\sigma}{dc_{\theta'}} - \int_{-1}^0 dc_{\theta'}\frac{d\sigma}{dc_{\theta'}}\right\} = \frac{3}{8}\frac{B}{A}, \tag{9.58}
$$

while the total cross section is given by

$$
\sigma = \int_{-1}^1 dc_{\theta'}\frac{d\sigma}{dc_{\theta'}} = \frac{4}{3}\pi\frac{\alpha^2}{s}A. \tag{9.59}
$$

When $s \sim M_0^2$ one derives the asymmetry $\Delta_{FB} = 3\left(\frac{av}{a^2+v^2}\right)^2$, which can be used to determine the Weinberg angle. The observables in (9.58) and (9.59) have been measured with very high precision at LEP.

9.4 Problem*: The W decay width

Compute the total decay width Γ_W of the W boson to massless fermions in terms of G_F, M_Z and M_W.

Chapter 10

The Fermi Lagrangian

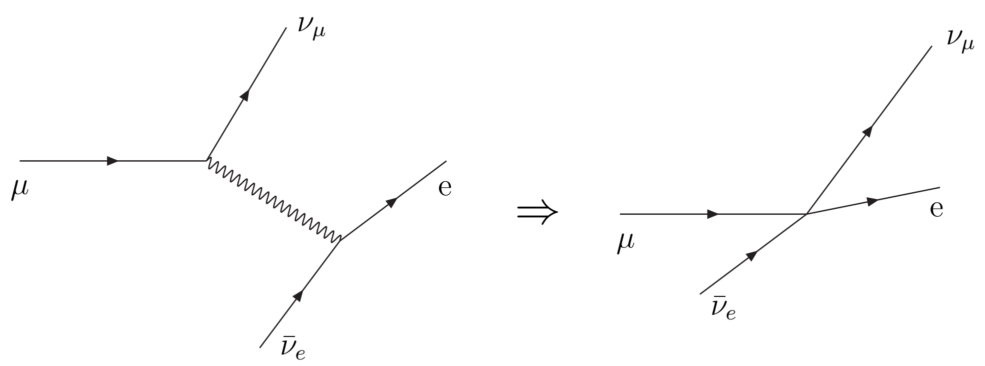
At low energies the electroweak Standard Model reduces to the 4-fermion contact interactions described by the Fermi Lagrangian. In this chapter, we discuss this limit in detail.

10.1 Charged currents

In the $M_W \to \infty$ limit, which is equivalent to the low energy regime we are interested in, the μ decay amplitude computed in chapter 9 can be also generated by an effective Lagrangian

$$\mathcal{L}^{eff} = \frac{G_F}{\sqrt{2}}\,\bar{\nu}_\mu \gamma_\alpha (1-\gamma_5)\mu\,\bar{e}\gamma^\alpha (1-\gamma_5)\nu_e, \tag{10.1}$$

in which the exchanged W is replaced by a contact interaction among four fermions,



By including all quarks and leptons, the effective 4-fermion Fermi Lagrangian involving charged currents reads

$$\mathcal{L}^c_{\rm F} = \frac{G_F}{\sqrt{2}} J^\dagger_{c\alpha} J^\alpha_c, \tag{10.2}$$

where

$$J^\alpha_c = \bar{\nu}_e \gamma^\alpha (1-\gamma_5) e + \bar{\nu}_\mu \gamma^\alpha (1-\gamma_5)\mu + \bar{\nu}_\tau \gamma^\alpha (1-\gamma_5)\tau + \sum_{i,j=1}^{3} \bar{u}_i \gamma^\alpha (1-\gamma_5) V_{ij} q_j$$

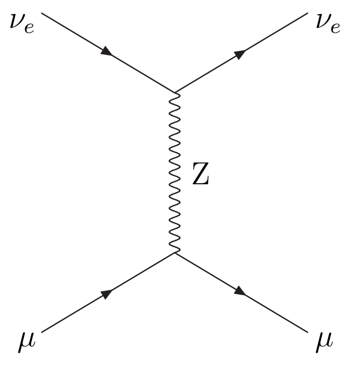
is the total charged current, with

$$\begin{array}{l} q_1 = d, \quad q_2 = s, \quad q_3 = b, \\ u_1 = u, \quad u_2 = c, \quad u_3 = t, \quad V_{ij} = \text{C.K.M. matrix}. \end{array}$$

Therefore, J^α_c contains 12 contributions.

10.2 Neutral currents

The effective 4-fermion Lagrangian involving neutral currents can be derived from the $M_0 \to \infty$ limit of the tree-level $\nu_e \mu \to \nu_e \mu$ amplitude



One obtains

$$\begin{aligned} \mathcal{M} &= \left(\frac{ig}{2c_\theta}\right)^2 (-i) \bar{\nu}_e \gamma_\alpha (v_{\nu_e} + a_{\nu_e}\gamma_5)\nu_e \, \bar{\mu}\gamma^\alpha (v_\mu + a_\mu \gamma_5)\mu \frac{1}{(-M_0^2)} \\ &= -i \frac{g^2}{4{c_\theta}^2 M_0^2} \bar{\nu}_e \gamma_\alpha (v_{\nu_e} + a_{\nu_e}\gamma_5)\nu_e \, \bar{\mu}\gamma^\alpha (v_\mu + a_\mu \gamma_5)\mu. \end{aligned} \tag{10.3}$$

Such an amplitude can be generated by the effective Lagrangian

$$\mathcal{L}^{eff} = \frac{G_F \rho}{\sqrt{2}} J_\mu J^\mu, \qquad (10.4)$$

where

$$J_\alpha = \bar{\nu}_e \gamma_\alpha (v_{\nu_e} + a_{\nu_e}\gamma_5)\nu_e + \bar{\mu}\gamma^\alpha (v_\mu + a_\mu \gamma_5)\mu \quad \text{and} \quad \rho = \frac{M_W^2}{c_\theta^2 M_0^2}. \qquad (10.5)$$

By including all quarks and leptons, the effective neutral 4-fermion Fermi Lagrangian reads

$$\mathcal{L}_{\mathrm{F}}^n = \frac{G_F \rho}{\sqrt{2}} J_{n\alpha} J_n^\alpha, \qquad (10.6)$$

where

$$J_n^\alpha = \sum_f \bar{f}\gamma^\alpha (v_f + a_f \gamma_5) f \qquad (10.7)$$

is the total neutral current containing 12 contributions. Note that $\rho = 1$, when M_W, c_θ, M_0 represent bare parameters of the Standard Model Lagrangian.

10.3 Problem: All possible interactions

Compute the total number of interactions described by the Fermi Lagrangian.

Solution

The complete Fermi Lagrangian is $\mathcal{L}_{\mathrm{F}} = \mathcal{L}_{\mathrm{F}}^c + \mathcal{L}_{\mathrm{F}}^n$, with

$$\mathcal{L}_{\mathrm{F}}^c = \frac{G_F}{\sqrt{2}} J_{c\alpha}^+ J_c^\alpha \quad \text{and} \quad \mathcal{L}_{\mathrm{F}}^n = \frac{G_F \rho}{\sqrt{2}} J_{n\alpha} J_n^\alpha. \qquad (10.8)$$

The current J_c^α contains $n = 12$ contributions. Thus $J_{c\alpha}^+ J_c^\alpha$ generates $\frac{n(n+1)}{2} = 78$ different 4-fermion interactions mediated by charged currents. Analogously, $J^{n\alpha}$ contains $n = 12$ terms, so that $\frac{n(n+1)}{2} = 78$ different neutral 4-fermion interactions are possible. In summary, the total number of interactions between leptons and quarks is 78+78 = 156.

10.4 Problem*: The electroweak interactions among leptons

Write down all possible 4-fermion interactions among leptons generated by the Lagrangian in (10.8).

Chapter 11

Gauge theories

In this chapter we show how the interaction between electrons and photons can be introduced by requiring abelian local gauge invariance. The resulting theory is called quantum electrodynamics (QED), and is described by the QED part of the Lagrangian in (4.7). Extending local gauge invariance to nonabelian transformations leads to the so-called *Yang-Mills* theories. Such theories contain generalizations of the photon called *gauge bosons*. The photon and gauge boson propagators cannot be defined without explicitly breaking gauge invariance. In the last part of this chapter we show how this difficulty can be circumvented.

11.1 Abelian local gauge invariance

Our starting point is the sum of the free Lagrangians describing non-interacting photons and electrons

$$\mathcal{L}^{\mathrm{QED}}_{\mathrm{FREE}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\not{\partial} - m)\Psi, \tag{11.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. From $\mathcal{L}^{\mathrm{QED}}_{\mathrm{FREE}}$, we aim at constructing a Lagrangian invariant under the following infinitesimal *abelian local transformation*

$$\Psi(x) \stackrel{\mathrm{LT}}{\to} (1 + ie\Lambda(x))\Psi(x). \tag{11.2}$$

The transformation $\stackrel{\mathrm{LT}}{\to}$ is *abelian* because it is the infinitesimal version of the U(1) group transformation $\Psi(x) \to \exp\{ie\Lambda(x)\}\Psi(x)$, and *local* because $\Lambda(x)$ is an arbitrary real function of the space-time.

One computes, at the first order in Λ,

$$\mathcal{L}^{\text{QED}}_{\text{FREE}} \stackrel{\text{LT}}{\to} \mathcal{L}^{\text{QED}}_{\text{FREE}} - e(\partial_\mu \Lambda(x))\bar{\Psi}\gamma^\mu \psi. \tag{11.3}$$

The extra piece in the r.h.s. is compensated if one adds to $\mathcal{L}^{\text{QED}}_{\text{FREE}}$ a term

$$\mathcal{L}^{\text{QED}}_{\text{INT}} = -eA_\mu \bar{\Psi}\gamma^\mu \Psi, \tag{11.4}$$

and assumes the following transformation law for the field A_μ. [1]

$$A_\mu \stackrel{\text{LT}}{\to} A_\mu - \partial_\mu \Lambda(x). \tag{11.5}$$

In summary, the Lagrangian

$$\mathcal{L}^{\text{QED}} = \mathcal{L}^{\text{QED}}_{\text{FREE}} + \mathcal{L}^{\text{QED}}_{\text{INT}} \tag{11.6}$$

is invariant under the changes in (11.2) and (11.5). The simultaneous transformations

$$\begin{aligned} \Psi(x) &\stackrel{\text{LT}}{\to} (1 + ie\Lambda(x))\Psi(x) \\ A_\mu &\stackrel{\text{LT}}{\to} A_\mu - \partial_\mu \Lambda(x) \end{aligned} \tag{11.7}$$

are called *local gauge transformations.*

11.2 Problem*: The covariant derivative

Show that the interaction in (11.4) can also be derived from (11.1) by replacing the derivative ∂_μ acting of ψ with a *covariant derivative* D_μ defined as

$$D_\mu = \partial_\mu + ieA_\mu. \tag{11.8}$$

11.3 Nonabelian local gauge invariance

Consider an element U of a unitary group G which acts on a multicomponent fields Ψ_i according to the following transformation

$$\Psi_i(x) \stackrel{\text{LT}}{\to} U_{ij}\Psi_j(x). \tag{11.9}$$

[1]Note that $F_{\mu\nu}$ is invariant under the change in (11.5).

The matrix U_{ij} can be written in terms of the group generators T^a_{ij} as follows

$$U = \exp\left(igT^a\lambda^a(x)\right), \tag{11.10}$$

where $\lambda^a(x)$ are arbitrary real functions of the space-time and

$$[T^a, T^b] = ic^{abc}T^c. \tag{11.11}$$

If the structure constants c^{abc} are different from zero, [2] the group G is nonabelian and the transformation in (11.9) is called *nonabelian local transformation.*

Starting from the free fermion Lagrangian and the generalization of (11.8), both written in matrix notation,

$$\mathcal{L}^{\text{ferm}}_{\text{FREE}} = \bar{\Psi}_j(i\partial\!\!\!/ - m)\Psi_j, := \bar{\Psi}(i\partial\!\!\!/ - m)\Psi, \tag{11.12}$$

$$(D_\mu)_{jk} = \delta_{jk}\partial_\mu + igA^a_\mu(T^a)_{jk} := \partial_\mu + igA_\mu, \tag{11.13}$$

we look for the nonabelian equivalent of (11.7), where the A^a_μ are called *gauge boson fields.*

The replacement $\partial\!\!\!/ \to D\!\!\!\!/$ in (11.12) gives

$$\mathcal{L}^{\text{ferm}}_{\text{FREE}} \to \mathcal{L}^{\text{ferm}}_{\text{INT}} = \bar{\Psi}(iD\!\!\!\!/ - m)\Psi. \tag{11.14}$$

The request of invariance of $\mathcal{L}^{\text{ferm}}_{\text{INT}}$ under the transformation in (11.9) implies the following transformation law

$$A_\mu \overset{\text{LT}}{\to} UA_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}. \tag{11.15}$$

As for the kinetic term of the gauge bosons, one adds to $\mathcal{L}^{\text{ferm}}_{\text{INT}}$ the invariant combination

$$\mathcal{L}^{\text{YM}} = -\frac{1}{4N_R}Tr(F_{\mu\nu}F^{\mu\nu}), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\nu, A_\mu], \tag{11.16}$$

where $Tr(T^aT^b) = N_R\delta^{ab}$. [3] In terms of fields, (11.16) is equivalent to

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gc^{abc}A^b_\mu A^c_\nu. \tag{11.17}$$

In summary, the Lagrangian

$$\mathcal{L}^{\text{INV}} = \mathcal{L}^{\text{ferm}}_{\text{INT}} + \mathcal{L}^{\text{YM}} \tag{11.18}$$

[2] We assume c^{abc} to be antisymmetric for exchanges of any two indices.

[3] N_R depends on the representation of the group G used for the matrices T.

is invariant under the simultaneous transformation in (11.9) and (11.15). This invariance is called *nonabelian local gauge invariance*.

The infinitesimal versions of (11.9) and (11.15) are

$$\begin{aligned} \Psi &\stackrel{\text{LT}}{\to} (1 + ig\Lambda(x))\Psi, \\ A_\mu &\stackrel{\text{LT}}{\to} A_\mu - \partial_\mu \Lambda(x) - ig[A_\mu, \Lambda(x)], \end{aligned} \tag{11.19}$$

where $\Lambda(x) := \lambda^a(x) T^a$. In terms of the fields A^a_μ, (11.19) gives

$$A^a_\mu \stackrel{\text{LT}}{\to} A^a_\mu - \partial_\mu \lambda^a(x) - g c^{abc} \lambda^b(x) A^c_\mu. \tag{11.20}$$

11.4 Problem: The nonabelian invariance

Prove that $\mathcal{L}^{\text{INV}}$ does not change under the transformations in (11.9) and (11.15).

11.5 Solution

We first consider $\mathcal{L}^{\text{ferm}}_{\text{INT}}$. The vector D_μ transforms as Ψ

$$D_\mu \Psi = [\partial_\mu + igA_\mu]\Psi \stackrel{\text{LT}}{\to} \left[\partial_\mu + ig\left(UA_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}\right)\right] U\Psi = U(D_\mu \Psi).$$

Thus

$$\mathcal{L}^{\text{ferm}}_{\text{INT}} = \bar{\Psi}(i\not{D} - m)\Psi \stackrel{\text{LT}}{\to} \bar{\Psi} U^{-1} U (i\not{D} - m)\Psi = \mathcal{L}^{\text{ferm}}_{\text{INT}}. \tag{11.21}$$

As for $\mathcal{L}^{\text{YM}}$, we first use the infinitesimal transformation in (11.19) to compute how $F^{\mu\nu}$ changes at the first order in $\Lambda(x)$

$$\begin{aligned} F^{\mu\nu} &\stackrel{\text{LT}}{\to} \partial_\mu\big(A_\nu - \partial_\nu\Lambda(x) - ig[A_\nu, \Lambda(x)]\big) - \partial_\nu\big(A_\mu - \partial_\mu\Lambda(x) - ig[A_\mu, \Lambda(x)]\big) \\ &\quad -ig\big([A_\nu, A_\mu] - \big[A_\nu, \partial_\mu\Lambda(x) + ig[A_\mu, \Lambda(x)]\big] - \big[\partial_\nu\Lambda(x) + ig[A_\nu, \Lambda(x)], A_\mu\big]\big) \\ &= F^{\mu\nu} - ig[F^{\mu\nu}, \Lambda(x)], \end{aligned} \tag{11.22}$$

where we have used the Jacobi identity. This gives

$$Tr(F_{\mu\nu}F^{\mu\nu}) \stackrel{\text{LT}}{\to} Tr\big(F_{\mu\nu}F^{\mu\nu} - igF_{\mu\nu}[F^{\mu\nu}, \Lambda(x)] - ig[F_{\mu\nu}, \Lambda(x)]F^{\mu\nu}\big) = Tr(F_{\mu\nu}F^{\mu\nu}),$$

where the last equality follows from the cyclic property of the trace. Therefore $\mathcal{L}^{\text{YM}} \stackrel{\text{LT}}{\to} \mathcal{L}^{\text{YM}}$.

11.6 The Physics content of $\mathcal{L}^{\rm YM}$

The Lagrangian $\mathcal{L}^{\rm YM}$ describes massless spin-1 self-interacting gauge bosons A^a_μ. In fact, the request of local gauge invariance led to the $F^{\mu\nu}$ in (11.16), which generates 3- and 4-particle interaction vertices among the A^a_μ. Note that it *is not* possible to insert *by hand* a mass term

$$\mathcal{L}^{\rm YM} \to \mathcal{L}^{\rm YM} - \frac{1}{2} M_A^2 A^a_\mu A^{a\mu}, \tag{11.23}$$

because the combination $A^a_\mu A^{a\mu}$ is not invariant under the local gauge transformations in (11.19),

$$A^a_\mu A^{a\mu} = \frac{1}{N_R} Tr(A_\mu A^\mu) \overset{\rm LT}{\to} \frac{1}{N_R} Tr(A_\mu A^\mu) - \frac{2}{N_R} Tr(A_\mu(\partial^\mu \Lambda(x))). \tag{11.24}$$

11.7 Gauge fixing and ghost fields

The part of $\mathcal{L}^{\rm YM}$ quadratic in the fields A^a_μ does not admit an inverse (see problem 4.4). Thus, the propagators of the gauge bosons cannot be defined. This can be understood because obtaining the equation of motions by imposing $\delta {\rm S} = 0$ does not make sense if S is invariant under a large class of transformations (the gauge transformations). To quantize the theory one has to break gauge invariance by introducing in the Lagrangian an explicit *gauge fixing* term $\mathcal{L}_{\rm GF}$ such that $\mathcal{L}_{\rm GF} \overset{\rm LT}{\to} \mathcal{L}'_{\rm GF} \neq \mathcal{L}_{\rm GF}$, but in a way that Physics do not depend on $\mathcal{L}_{\rm GF}$. This is obtained by choosing [4]

$$\mathcal{L}_{\rm GF} = -\frac{1}{2} C^a C^a, \tag{11.25}$$

where C^a is any non-singular expression which transforms as

$$C^a \overset{\rm LT}{\to} C^a + M^{ab} \lambda^b(x), \tag{11.26}$$

and adding an additional *ghost* term $\mathcal{L}_{\rm Ghost}$ defined as

$$\mathcal{L}_{\rm Ghost} = \bar{\eta}^a M^{ab} \eta^b, \tag{11.27}$$

where η and $\bar{\eta}$ are anticommuting fields [4] that can only appear in loops. In summary, a good Lagrangian to start the quantization is obtained from $\mathcal{L}^{\rm YM}$ as follows

$$\mathcal{L}^{\rm YM} \to \mathcal{L}^{\rm YM} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm Ghost}. \tag{11.28}$$

[4]This means that a minus sign has to be associated to each ghost loop.

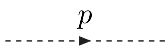
11.8 Problem: $\mathcal{L}_{\text{Ghost}}$ in QED

Show that the choice $C^a = \partial_\mu A^\mu$ in QED implies that one can safely take $\mathcal{L}_{\text{Ghost}} = 0$.

Solution

In QED $a = 1$, so that we relabel $C := C^a$. Under the abelian transformation in (11.5) one has $C \overset{\text{T}}{\to} C - \partial^2\Lambda(x)$, which gives $M := M^{ab} = -\partial^2$. Hence

$$\mathcal{L}_{\text{Ghost}} = -\bar{\eta}\partial^2\eta. \tag{11.29}$$

The only content of (11.29) is a ghost propagator 

$= i/p^2$, which does not interact with any field. Thus, $\mathcal{L}_{\text{Ghost}}$ can be neglected.

11.9 Problem: $\mathcal{L}_{\text{Ghost}}$ in QCD

Derive the ghost Lagrangian in (4.22) from the gauge fixing Lagrangian in (4.21).

Solution

In QCD the gauge fields $A^{\mu a}$ are the gluon fields $G^{\mu a}$ and the gauge group is SU(3). Equation (4.21) corresponds to the choice $C^a = \partial_\mu G^{\mu a}$. The nonabelian transformation in (11.20) gives $C^a \overset{\text{LT}}{\to} C^a - \partial^2\delta^{ab}\lambda(x) - gc^{abc}\partial_\mu(G^{\mu c}\lambda^b(x))$. Thus

$$M^{ab} = -\partial^2\delta^{ab} - gc^{abc}\partial_\mu G^{\mu c}, \tag{11.30}$$

which gives the ghost Lagrangian of (4.22).

Chapter 12

The electroweak Standard Model

In this chapter we write down the full Standard Model electroweak Lagrangian $\mathcal{L}^{\text{SM}}$. After introducing the Higgs mechanism, we keep track of the terms which produce the W and Z masses. As for the fermionic part of $\mathcal{L}^{\text{SM}}$, we explicitly deduce the couplings between gauge bosons and fermions, and explain how fermion masses are generated by interactions among Higgs doublets and fermions. Finally, we discuss the gauge fixing needed to quantize $\mathcal{L}^{\text{SM}}$.

12.1 The bosonic part of the Lagrangian

The bosonic part of $\mathcal{L}^{\text{SM}}$ reads as follows

$$\mathcal{L}_{\text{Bos}} = -\frac{1}{4}F^0_{\alpha\beta}F^{0\,\alpha\beta} - \frac{1}{4}F^a_{\alpha\beta}F^{a\,\alpha\beta} + (D^\alpha K)^\dagger(D_\alpha K) - \mu^2(K^\dagger K) - \lambda(K^\dagger K)^2. \quad (12.1)$$

The field strength tensors in (12.1) are defined in terms of a U(1) singlet vector field B^0_α and an SU(2) triplet B^a_α ($a = 1 \div 3$),

$$\begin{aligned} F^0_{\alpha\beta} &= \partial_\alpha B^0_\beta - \partial_\beta B^0_\alpha, \\ F^a_{\alpha\beta} &= \partial_\alpha B^a_\beta - \partial_\beta B^a_\alpha - g\epsilon^{abc}B^b_\alpha B^c_\beta, \end{aligned} \quad (12.2)$$

where ϵ^{abc} is the SU(2) structure constant (see section 13.1), and g the SU(2) coupling. The field K is an SU(2) complex doublet

$$K = \begin{pmatrix} \phi^+ \\ \phi_0 + \frac{i}{\sqrt{2}}\phi_3 \end{pmatrix}, \quad (12.3)$$

and the covariant derivative acting on K is

$$D_\alpha K = \left(\partial_\alpha + ig\frac{\tau^a}{2}B^a_\alpha + ig'\frac{Y(K)}{2}B^0_\alpha\right)K. \tag{12.4}$$

The *hypercharge* Y is defined as

$$Y = 2(Q - I_3), \tag{12.5}$$

where Q is the electric charge and I_3 the third *isospin* component. Thus, $Y(K) = 1$. The constant g' is the U(1) coupling and $\tau^a := \sigma^a$ are the three Pauli matrices defined in (13.1).

By construction, $\mathcal{L}_{\text{Bos}}$ is invariant under the following infinitesimal SU(2)×U(1) local gauge transformations [1]

$$\begin{aligned} B^0_\alpha &\overset{\text{LT}}{\to} B^0_\alpha - \partial_\alpha\lambda^0(x), \\ B^a_\alpha &\overset{\text{LT}}{\to} B^a_\alpha - \partial_\alpha\lambda^a(x) - g\epsilon^{abc}\lambda^b(x)B^c_\alpha, \\ K &\overset{\text{LT}}{\to} \left(1 + ig\frac{\tau^a}{2}\lambda^a(x) + ig'\frac{Y(K)}{2}\lambda^0(x)\right)K. \end{aligned} \tag{12.6}$$

12.2 The Higgs mechanism

Consider the potential given by the last two terms of (12.1),

$$V(K) := \mu^2(K^\dagger K) + \lambda(K^\dagger K)^2. \tag{12.7}$$

If the field ϕ_0 in (12.3) develops a *vacuum expectation value* v, namely

$$\phi_0 = \frac{1}{\sqrt{2}}(H + v) \quad \text{with} \quad v = \text{constant}, \tag{12.8}$$

one rewrites

$$K \sim \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ H + v \end{pmatrix}, \tag{12.9}$$

where the symbol $\sim$ means that we neglect contributions proportional to the fields $\phi^\pm$ or ϕ_3. [2] Inserting this in (12.7) gives

$$V(K) \sim \frac{v^2}{2}\left(\mu^2 + \frac{\lambda v^2}{2}\right) + \left(\mu^2 v + \lambda v^3\right)H + \left(\mu^2 + 3\lambda v^2\right)\frac{H^2}{2} + \lambda v H^3 + \frac{\lambda}{4}H^4. \tag{12.10}$$

[1] Cfr. (11.19) and (11.20).

[2] They play the role of the longitudinal polarizations of $W^\pm$ and Z (see section 12.5). To simplify our discussion we do not include them here.

In the following, we discuss, in turn, the five contributions in (12.10). The first term is an irrelevant constant. As for the second one, the field H is physical if the coefficient of H vanish. [3] This happens when

$$\begin{aligned} v &= 0, \quad \text{or} & (12.11) \\ \mu^2 &= -\lambda v^2. & (12.12) \end{aligned}$$

The solution with $v \neq 0$ drives the so called *spontaneous symmetry breaking*, [4] and H is the Higgs field. Inserting (12.12) in the third piece gives the Higgs mass

$$M_H^2 = 2\lambda v^2. \tag{12.13}$$

This implies $\lambda > 0$, so that $\mu^2 < 0$ in (12.12). Finally, the last two contributions are the trilinear and quartic Higgs boson self couplings, respectively. Note that the whole Higgs potential $V(K)$ depends on two free parameters, which can be taken to be v and M_H,

$$V(K) \sim \frac{1}{2}M_H^2 H^2 + \frac{M_H^2}{2v}H^3 + \frac{M_H^2}{8v^2}H^4. \tag{12.14}$$

12.3 The W and Z masses

The masses of the $W^\pm$ and Z bosons are generated by the $(D^\alpha K)^\dagger(D_\alpha K)$ term in (12.1). One computes

$$\begin{aligned} (D_\alpha K) &\sim \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \partial_\alpha H \end{pmatrix} \\ &+ \frac{ig}{2\sqrt{2}}(H+v)\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B_\alpha^1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} B_\alpha^2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} B_\alpha^3 + \frac{g'}{g}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} B_\alpha^0\right]\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \partial_\alpha H \end{pmatrix} + \frac{ig}{2\sqrt{2}}(H+v)\begin{pmatrix} \sqrt{2}W_\alpha^+ \\ \frac{g'}{g}B_\alpha^0 - B_\alpha^3 \end{pmatrix}, \end{aligned} \tag{12.15}$$

where the $W^\pm$ fields are defined as

$$W_\alpha^\pm = \frac{1}{\sqrt{2}}(B_\alpha^1 \mp iB_\alpha^2). \tag{12.16}$$

[3] Otherwise H particles could be generated and/or absorbed by the vacuum.

[4] The symmetry that is broken is the minimum of the potential $V(K)$, which is not any longer in $K = 0$.

The structure of the last term in (12.15) suggests to introduce the Z and A fields as rotations of the B^0 and B^3 fields. This is achieved by defining

$$\frac{g'}{g} = \frac{s_\theta}{c_\theta}, \tag{12.17}$$

in which s_θ (c_θ) is the sine (cosine) of an angle dubbed *weak mixing angle*. Thus $\frac{g'}{g} B^0_\alpha - B^3_\alpha = -\frac{1}{c_\theta} Z_\alpha$, where

$$\begin{pmatrix} Z_\alpha \\ A_\alpha \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} B^3_\alpha \\ B^0_\alpha \end{pmatrix}, \quad \begin{pmatrix} B^3_\alpha \\ B^0_\alpha \end{pmatrix} = \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} Z_\alpha \\ A_\alpha \end{pmatrix}, \tag{12.18}$$

so that

$$(D^\alpha K)^\dagger (D_\alpha K) \sim \frac{1}{2}(\partial_\alpha H)(\partial^\alpha H) + \frac{g^2}{4}(H+v)^2 W^+_\alpha W^{-\,\alpha} + \frac{g^2}{8c_\theta^2}(H+v)^2 Z_\alpha Z^\alpha. \tag{12.19}$$

Hence, the gauge boson masses are

$$M_W^2 = \frac{g^2 v^2}{4} \quad \text{and} \quad M_Z^2 = \frac{M_W^2}{c_\theta^2}, \tag{12.20}$$

while the photon field A remains massless. In summary, the part of the bosonic Lagrangian quadratic in the gauge fields reads

$$\begin{aligned} \mathcal{L}^{(2)}_{\text{Bos}} &= -\frac{1}{4}\sum_{j=0}^{3}(\partial_\alpha B^j_\beta - \partial_\beta B^j_\alpha)(\partial^\alpha B^{j\beta} - \partial^\beta B^{j\alpha}) + M_W^2 W^+_\alpha W^{-\alpha} + \frac{M_Z^2}{2} Z_\alpha Z^\alpha \\ &= \mathcal{L}_{\text{YM,A}} + \mathcal{L}^{(2)}_{\text{YM,Z}} + \mathcal{L}^{(2)}_{\text{YM,W}} + M_W^2 W^+_\alpha W^{-\alpha} + \frac{M_Z^2}{2} Z_\alpha Z^\alpha, \end{aligned} \tag{12.21}$$

with $\mathcal{L}_{\text{YM,A}}$, $\mathcal{L}^{(2)}_{\text{YM,Z}}$ and $\mathcal{L}^{(2)}_{\text{YM,W}}$ listed in (4.11).

Finally, note that the model predicts the couplings HW^+W^-, HHW^+W^-, HZZ, $HHZZ$, and that the first terms in (12.14) and (12.19) give the H propagator.

12.4 The fermionic part of the Lagrangian

The part of $\mathcal{L}^{\text{SM}}$ generating the couplings between fermions [5]

$$\Psi_L = \begin{pmatrix} f_L \\ f'_L \end{pmatrix}, \quad f_R, \quad f'_R, \quad f^{(\prime)}_{L,R} := \frac{1}{2}(1 \mp \gamma_5) f^{(\prime)} \tag{12.22}$$

[5]The fields f_L and f'_L denote the isospin 1/2 and -1/2 left-handed components of the SU(2) doublet, while f_R and f'_R are right-handed singlets.

and gauge boson fields reads

$$\mathcal{L}_f = \bar{\Psi}_L(i\not{D})\Psi_L + \bar{f}_R(i\not{D})f_R + \bar{f}'_R(i\not{D})f'_R. \tag{12.23}$$

The covariant derivatives which makes $\mathcal{L}_f$ invariant under SU(2)×U(1) local gauge transformations are

$$\begin{aligned} D_\alpha \Psi_L &= \left(\partial_\alpha + ig\frac{\tau^a}{2}B^a_\alpha + ig'\frac{Y(\Psi_L)}{2}B^0_\alpha\right)\Psi_L, \\ D_\alpha f^{(\prime)}_R &= \left(\partial_\alpha + ig'\frac{Y(f^{(\prime)})}{2}B^0_\alpha\right) f^{(\prime)}_R. \end{aligned} \tag{12.24}$$

Using (12.16), (12.17) and (12.18) in (12.23) gives

$$\begin{aligned} \mathcal{L}_f = \bar{f}(i\not{\partial})f + \bar{f}'(i\not{\partial})f' - \frac{g}{2\sqrt{2}}W^+_\alpha \bar{f}\gamma^\alpha(1-\gamma_5)f' - \frac{g}{2\sqrt{2}}W^-_\alpha \bar{f}'\gamma^\alpha(1-\gamma_5)f \\ -gs_\theta Q_f A_\alpha \bar{f}\gamma^\alpha f - gs_\theta Q_{f'} A_\alpha \bar{f}'\gamma^\alpha f' - \frac{g}{2c_\theta}Z_\alpha \bar{f}\gamma^\alpha(v_f + a_f\gamma_5)f, \end{aligned} \tag{12.25}$$

with v_f and a_f in (4.9). Inserting color indices and summing over all fermions leads to the couplings in (4.8).

Fermion masses are generated by adding to $\mathcal{L}_f$ a contribution $\mathcal{L}_Y$ containing gauge invariant Yukawa interactions between K and the fields in (12.22), [6]

$$\mathcal{L}_Y = -\lambda_{f'}\bar{\Psi}_L K f'_R - \lambda_f \bar{\Psi}_L \tilde{K} f_R + \text{h.c.} \tag{12.26}$$

where

$$\tilde{K} := i\tau^2 K^* \quad (\tau^2 \text{ is the second Pauli matrix and } Y(\tilde{K}) = -1). \tag{12.27}$$

Using (12.9) gives

$$\mathcal{L}_Y \sim -\frac{H+v}{\sqrt{2}}\left(\lambda_{f'}\bar{f}'f' + \lambda_f \bar{f}f\right). \tag{12.28}$$

Hence

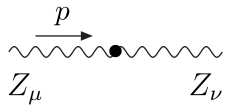
$$m_{f'} = \frac{v\lambda_{f'}}{\sqrt{2}} \quad \text{and} \quad m_f = \frac{v\lambda_f}{\sqrt{2}}. \tag{12.29}$$

Note that $\mathcal{L}_Y$ predicts interactions between H and massive fermions.

[6] Inserting by hand fermion masses in $\mathcal{L}_f$ would break gauge invariance.

12.5 Fixing the gauge

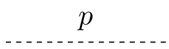
Here we consider the problem of defining the gauge boson propagators by discussing in detail the case of the Z. The 2-point vertex one reads from (12.21) is non singular



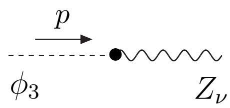
$$= -i\big[(p^2 - M_Z^2)\, g^{\mu\nu} - p^\mu p^\nu\big]. \tag{12.30}$$

However, the last three terms of (12.1) produce zero-order interactions between the gauge bosons and the fields $\phi^\pm$ and ϕ_3 in (12.3). In particular, the part of $\mathcal{L}_{\text{Bos}}$ quadratic in the ϕs or in their products with the gauge fields reads

$$\begin{aligned} \mathcal{L}_\phi^{(2)} &= (\partial_\mu \phi^+)(\partial^\mu \phi^-) - M_W\big[(\partial_\mu \phi^+)W^{-\mu} + (\partial_\mu \phi^-)W^{+\mu}\big] \\ &+\frac{1}{2}(\partial_\mu \phi_3)(\partial^\mu \phi_3) - M_Z(\partial_\mu \phi_3)Z^\mu. \end{aligned} \tag{12.31}$$

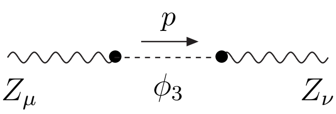
The last two terms give rise to the ϕ_3 propagator 

$= i/p^2$ and to the vertex



$$= -M_Z p^\nu. \tag{12.32}$$

This generates a further contribution,



$$= -ip^\mu p^\nu M_Z^2/p^2,$$

to be added to (12.30). The resulting 2-point Z vertex,

$$V_Z^{\mu\nu} = -i(p^2 - M_Z^2)\left[g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right], \tag{12.33}$$

is singular and requires the addition of a gauge fixing term. A convenient choice is

$$\begin{aligned} \mathcal{L}_{\text{GF,Z}} &= -\frac{1}{2}(\partial^\mu Z_\mu + M_Z\phi_3)^2 \\ &= -\frac{1}{2}(\partial^\mu Z_\mu)^2 - (\partial^\mu Z_\mu)M_Z\phi_3 - \frac{1}{2}M_Z^2\phi_3^2. \end{aligned} \tag{12.34}$$

The first two terms cancel the $p^\mu p^\nu$ and p^ν contributions in (12.30) and (12.32), respectively. [7] As a consequence, the final result for the 2-point Z vertex is

$$V_Z'^{\mu\nu} = -i(p^2 - M_Z^2)g^{\mu\nu}, \tag{12.35}$$

[7] The third term generates, instead, a mass M_Z for the ϕ_3 field. This is why ϕ_3 represents the longitudinal polarization of the massive Z boson.

which gives the propagator in (4.17). In an analogous way, adding

$$\mathcal{L}_{\text{GF,W}} = -\frac{1}{2}(\partial^\mu W_\mu^- + M_W \phi^+)^2 - \frac{1}{2}(\partial^\mu W_\mu^+ + M_W \phi^-)^2 \qquad (12.36)$$

produces the W propagator of (4.16).

12.6 Problem∗: The Standard Model ghost Lagrangian

Construct the ghost Lagrangian corresponding to the gauge fixing terms in (12.34) and (12.36).

Chapter 13

The Flavour SU(N) symmetries

In this chapter, we discuss the group SU(N) and its role in the classification of mesonic and baryonic states [5]. Our approach is, once again, a very practical one. Firstly, we recall the SU(2) and SU(3) group algebras and prove them by means of explicit matrix calculus. Secondly, we introduce the representation theory of SU(N), and the Young-Tableaux as a convenient tool for manipulating it. At every step, a few problems are proposed that serve as a link between the introduced mathematical objects and the physical description of the hadrons.

13.1 The SU(2) Algebra

The fundamental representation of SU(2) is given in terms of the 3 Pauli Matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13.1}$$

By introducing $J_i = \frac{\sigma_i}{2}$, the SU(2) algebra can be written as

$$[J_i, J_j] = i\epsilon_{ijk} J_k.$$

13.2 The SU(3) Algebra

The fundamental representation of the SU(3) algebra is given in terms of 8 the Gell-Mann Matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

In this case $t_a = \frac{\lambda_a}{2}$ and the SU(3) algebra is

$$[t_a, t_b] = i f^{abc} t_c,$$

where f^{abc} is totally asymmetric and can only have one of the following values $(0, 1, \frac{1}{2}, -\frac{1}{2}, \sqrt{\frac{3}{2}})$.

13.3 Problem: The SU(2) and SU(3) algebras

a) Verify the SU(2) algebra explicitly.
b) Verify the SU(3) algebra explicitly.
c) Prove that $Tr[\sigma_i] = 0$.
d) Prove that $Tr[\lambda_i] = 0$.
e) Prove that $Tr[t^a t^b] = \frac{\delta^{ab}}{2}$ explicitly.
f) Prove that $f^{abc} = -2i[Tr(t^a t^b t^c) - Tr(t^a t^c c^b)]$.

Note that the relation f) allows one to compute f^{abc}.

Solution

a) To verify the SU(2) algebra explicitly, we only have to expand the commutator:

$$\begin{aligned}[J_1, J_2] &= J_1 J_2 - J_2 J_1 = \frac{1}{4}\left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} - \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\} \\ &= \frac{1}{4}\left\{\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} - \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\right\} = \frac{i}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \frac{i}{2}\sigma_3 = iJ_3 \qquad etc.\end{aligned}$$

b) To prove this, we do exactly the same, but with the Gell-Mann matrices:

$$[t^1, t^2] = [\frac{\lambda^1}{2}, \frac{\lambda^2}{2}] = \frac{1}{4}\left[2i\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\right] = it^3 \qquad etc.$$

c) We can see, by simple inspection, that the trace of the Pauli matrices is zero.

d) As in the above case, by simple inspection we see that the trace of the Gell-Mann matrices is always 0.

e)

$$Tr[t^1 t^1] = Tr[(t^1)^2] = \frac{1}{4}Tr\left[\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right] = \frac{1}{4}Tr\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} = \frac{1}{2}$$

On the other hand:

$$Tr[t^1 t^2] = \frac{1}{4}Tr\left[\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right] = \frac{1}{4}Tr\begin{pmatrix}i & 0 & 0\\ 0 & -i & 0\\ 0 & 0 & 0\end{pmatrix} = 0 \qquad etc.$$

f) We start by multiplying the original expression by t^c on the right:

$$[t^a, t^b]t^c = if^{abd}t^d t^c$$

$$[t^a t^b t^c - t^b t^a t^c] = if^{abd}t^d t^c$$

Now, by taking traces:

$$Tr[t^a t^b t^c - t^b t^a t^c] = Tr[t^a t^b t^c - t^a t^c t^b] = i f^{abd} Tr[t^d t^c]$$

Using what we obtained in the previous section : $Tr[t^a t^b] = \frac{\delta^{ab}}{2}$

$$Tr[t^a t^b t^c - t^a t^c t^b] = i f^{abd} \frac{\delta^{dc}}{2} = \frac{i}{2} f^{abc}$$

So finally, we obtain the expression we where looking for:

$$f^{abc} = -2i \left[Tr(t^a t^b t^c) - Tr(t^a t^c c^b) \right].$$

13.4 Problem: The SU(2) symmetry for protons and neutrons

Prove that the isospin SU(2) symmetry is a good approximated symmetry for protons and neutrons.

Solution

We can put p and n together to form a SU(2) isospin doublet ($T = \frac{1}{2}$):

$$\begin{pmatrix} p \\ n \end{pmatrix},$$

so that they only differ by their T_3 projections:

$$T_3 \begin{pmatrix} p \\ 0 \end{pmatrix} = \frac{\sigma_3}{2} \begin{pmatrix} p \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p \\ 0 \end{pmatrix} = +\frac{1}{2} \begin{pmatrix} p \\ 0 \end{pmatrix}$$

$$T_3 \begin{pmatrix} 0 \\ n \end{pmatrix} = \frac{\sigma_3}{2} \begin{pmatrix} 0 \\ n \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ n \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 \\ n \end{pmatrix},$$

meaning that the proton has third isospin component $= \frac{1}{2}$, while the neutron has third isospin component $= -\frac{1}{2}$. p and n are then related by the "step" operator $T^{\pm} = T_1 \pm iT_2$ as follows

$$|p\rangle = T_+|n\rangle. \tag{13.2}$$

Suppose now $H|n\,\rangle = E|n\,\rangle$, and that $[H, T_i] = 0$, namely that T_i commutes with the Hamiltonian of the system. Then

$$H|p\rangle = HT_+|n\rangle = T_+H|n\rangle = T_+E|n\rangle = E|p\rangle.$$

That means that, if $[H, T_i] = 0$, all the members of an isomultiplet should be *degenerated in mass*. Let us check whether this is true for the isodoublet of p and n:

$$\frac{m_n - m_p}{m_n + m_p} = 0.7 \times 10^{-3}.$$

The SU(2) isospin symmetry is therefore a rather a good symmetry for protons and neutrons.

13.5 Problem: The SU(2) symmetry for pions

Show that the isospin SU(2) symmetry is a good approximated symmetry for the pions $\pi^{\pm}$ and π^0.

Solution

Now the π^0 , π^+ ,π^- can be put into a SU(2) isotriplet ($T = 1$)

$$\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$$

and we can test the symmetry in the same way as in the previous problem by calculating

$$\frac{m_{\pi^{\pm}} - m_{\pi^0}}{m_{\pi^{\pm}} + m_{\pi^0}} = 1.7 \times 10^{-2}.$$

The SU(2) isospin symmetry is therefore still a rather a good symmetry for the 3 pions.

13.6 Products of representations

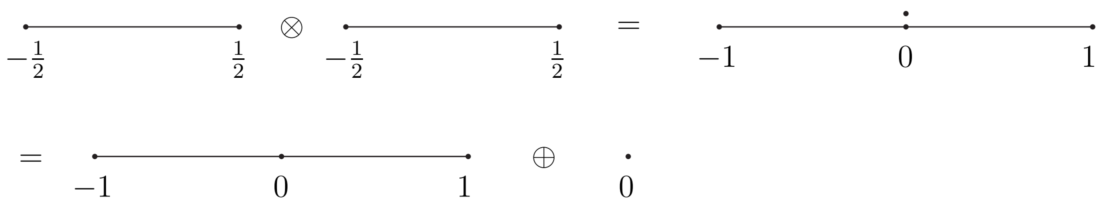
As it should be clear from the two previous examples, representations of different dimensionality of the SU(N) groups exist. Representations of higher dimensionality can be obtained by performing the tensor product of 2 representations of lower dimensionality. This can be seen both graphically and with the help of Young Tableaux.

13.7 Problem: Graphical product of representations

Perform the tensor product $\frac{1}{2} \otimes \frac{1}{2}$ graphically.

Solution

The graphical tensor product of 2 representations is performed by putting the *center* of one representation to coincide with all possible values of the other representation. In the case of SU(2) we obtain



so that

$$\frac{1}{2} \otimes \frac{1}{2} = \mathbf{1} \oplus 0.$$

13.8 Problem: The tensor product $3 \otimes 3^*$

Perform, for SU(3), the tensor product $3 \otimes 3^*$ graphically.

Solution

In this case the fundamental representation is *bi-dimensional* since in SU(3) there are 2 diagonal matrices, namely λ_3 and λ_8.

The two diagonal generators are therefore (remember $F_a = \frac{\lambda_a}{2}$)

$$F_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad F_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

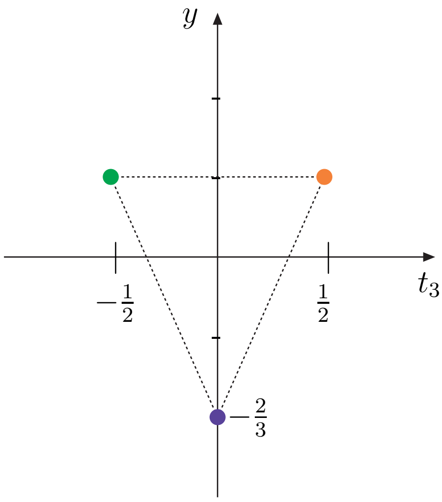
We can define 2 additive quantum numbers:
$T_3 = F_3 \rightarrow$ *Isospin*, and $Y = \frac{2}{\sqrt{3}}F_8 \rightarrow$ *Hypercharge.*

All states can be represented in the (t_3, y) plane, where t_3 and y are the eigenvalues of T_3 and Y, respectively. [1] The possible states are

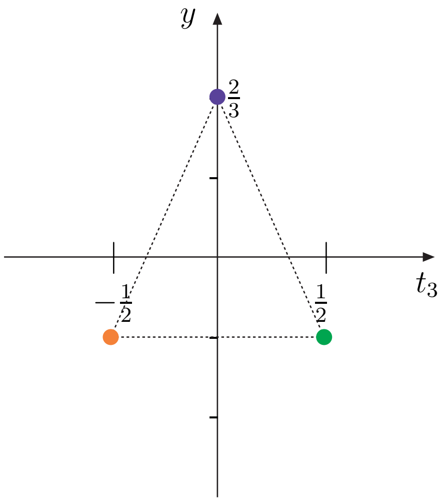
$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{with} \quad t_3 = \frac{1}{2} \quad \text{and} \quad y = \frac{1}{3},$$
$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{with} \quad t_3 = -\frac{1}{2} \quad \text{and} \quad y = \frac{1}{3},$$
$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{with} \quad t_3 = 0 \quad \text{and} \quad y = -\frac{2}{3}. \qquad (13.3)$$

Once we have the eigenvalues, we can represent them graphically in a t_3, y plane as follows

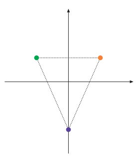
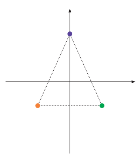
[1] The relations of t_3 and y with baryon number B, strangeness S and charge Q are $Q = t_3 + \frac{y}{2}$ and $y = B + S$.

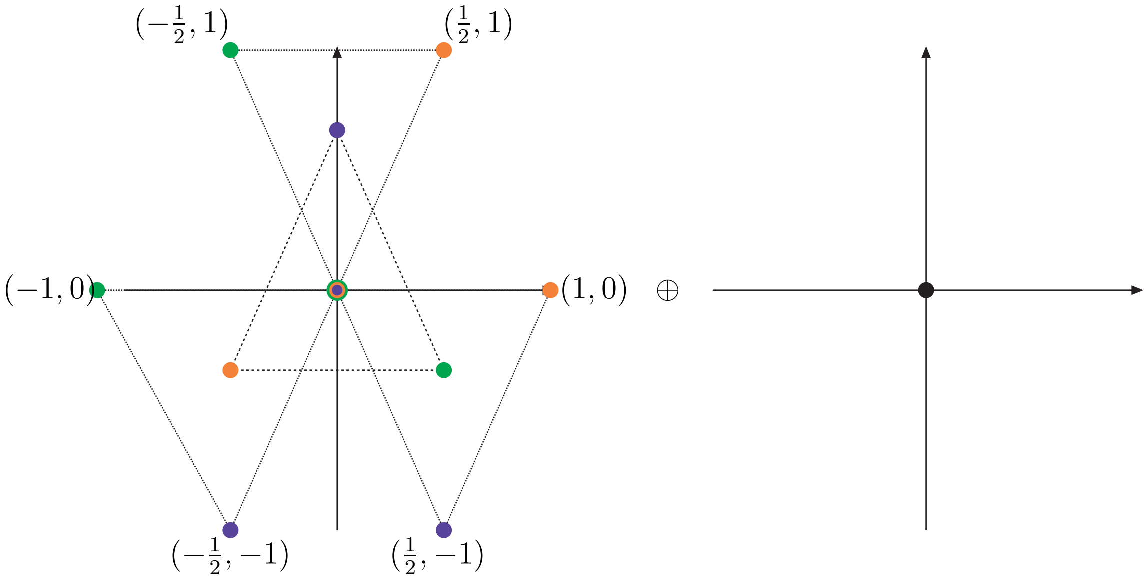


Representation of 3
for quarks



Representation of 3∗
for antiquarks

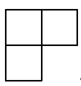
The composition will then be  $\otimes$  $= 3 \otimes 3^* = \mathbf{8} \oplus 1$:



13.9 Young Tableaux

A Young Tableau is a combinatorial object useful in representation theory. It provides a convenient way for describing the group representations of the symmetric and general linear groups and to study their properties. As we will see now the tableau is a finite collection of boxes, or cells, arranged in left-justified rows, with the row sizes weakly decreasing.
When working with the Young Tableaux one has to keep in mind this rules:

- For SU(N) the tableau has *no more* than **N-1** rows
- The length of the lower rows cannot exceed the upper ones.
- The numbers inside the boxes are no decreasing from **left to right** and increasing **top to bottom**

This is an example of a Young Tableau  .

Some important definitions are

- f_1: Length of the first row,
- f_2: Length of the second row,
- $\lambda_1 = f_1 - f_2$,
- $\lambda_2 = f_2 - f_3$.

The dimension of the representation is given by the formula

$$d(\lambda_1, \lambda_2) = (1+\lambda_1)(1+\lambda_2)\left(1+\frac{\lambda_1+\lambda_2}{2}\right).$$

13.10 Problem: *3* and *3** of SU(3)

Find the representations *3* and *3** of SU(3) in terms of Young Tableaux

Solution

The *3* representation of SU(3)

Since we are working in SU(3) the tableau has a maximum of 2 rows. In this case:

$\square \rightarrow \quad f_1 = 1 \quad f_2 = 0 \quad f_3 = 0$

$\Rightarrow \quad \lambda_1 = 1 \quad \lambda_2 = 0$

and the dimension is: $d(1,0) = (1+1)(1)(1+\frac{1}{2}) = 3$.

In fact there are the following 3 possibilities $\boxed{1}$ $\boxed{2}$ $\boxed{3}$.

The *3** representation of SU(3)

The representing Young Tableau is

$\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \rightarrow \quad f_1 = 1 \quad f_2 = 1 \quad f_3 = 0$

$$\Rightarrow \qquad \lambda_1 = 0 \qquad \lambda_2 = 1$$

and the dimension is: $d(0,1) = (1)(1+1)(1+\frac{1}{2}) = 3$.

In fact all the possible combinations are: $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}$.

13.11 Problem: The octet of SU(3)

Show that $\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ is an octet of SU(3).

Solution

The length of the rows is:

$$f_1 = 2 \quad f_2 = 1 \quad f_3 = 0 \quad \Rightarrow \quad \lambda_1 = 1 \qquad \lambda_2 = 1 \tag{13.4}$$

and the dimension is

$$d(\lambda_1, \lambda_2) = d(1,1) = (1+1)(1+1)(1+1) = 8.$$

Explicitly, the eight different possibilities are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

13.12 Problem: The decouplet of SU(3)

Show that $\begin{array}{|c|c|c|}\hline & & \\ \hline\end{array}$ is a decouplet of SU(3).

Solution

As we did before, we start by looking at the row's length

$$f_1 = 3 \quad f_2 = 0 \quad f_3 = 0 \quad \Rightarrow \quad \lambda_1 = 3 \qquad \lambda_2 = 0. \tag{13.5}$$

The dimension of the representation is then

$$d(\lambda_1, \lambda_2) = d(3,0) = (1+3)(1+0)(1+\frac{3}{2}) = 10.$$

Explicitly, the ten different possibilities are

1 1 1	1 1 2	1 1 3	1 2 2	1 2 3
1 3 3	2 2 2	2 2 3	2 3 3	3 3 3.

13.13 Problem: A representation of SU(3) with dimension 6

Show that $\square\square$ has d=6.

Solution

The length of the row is:

$$f_1 = 2 \quad f_2 = 0 \quad f_3 = 0 \quad \Rightarrow \quad \lambda_1 = 2 \qquad \lambda_2 = 0, \tag{13.6}$$

and the dimension is

$$d(\lambda_1, \lambda_2) = d(2,0) = (1+2)(1+0)(1+1) = 6.$$

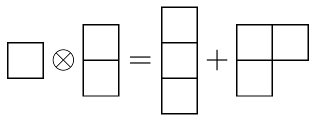
Explicitly:

1 1	1 2	1 3	2 2	2 2	2 3	3 3.

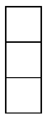
13.14 Prod. of representations and Young Tableaux

The product of representations can be obtained by *adding* one representation to the other in all possible ways that still generate a Young Tableau. For example

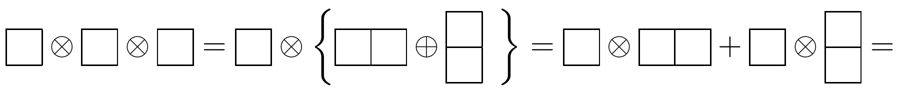
- Mesons : $3 \otimes 3^*$

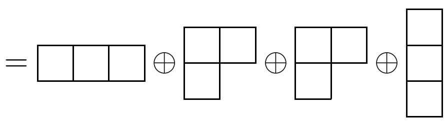
   $\Rightarrow \quad 8 \oplus 1 \rightarrow$

  Meson nonet.

  Note that  is a singolet of SU(3), because it corresponds to the only possibility $\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}$.

- Baryons : $3 \otimes 3 \otimes 3$

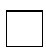
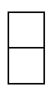
  

   $= 10 \oplus 8 \oplus 8 \oplus 1 \rightarrow$

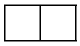
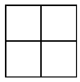
  Baryon decuplet, octets, and singlet.

13.15 Conj. representation and Young Tableaux

To construct the conjugate representation one rotates of 180º the complementary part, namely the part one has to add to obtain N rows in SU(N). For example:

$3 \rightarrow$  $\quad \Rightarrow \quad 3^* \rightarrow$ 

or

$6 \rightarrow$  $\Rightarrow$ $6^* \rightarrow$ 

13.16 Problem: The 0^- mesons

Given $q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$ and $\bar{q} = \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix}$ show that the 0^- mesons form the representation $8 \oplus 1$ of SU(3) in the (t_3, y) plane.

Solution

We know that the constituent of the 0^- meson nonet are

$$\begin{array}{ccc} \pi^+ \sim \bar{d}u & \pi^- \sim \bar{u}d & \pi^0 \sim \frac{1}{\sqrt{2}}\left(\bar{u}u - \bar{d}d\right) \\ K^+ \sim \bar{s}u & K^- \sim \bar{u}s & K^0 \sim \bar{s}d \\ \bar{K^0} \sim \bar{d}s & \eta_0 \sim \frac{1}{\sqrt{6}}\left[\bar{u}u + \bar{d}d - 2\bar{s}s\right] & \eta' \sim \dfrac{\bar{u}u + \bar{d}d + \bar{s}s}{\sqrt{3}} \end{array} . \tag{13.7}$$

We start by looking for the t_3 and y eigenvalues of the quarks:

$$t_3(u) = \frac{1}{2} \quad y(u) = \frac{1}{3} \quad ; \quad t_3(d) = -\frac{1}{2} \quad y(d) = \frac{1}{3} \quad ; \quad t_3(s) = 0 \quad y(s) = -\frac{2}{3}.$$

And for the antiquarks:

$$t_3(\bar{u}) = -\frac{1}{2} \quad y(\bar{u}) = -\frac{1}{3} \quad ; \quad t_3(\bar{d}) = \frac{1}{2} \quad y(\bar{d}) = -\frac{1}{3} \quad ; \quad t_3(\bar{s}) = 0 \quad y(\bar{s}) = \frac{2}{3}.$$

Now, we can obtain the eigenvalues for the mesons since they're additive numbers.

$$\pi^+ \to \left(\frac{1}{2}+\frac{1}{2},0\right) \qquad ; \qquad K^+ \to \left(\frac{1}{2},\frac{2}{3}+\frac{1}{3}\right) = \left(\frac{1}{2},1\right)$$

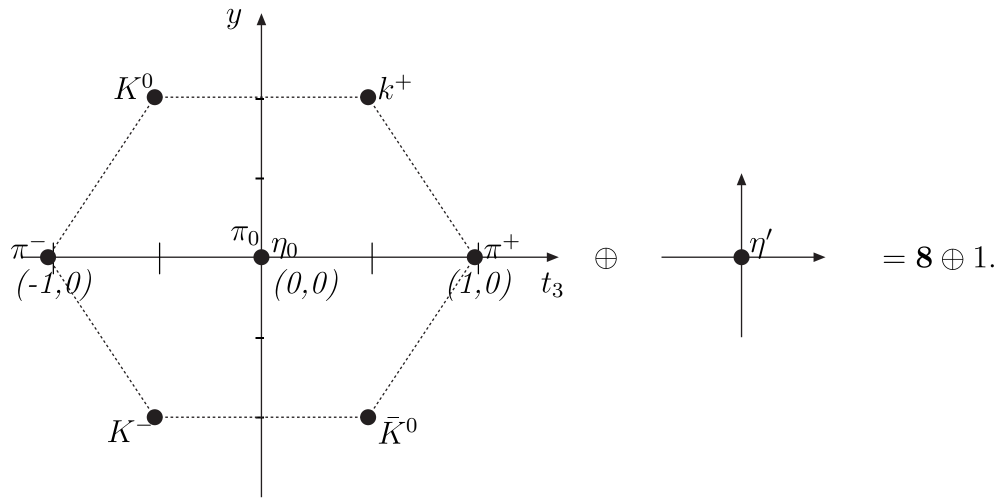
$$\pi^- \to (-1,0) \qquad ; \qquad K^- \to \left(-\frac{1}{2},-\frac{1}{3}-\frac{2}{3}\right) = \left(-\frac{1}{2},-1\right)$$

$$\pi^0 \to \frac{1}{\sqrt{2}}(0,0) \qquad ; \qquad K^0 \to \left(-\frac{1}{2},\frac{2}{3}+\frac{1}{3}\right) = \left(-\frac{1}{2},1\right)$$

$$\bar{K^0} \to \left(\frac{1}{2},-1\right)$$

$$\eta_0 \to (0,0) \qquad ; \qquad \eta' \to (0,0)$$

And finally, we represent the eigenvalues for the nine mesons in the (t_3, y) plane:



We can see that 3 out of the 9 states have quantum numbers $t_3 = y = 0$. These are linear combinations of $u\bar{u}$, $d\bar{d}$, and $s\bar{s}$. The singlet combination must contain each quark flavour on a equal footing, so after normalization we have:

$$\eta' = \frac{1}{\sqrt{3}} \left(u\bar{u} + d\bar{d} + s\bar{s} \right)$$

another one is a member of the isospin triplet, and so:

$$\pi^0 = \frac{1}{\sqrt{2}} \left(u\bar{u} - d\bar{d} \right)$$

By requiring orthogonality to both π^0 and η' we found that the isospin singlet $(T_3 = 0)$ is:

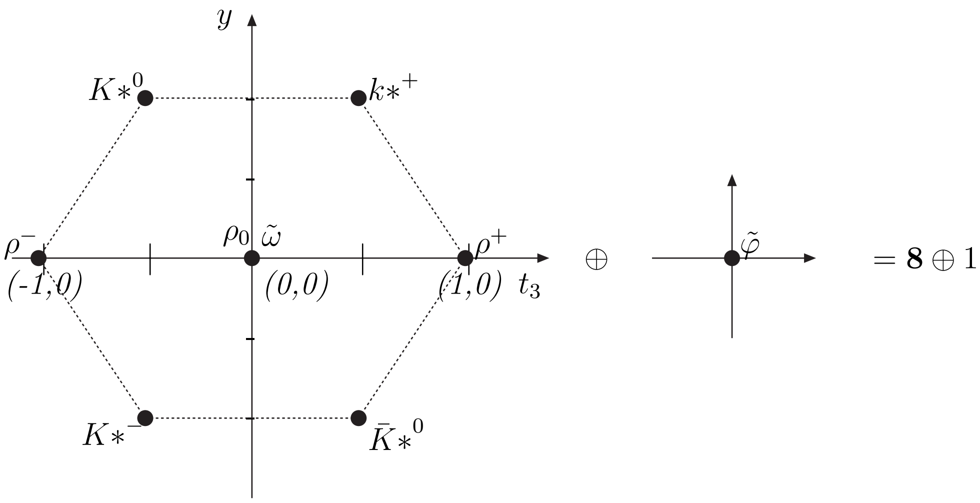
$$\eta_0 = \frac{1}{\sqrt{6}} \left(u\bar{u} + d\bar{d} - 2s\bar{s} \right) .$$

13.17 Problem: The 1^- mesons

Repeat what we did in the previous problem in the case of the 1^- mesons.

Solution

It is possible to have excited states of the constituent quarks of the 0^- nonet, giving particles with the same quark composition, but higher J. The 1^- nonet is an example. We can draw the (t_3, y) representation in the same way



Since states with the same quantum numbers can mix, and since both $\tilde{\omega}$ and $\tilde{\varphi}$ have $t_3 = y = 0$, one has

$$\varphi = cos(\theta)\tilde{\varphi} + sin(\theta)\tilde{\omega}$$
$$\omega = -sin(\theta)\tilde{\varphi} + cos(\theta)\tilde{\omega}$$

where θ is a mixing angle. The physical states ω and φ are then a combination of the isospin singlet $\tilde{\omega}$ and the singlet $\tilde{\varphi}$.

13.18 Problem: The $\frac{1}{2}^+$ baryon octet

Given that the tensor product $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$ show that p, n, $\Sigma^{\pm}$, Σ^0, Ξ^-, Ξ^0, and Λ^0 form an octet of SU(3).

Solution

First, we have to know the quark composition of the particles

$$p \sim udu \quad ; \quad n \sim udd \quad ; \quad \Xi^0 \sim ssu \quad ; \quad \Xi^- \sim ssd$$

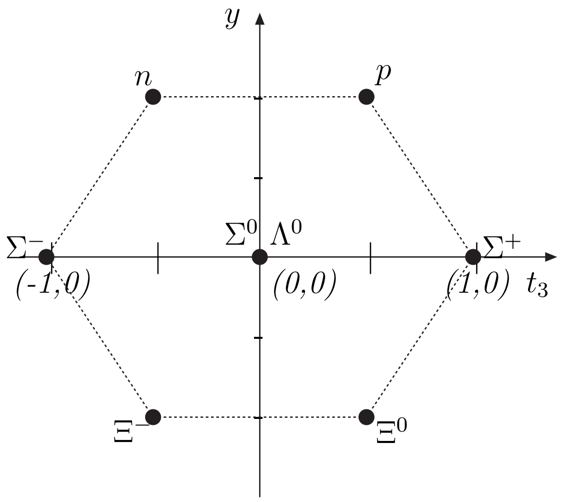
$$\Sigma^+ \sim suu \quad ; \quad \Sigma^- \sim sdd \quad ; \quad \Sigma^0 \sim \frac{s(ud+du)}{\sqrt{2}} \quad ; \quad \Lambda^0 \sim \frac{s(ud-du)}{2}.$$

Then one obtains the following t_3 and y eigenvalues (remember the eigenvalues of the constituent quarks from Problem 13.16)

$$p \to (\frac{1}{2}, 1) \quad ; \quad n \to (-\frac{1}{2}, 1) \quad ; \quad \Xi^0 \to (\frac{1}{2}, -1) \quad ; \quad \Xi^- \to (-\frac{1}{2}, -1)$$

$$\Sigma^+ \to (1, 0) \quad ; \quad \Sigma^- \to (-1, 0) \quad ; \quad \Sigma^0 \to (0, 0) \quad ; \quad \Lambda^0 \to (0, 0).$$

Therefore, the representation in the t_3, y plane for the baryons is



This is the $\frac{1}{2}^+$ baryon octet.

13.19 Problem: The baryonic SU(3) symmetry

Is the baryonic SU(3) symmetry a good one?

Solution

SU(3) is a good symmetry if $m_s = m_d = m_u$, implying that all particles in the same octet should have the same mass. In the particle data book one can find the mass for Σ particles and nucleons N:

$$M_N = 938.27203 \text{ MeV}, \qquad M_\Sigma = 1189.37 \text{ MeV}. \tag{13.8}$$

giving

$$\frac{M_\Sigma - M_N}{M_\Sigma + M_N} = 0.12. \tag{13.9}$$

Therefore SU(3) is not as good as SU(2), since it's broken up to the 10%. However, as we will see later, we can use this model to obtain some relations among the particle's masses, namely the *Gell-Man Okubo formula*.

13.20 Problem: The $\frac{3}{2}^+$ baryonic decuplet of SU(3)

Show that the $\frac{3}{2}^+$ baryons form a decuplet of SU(3).

Solution

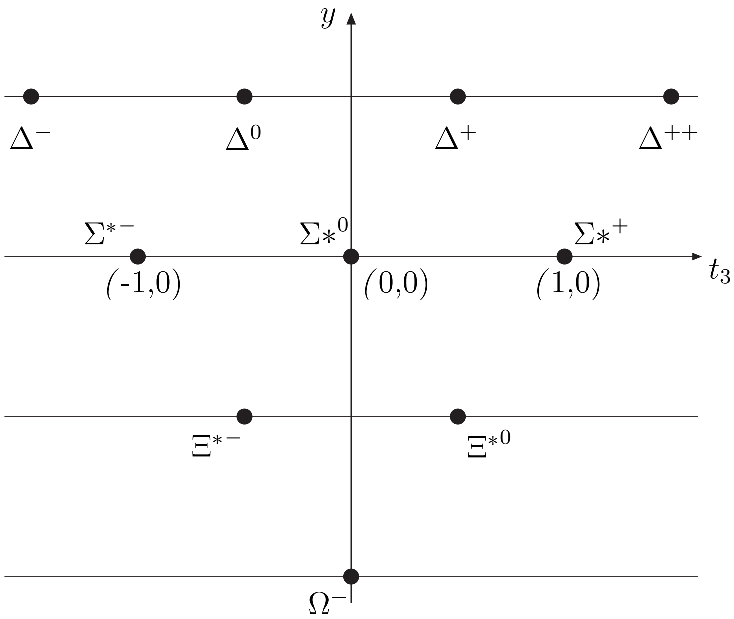
The quark composition for the $\frac{3}{2}^+$ baryons is:

$$\Delta^{++} \sim uuu \quad ; \quad \Delta^{+} \sim uud \quad ; \quad \Delta^{0} \sim udd \quad ; \quad \Delta^{-} \sim ddd$$

$$\Sigma^{*+} \sim suu \quad ; \quad \Sigma^{+0} \sim sud \quad ; \quad \Sigma^{*-} \sim sdd$$

$$\Xi^{*0} \sim ssu \quad ; \quad \Xi^{*-} \sim ssd \quad ; \quad \Omega^{-} \sim sss$$

And representing the eigenvalues in the t_3, y plane, we have:



That is the representation of the $\frac{3}{2}^{+}$ baryon decuplet.

13.21 Problem: The Gell-Mann Okubo mass formula

By using the quark model derive relations among the masses of π, K, and η.

Solution

We suppose that the SU(2) symmetry is exact, so that $m_u = m_d$. Then, the masses are:

$$\pi \sim ud \longrightarrow {m_\pi}^2 = m_0 + m_d + m_u = m_0 + 2m_u$$

$$K \sim su \longrightarrow {m_K}^2 = m_0 + m_u + m_s$$

$$\eta_0 \sim \frac{1}{\sqrt{6}}\left(u\bar{u} + d\bar{d} - 2s\bar{s}\right) \longrightarrow {m_\eta}^2 = m_0 + \frac{2m_u}{3} + 2m_s\frac{4}{6} = m_0 + \frac{2}{3}\left(m_u + 2m_s\right),$$

where m_0 is the binding energy, $m_u = m_d$ and the mesons masses are squared because it can be proved that the relation works better this way. We have to work with these 3 expressions so we can obtain one relation among the three mesons masses. We start with

$$4{m_k}^2 - 3{m_\eta}^2 = 4m_0 + 4m_u + 4m_s - 3m_0 - 2m_u - 4m_s = m_0 + 2m_u = {m_\pi}^2$$

then

$$4{m_k}^2 = {m_\pi}^2 + 3{m_\eta}^2, \tag{13.10}$$

This last relation is known as the *Gell-Mann Okubo* masses formula for mesons. Numerically, the l.h.s. of (13.10) gives 0.98 GeV^2 while the r.h.s. is 0.92 GeV^2.

13.22 Problem: A mass formula for the $\frac{1}{2}^+$ baryons

By using the quark model derive the mass formula for the $\frac{1}{2}^+$ baryons

$$\frac{m_\Sigma + 3m_\Lambda}{2} = m_n + m_\Xi.$$

Solution

As we did in the previous problem, the first thing to do is knowing the quark composition of the baryons (remember that we are considering $m_u = m_d$ since the SU(2) symmetry is supposed to be exact)

$m_n = m_0 + 3m_u\,;\; m_\Sigma = m_0 + 2m_u + m_s\,;\; m_\Xi = m_0 + m_u + 2m_s\,;\; m_\Lambda = m_0 + 2m_u + m_s.$

The left part of the relation gives

$$\begin{aligned}\frac{m_\Sigma + 3m_\Lambda}{2} &= \frac{1}{2}\left(m_0 + 2m_u + m_s + 3m_0 + 6m_u + 3m_s\right) \\ &= 2m_0 + 4m_u + 2m_S, \end{aligned} \tag{13.11}$$

while the right part reads

$$m_n + m_\Xi = 2m_0 + 4m_u + 2m_s. \tag{13.12}$$

Since they're equal, we have indeed proved the mass relation

$$\frac{m_\Sigma + 3m_\Lambda}{2} = m_N + m_\Xi. \tag{13.13}$$

This is called the *Gell-Mann Okubo* mass formula for $\frac{1}{2}^+$ baryons. Numerically, the l.h.s. of (13.13) gives 2.23 GeV while the r.h.s. is 2.25 GeV.

13.23 Problem: A mass formula for the $\dfrac{3}{2}^+$ baryon decuplet

For the $\dfrac{3}{2}^+$ baryon decuplet derive the rule: $m_{\Omega^-} - m_{\Xi^*} = m_{\Xi^*} - m_{\Sigma^*} = m_{\Sigma^*} - m_\Delta$.

Solution

By looking at the quark composition of the members of the decuplet (see Problem 13.20), by keeping in mind that we consider $m_u = m_d$, and consequently that particles with the same isospin are degenerated in mass, and by decomposing the mass of the particles into the binding energy (m_0) plus the masses of the constituent quarks, one obtains:

$$\Omega^- \sim sss \longrightarrow m_{\Omega^-} = m_0 + 3m_s$$

$$\Xi^* \sim ssu \longrightarrow m_{\Xi^*} = m_0 + 2m_s + m_u$$

$$\Sigma^* \sim suu \longrightarrow m_{\Sigma^*} = m_0 + m_S + 2m_u$$

$$\Delta \sim uuu \longrightarrow m_\Delta = m_0 + 3m_u.$$

By subtracting, as suggested by the statement of the problem, one obtains

$$m_{\Omega^-} - m_{\Xi^*} = m_s - m_u \quad ; \quad m_{\Xi^*} - m_{\Sigma^*} = m_s - m_u \quad ; \quad m_{\Sigma^*} - m_\Delta = m_s - m_u,$$

proving indeed the relation we where looking for:

$$m_{\Omega^-} - m_{\Xi^*} = m_\Xi^* - m_{\Sigma^*} = m_{\Sigma^*} - m_\Delta.$$

13.24 Problem*: A representation of SU(3)

Compute the dimensionality of the representation $\Box\!\Box\!\Box\!\Box$ of SU(3) and list explicitly all possible states in the language of the Young Tableaux.

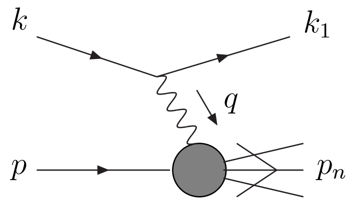
Chapter 14

Collisions involving hadrons

In the previous chapters, we dealt with the computation of cross sections and decay rates for initial states involving fundamental point-like particles, such as (anti)electrons or gauge bosons. The perfect knowledge of the initial state allows one to derive very precise predictions. On the contrary, (anti)protons are not point like particles, since they can be interpreted as bound states of quarks and gluons (partons). Despite of this fact, since from an experimental point of view it is much easier accelerating heavy objects, in modern high energy accelerator collisions are studied between protons (or between protons and anti-protons, or protons and leptons). In this chapter, we briefly illustrate the complications that arise in this kind of processes and also introduce computational tools that can be used to obtain physical predictions.

14.1 The deep inelastic scattering

The simplest possible process involving hadrons is the scattering of an electron with four-momentum k against a proton with four-momentum p. The kinematics is given by



$k = (E, \bar{k}) \qquad k_1 = (E_1, \bar{k}_1)$

$q = (k - k_1)$

$Q^2 = -q^2$

In the center-of-mass frame of the proton $p = (M, \vec{0})$ one has

$$E - E_1 = \frac{q \cdot p}{M} = \nu \quad \text{and} \quad x = \frac{Q^2}{2M\nu} = \frac{EE_1(1 - \cos\theta)}{M(E - E_1)}, \tag{14.1}$$

therefore the kinematics is completely determined by E, E_1 and $\cos\theta$.

The inclusive cross section can be written as

$$\frac{d\sigma^2}{dE_1 d\Omega} = \frac{\alpha^2}{q^4}\left(\frac{E_1}{E}\right) L_{\mu\nu} W^{\mu\nu}, \tag{14.2}$$

where $L_{\mu\nu}$ and $W^{\mu\nu}$ are leptonic and hadronic tensors, respectively, whose form can be determined by calculating the fully differential cross section

$$d\sigma = \frac{(2\pi)^4}{4EM} \sum_n \prod_{i=1}^{n} \left[\frac{d^3 p_i}{(2\pi)^3 2\omega_i}\right] |\bar{M}|^2 \delta^4(p + k - k_1 - p_n) \frac{d^3 k_1}{(2\pi)^3 2E_1}, \tag{14.3}$$

where M is the amplitude of the process

$$M = -\frac{e^2}{q^2} \bar{u}_\lambda(k_1) \gamma_\mu u_\lambda(k) T^\mu(\sigma). \tag{14.4}$$

In the previous equation $T^\mu(\sigma)$ is the hadronic current and λ, σ denote spin polarizations.

The matrix element squared, summed over the final state polarizations and averaged over the initial state ones, is given by

$$\begin{aligned} |\bar{M}|^2 &= \frac{1}{4}\frac{e^4}{q^4} Tr[\not{k}_1 \gamma_\mu \not{k} \gamma_\nu] \sum_\sigma T^\mu(\sigma) T^{*\nu}(\sigma) \\ &= \frac{e^4}{q^4}\left\{k_1^\mu k^\nu + k_1^\nu k^\mu + \frac{q^2}{2} g^{\mu\nu}\right\} \sum_\sigma T_\mu(\sigma) T^*_\nu(\sigma), \end{aligned} \tag{14.5}$$

therefore

$$d\sigma = \frac{(2\pi)^4}{4EM}\frac{(4\pi\alpha)^2}{q^4}\frac{E_1^2\,d\Omega\,dE_1}{(2\pi)^3 2E_1}\{...\}\sum_{n,\sigma}\prod_{i=1}^{n}\left[\frac{d^3p_i}{(2\pi)^3 2\omega_i}\right]T_\mu(\sigma)T_\nu^*(\sigma)\delta^4(p+q-p_n) \quad (14.6)$$

By comparing the previous equation with (14.2), one obtains

$$\begin{aligned} L^{\mu\nu} &= 2\left\{k_1^\mu k^\nu + k_1^\nu k^\mu + \frac{q^2}{2}g^{\mu\nu}\right\} \equiv \frac{1}{2}Tr[\not{k}_1\gamma_\mu\not{k}\gamma_\nu], \\ W^{\mu\nu} &= \frac{1}{4M}\sum_{n,\sigma}\int\prod_{i=1}^{n}\frac{d^3p_i}{(2\pi)^3 2\omega_i}(2\pi)^3\delta^4(p+q-p_n)T_\mu(\sigma)T_\nu^*(\sigma). \end{aligned} \quad (14.7)$$

The tensor $W^{\mu\nu}$ is unknown but conserved (namely $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$) and can be written in terms of two form factors

$$W_{\mu\nu} = \frac{1}{M}\left\{-F_1\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \frac{F_2}{M\nu}\left(p^\nu - \frac{(p\cdot q)}{q^2}q^\nu\right)\left(p^\mu - \frac{(p\cdot q)}{q^2}q^\mu\right)\right\}, \quad (14.8)$$

yielding

$$\begin{aligned} \frac{d^2\sigma}{d\Omega dE_1} &= \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\left(\frac{2F_1}{M}\sin^2\frac{\theta}{2} + \frac{F_2}{\nu}\cos^2\frac{\theta}{2}\right) \\ &= \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\frac{1}{\nu}\left[\cos^2\frac{\theta}{2}F_2 + \sin^2\frac{\theta}{2}\frac{Q^2}{xM^2}F_1\right]. \end{aligned} \quad (14.9)$$

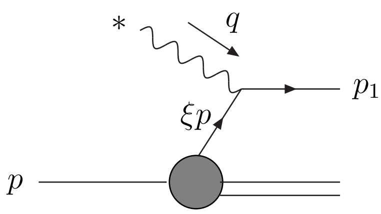
Experimentally one observes a *scaling* phenomenon, namely

$$\underbrace{\lim_{Q^2\to\infty}}_{at\ x\ fixed} F_j(x, \frac{Q^2}{M^2}) = F_j(x), \quad (14.10)$$

and the following *Callan-Gross* relation between the two form factors

$$F_2 = 2xF_1. \quad (14.11)$$

The two equations above can be explained by assuming a quark structure for the proton. In fact, in the limit of four-momenta without any transversal component (infinite momentum frame), the point-like scattering of the photon with a quark carrying a fraction ξ of the momentum of the proton



allows to compute

$$T_\mu(\sigma) = \bar{u}(p_1)\gamma_\mu u(\xi p) \tag{14.12}$$

and its contribution to the hadronic tensor $W^{\mu\nu}$

$$\begin{aligned} W^{\mu\nu}_\xi &= \underbrace{\frac{1}{4M\xi}}_{from\ flux} \int \frac{d^3p_1}{(2\pi)^3 2p_1^0}(2\pi)^3\delta^4(\xi p + q - p_1)\, Tr\,\{\not{p}_1\gamma_\mu\not{p}\gamma_\nu\}\,\xi \\ &= \frac{1}{4M}\int\frac{d^3p_1}{2p_1^0}\delta^4(q+\xi p - p_1)\,Tr\{\cdots\} = \frac{1}{4M}\frac{1}{2p_1^0}\delta(q_0+\xi p_0 - p_1^0)\,Tr\{\cdots\}. \end{aligned} \tag{14.13}$$

Note that the final state integration is performed over a 1-body phase space, because only one parton collides. But in the infinite momentum frame one can rewrite

$$\begin{aligned} \frac{\delta(q_0+\xi p_0 - p_1^0)}{2p_1^0} &= \delta[(p_1^0)^2 - (q_0+\xi p_0)^2]\,\theta(p_1^0) \;=\; \delta[p_1^2 - (q+\xi p)^2]\,\theta(q_0+\xi p_0) \\ &= \delta[q^2 + 2(q\cdot p)\xi]\,\theta(\cdots) \;=\; \delta[-2x(q\cdot p) + 2(q\cdot p)\xi]\,\theta(\cdots) \\ &= \frac{1}{2(q\cdot p)}\delta(x-\xi)\,\theta(\cdots) \;=\; \frac{1}{2M\nu}\delta(x-\xi)\,\theta(q_0+\xi p_0). \end{aligned} \tag{14.14}$$

Therefore

$$W^{\mu\nu}_\xi = \frac{1}{8M^2\nu}\delta(x-\xi)\,Tr\{(\not{q}+\xi\not{p})\gamma^\mu\not{p}\gamma^\nu\}, \tag{14.15}$$

and

$$\begin{aligned} W^{\mu\nu} &= \frac{1}{8M^2\nu}\int_0^1 d\xi\, f(\xi)\,\delta(x-\xi)\,Tr\,\{(\not{q}+\xi\not{p})\gamma^\mu\not{p}\gamma^\nu\} \\ &= \frac{1}{2M^2\nu}f(x)\Big\{(q+xp)^\mu p^\nu + (q+xp)^\nu p^\mu - g^{\mu\nu}(q\cdot p)\Big\} \\ &= \frac{f(x)}{2M^2\nu}\Big\{p^\mu p^\nu\,(2x) + \cdots - g^{\mu\nu}(q\cdot p)\Big\} \\ &= f(x)\left\{p^\mu p^\nu\left(\frac{x}{M^2\nu}\right) - g^{\mu\nu}\frac{(q\cdot p)}{2M^2\nu}\right\}, \end{aligned} \tag{14.16}$$

yielding

$$\begin{aligned} \frac{F_2}{M^2\nu} &= \frac{f(x)x}{M^2\nu} \qquad \Longrightarrow \qquad F_2 = xf(x) \\ -\frac{F_1}{M} &= -\frac{f(x)}{2M} \qquad \Longrightarrow \qquad F_1 = \frac{f(x)}{2}. \end{aligned} \tag{14.17}$$

Therefore

$$F_2 = 2x\, F_1, \tag{14.18}$$

which is the Callan-Gross relation, and $F_{1,2}$ depend on x, but not on Q^2, as promised.

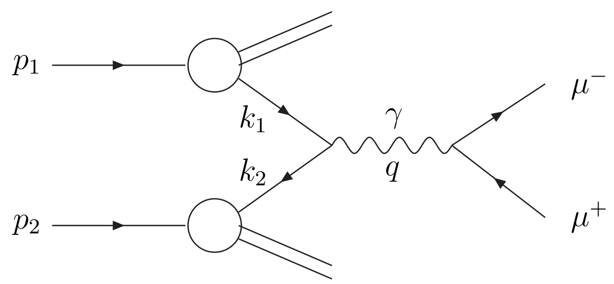
$f(x)$ is interpreted as a *parton density*. For example, in a proton

$$f(x) = \frac{4}{9}(u+\bar{u}) + \frac{1}{9}(d+\bar{d}) + \frac{1}{9}(s+\bar{s}). \tag{14.19}$$

Sum-rules exist which reproduce the proton quantum numbers. For example, the electric charge of the proton implies

$$\int_0^1 dx \left\{ \frac{2}{3}[u-\bar{u}] - \frac{1}{3}[d-\bar{d}] - \frac{1}{3}[s-\bar{s}] \right\} = 1. \tag{14.20}$$

Furthermore $u = u_v + u_s$ and $d = d_v + d_s$, with u_s and d_s contributions due to the see of gluons. Analogously in Drell-Yan processes (to produce, for example, a $\mu^+\mu^-$ pair) one has the following picture



with

$$\begin{aligned} k_1 &= x_1 p_1, \quad k_2 = x_2 p_2, \\ q^2 &= (k_1+k_2)^2 = 2x_1x_2(p_1\cdot p_2) = s x_1 x_2. \end{aligned} \tag{14.21}$$

By denoting $\frac{d\hat{\sigma}_{ii}}{dq^2}$ the parton level cross section for the process $q_i\bar{q}_i \to \mu^+\mu^-$, the hadron level cross-section can be written as follows

$$\frac{d\sigma}{dq^2} = \sum_i \int dx_1 dx_2 \left[q_i(x_1)\bar{q}_i(x_2) + \bar{q}_i(x_1)q_i(x_2)\right] \frac{d\hat{\sigma}_{ii}}{dq^2}\delta(q^2 - s x_1 x_2). \tag{14.22}$$

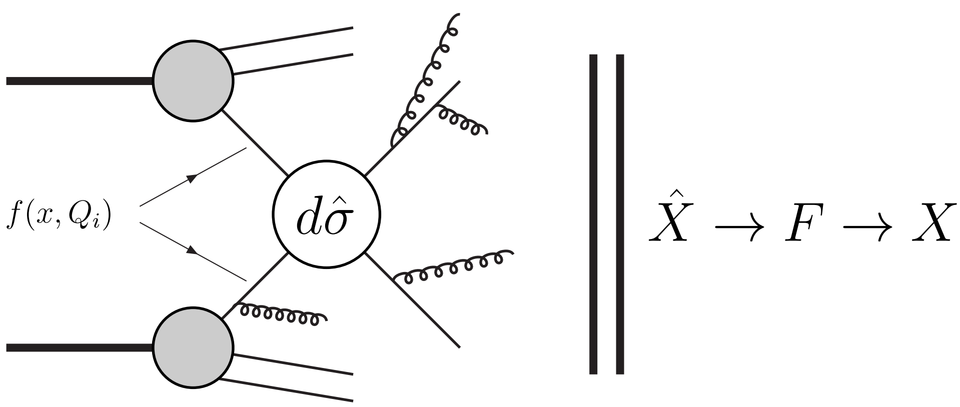
The previous equation is an example of factorization formula, which will be discussed in more detail in the next section.

14.2 The factorization formula

The key ingredient to study hadronic collisions is the so called *factorization theorem*

$$\frac{d\sigma}{dX} = \sum_{j,k} \int_{\hat{X}} f_j(x_1, Q_i) f_k(x_2, Q_i) \frac{d\hat{\sigma}_{j,k}(Q_i, Q_f)}{d\hat{X}} F(\hat{X} \to X; Q_i, Q_f). \qquad (14.23)$$

The l.h.s. of (14.23) represents an observable at the hadronic level, that can be obtained by convoluting the predictions for the hard collisions at the parton level, $\frac{d\hat{\sigma}_{j,k}(Q_i,Q_f)}{d\hat{X}}$,with the so called parton densities ,$f_j(x_i, Q_i)$, that represent the probability of finding the partons inside the (anti)proton. The sum $\sum_{j,k}$ is over all possible partons in the (anti)proton. Finally, $F(\hat{X} \to X; Q_i, Q_f)$ represents the transition of the partons in the final state to observable hadrons (the hadronization process). In the previous formula, the quantities computable in Quantum Field Theory are denoted by a $\hat{}$. They correspond to hard collisions between point-like, elementary objects (the partons). The other quantities, such as the parton densities, can be measured, and this experimental information, together with the computation of the hard part of the process, can be used to calculate physical observables at hadron colliders. In summary, (14.23) tells us that the Physical description of the hadronic collisions can be factorized into a short distance, perturbatively computable part, $\frac{d\hat{\sigma}_{j,k}(Q_i,Q_f)}{d\hat{X}}$, and a long distance non perturbative piece, represented by the parton densities. The correctness of such an assumptions relies, in turn, to the asymptotic freedom of QCD, that we will discuss in chapter 17. Equation (14.23) can be represented, pictorially, as follows



jp	subprocess	jp	subprocess	jp	subprocess
1	$q\bar{q}' \to WQ\overline{Q}$	2	$qg \to q'WQ\overline{Q}$	3	$gq \to q'WQ\overline{Q}$
4	$gg \to q\bar{q}'WQ\overline{Q}$	5	$q\bar{q}' \to WQ\overline{Q}q''\bar{q}''$	6	$qq'' \to WQ\overline{Q}q'q''$
7	$q''q \to WQ\overline{Q}q'q''$	8	$q\bar{q} \to WQ\overline{Q}q'\bar{q}''$	9	$q\bar{q}' \to WQ\overline{Q}q\bar{q}$
10	$\bar{q}'q \to WQ\overline{Q}q\bar{q}$	11	$q\bar{q} \to WQ\overline{Q}q\bar{q}'$	12	$q\bar{q} \to WQ\overline{Q}q'\bar{q}$
13	$qq \to WQ\overline{Q}qq'$	14	$qq' \to WQ\overline{Q}qq$	15	$qq' \to WQ\overline{Q}q'q'$
16	$qg \to WQ\overline{Q}q'q''\bar{q}''$	17	$gq \to WQ\overline{Q}q'q''\bar{q}''$	18	$qg \to WQ\overline{Q}qq\bar{q}'$
19	$qg \to WQ\overline{Q}q'q\bar{q}$	20	$gq \to WQ\overline{Q}qq\bar{q}'$	21	$gq \to WQ\overline{Q}q'q\bar{q}$
22	$gg \to WQ\overline{Q}q\bar{q}'q''\bar{q}''$	23	$gg \to WQ\overline{Q}q\bar{q}q\bar{q}'$		

Table 14.1: Subprocesses contributing to $WQ\bar{Q}\ +\ n\ jets$ final states.

14.3 Problem: Summing over subprocesses

Classify the possible subprocesses contributing to the process

$$pp \to WQ\bar{Q}\ +\ n\ jets,$$

where Q is a heavy quark (b or c), not present in the initial state.

Solution

According to the number of jets (n), the situation can be summarized as in table 14.1, where gluons in the final state are understood. So, for example, when $n = 0$ there is just one contributing process ($jp = 1$ in the table), while when $n = 2$ there are 3 contributing processes, namely $jp = 1$ with a final state gluon, $jp = 2$ and $jp = 3$.

14.4 Problem: The number of Feynman diagrams

Find the number of Feynman diagrams contributing to the subprocesses $g\ g \to n\ g$ and $q\bar{q} \to n\ g$ with $n = 7, 8, 9$.

Solution

It is clear that only for small values of n one can find the solution by actually drawing the diagrams. For example $q\bar{q} \to 2g$ receive contributions from 3 Feynman diagrams. The situation for large values of n is summarized in table 14.2.

Process	$n = 7$	$n = 8$	$n = 9$	$n = 10$
$g\ g \to n\ g$	559,405	10,525,900	224,449,225	5,348,843,500
$q\bar{q} \to n\ g$	231,280	4,016,775	79,603,720	1,773,172,275

Table 14.2: Number of Feynman diagrams corresponding to amplitudes with different numbers of quarks and gluons. From F. Caravaglios, M. L. Mangano, M. Moretti and R. P., NPB 539 (1999) 215.

14.5 ALPGEN

From the 2 previous problems, one can convince himself that a very little space is left for analytic work in the case of Hadronic Collisions. On the other hand, the LHC at CERN *is* a pp collider, and TEVATRON at FERMILAB *is* a $p\bar{p}$ collider, so that, for both of them, theoretical predictions *are* necessary. For these reasons, public numerical codes are available, which one can use to obtain predictions, such as ALPGEN [1].

14.6 Problem: Downloading ALPGEN

Download ALPGEN in your Personal Computer.

Solution

ALPGEN can be downloaded from the URL

http://mlm.home.cern.ch/mlm/alpgen/

14.7 Problem: Estimating $W+2j$ production with ALPGEN

By Using ALPGEN, estimate the number of produced events in the processes:

1. $p\bar{p} \to W^{\pm} + 2\ jets$ with $W^+ \to e^+\nu_e$ and $W^- \to e^-\bar{\nu}_e$
   at TEVATRON with integrated luminosity $L = 10 fb^{-1}$ and energy $\sqrt{s} = 2E_b =$ 1960 GeV.
2. $pp \to W^{\pm} + 2\ jets$ with $W^+ \to e^+\nu_e$ and $W^- \to e^-\bar{\nu}_e$
   at the LHC with integrated luminosity $L = 600 fb^{-1}$ and energy $\sqrt{s} = 2E_b =$ 14000 GeV.

Use the following set of cuts

- $p_T(jets) > 20$ GeV,
- $|\eta(jets)| < 2.5$,
- $\Delta R(jet, jet) > 0.7$.

Solution

By running ALPGEN, one finds the following estimates for the cross sections at TEVATRON and at the LHC

$$\begin{aligned} \sigma_{TEV}(W2j) &= 34.0(2)\,\text{pb} \\ \sigma_{LHC}(W2j) &= 1075(4)\,\text{pb} \end{aligned} \tag{14.24}$$

The numbers of expected events can therefore be found after multiplication by the integrated luminosities

$$\begin{aligned} N_{TEV}(W2j) &= 340000 \\ N_{LHC}(W2j) &= 645 \times 10^6. \end{aligned} \tag{14.25}$$

14.8 Problem: Estimating e^+e^-+2j production with ALPGEN

By Using ALPGEN, estimate the number of produced events in the processes:

1. $p\bar{p} \to Z/\gamma^* + 2\ jets$ with $Z/\gamma^* \to e^+e^-$
   at TEVATRON with integrated luminosity $L = 10fb^{-1}$ and energy $\sqrt{s} = 2E_b =$ 1960 GeV.
2. $pp \to Z/\gamma^* + 2\ jets$ with $Z/\gamma^* \to e^+e^-$
   at the LHC with integrated luminosity $L = 600fb^{-1}$ and energy $\sqrt{s} = 2E_b =$ 14000 GeV.

Use the following set of cuts

- $p_T(jets) > 20$ GeV,
- $|\eta(jets)| < 2.5$,
- $\Delta R(jet, jet) > 0.7$,
- 40 GeV $< m(e^+e^-) <$ 200 GeV,

where $m(e^+e^-)$ is the invariant mass of the e^+e^- system.

Solution

By running ALPGEN, one finds the following estimates for the cross sections at TEVATRON and at the LHC

$$\begin{aligned} \sigma_{TEV}(Z2j) &= 3.61(4)\,\text{pb} \\ \sigma_{LHC}(Z2j) &= 116(1)\,\text{pb} \end{aligned} \tag{14.26}$$

The numbers of expected events can therefore be found after multiplication by the integrated luminosities

$$\begin{aligned} N_{TEV}(Z2j) &= 36100 \\ N_{LHC}(Z2j) &= 69600000. \end{aligned} \tag{14.27}$$

14.9 Problem: Estimating $H+2j$ production with ALPGEN

By Using ALPGEN, estimate the number of produced events in the processes:

1. $p\bar{p} \to H + 2\ jets$ (coming from an effective ggH coupling)

   at TEVATRON with integrated luminosity $L = 10fb^{-1}$ and energy $\sqrt{s} = 2E_b = 1960$ GeV.

2. $pp \to H + 2\ jets$ (coming from an effective ggH coupling)

   at the LHC with integrated luminosity $L = 600fb^{-1}$ and energy $\sqrt{s} = 2E_b = 14000$ GeV.

Use the following set of cuts

- $p_T(jets) > 20$ GeV,
- $|\eta(jets)| < 2.5$,
- $\Delta R(jet, jet) > 0.7$.

Solution

By running ALPGEN, one finds the following estimates for the cross sections at TEVATRON and at the LHC

$$\begin{aligned} \sigma_{TEV}(H2j) &= 0.0273(2)\,\mathrm{pb} \\ \sigma_{LHC}(H2j) &= 4.99(4)\,\mathrm{pb} \end{aligned} \tag{14.28}$$

The numbers of expected events can therefore be found after multiplication by the integrated luminosities

$$\begin{aligned} N_{TEV}(H2j) &= 273 \\ N_{LHC}(H2j) &= 2994000. \end{aligned} \tag{14.29}$$

14.10 Problem*: Estimating WW production with ALPGEN

By Using ALPGEN, estimate the cross section for the processes:

1. $p\bar{p} \to WW$ at TEVATRON,
2. $pp \to WW$ at the LHC.

Chapter 15

Accelerating particles

In this chapter, we list a few problems on the main quantities one has to take into account in particle accelerator Physics.

15.1 Parameters for accelerating particles

The following fundamental relation holds among the radius of the orbit, the momentum, the charge and the magnetic field:

$$R = \frac{pc}{qB} \overset{\text{mixed units}}{\overset{\uparrow}{\simeq}} \frac{p}{0.3B} \begin{cases} [p] = \text{GeV/c} \\ [B] = \text{Tesla} \\ [R] = \text{meters.} \end{cases}$$

15.2 Problem: Accelerating protons

Calculate the radius needed to accelerate protons to momenta of about 30 GeV/c with a magnetic fields of 2 Tesla.

Solution

$$R = \frac{p}{0.3B} = \frac{30}{(0.3)(2)} = 50\,\mathrm{m}. \tag{15.1}$$

15.3 Problem: The magnetic field of the LHC

At the LHC, whose circumference $\mathcal{C}$ is 27 km, pp collisions are going to be produced at a center-of-mass energy of $\sqrt{s}$= 14 TeV. Compute the magnetic field that should have the magnets to keep the protons in the orbit.

Solution

$$\frac{p}{0.3R}, \quad \mathcal{C} = 2\pi R, \tag{15.2}$$

$$p = \frac{\sqrt{s}}{2}\frac{1}{c} = 7000\,\mathrm{GeV/c}. \tag{15.3}$$

Then

$$B = \frac{7000 \cdot 2\pi}{0.3 \cdot \mathcal{C}} = \frac{7000 \cdot 6.28}{0.3 \cdot 27000} = \frac{7 \cdot 6.28}{0.3 \cdot 27} = 5.4\,\mathrm{Tesla}. \tag{15.4}$$

Since, in normal magnets, $B \leq 2T$, superconducting magnets have to be used at the LHC.

15.4 Problem: The luminosity of the LHC

Compute the instantaneous luminosity of the LHC by knowing that protons bunches contain 10^{11} particles, have a transverse radius of $15\mu m$ and that there are 2600 bunches for each beam.

Solution

The instantaneous luminosity is given by the formula

$$\mathcal{L} = f \cdot n \frac{N_1 N_2}{A}. \tag{15.5}$$

By using

$$\begin{aligned} n &= 2600, \quad N_1 = N_2 = 10^{11}, \quad f = \frac{300000}{27\, s} = 11111 s^{-1}, \\ A &= 4\pi R^2, \quad R = 15 \cdot 10^{-6}\,\mathrm{m} = 15 \cdot 10^{-4}\,\mathrm{cm}, \end{aligned} \tag{15.6}$$

one obtains

$$\mathcal{L} = \frac{(11111)(2600)(10^{22})}{(4)(3.14)(15^2)(10^{-8})} \frac{1}{\mathrm{cm}^2 \cdot s} = 10222 \cdot 10^{30}\,\mathrm{cm}^{-2} s^{-1} = 1 \cdot 10^{34}\,\mathrm{cm}^{-2} s^{-1}. \tag{15.7}$$

15.5 Problem: The integrated luminosity of the LHC

Compute, in fb^{-1}, the integrated luminosity $\int dt\ \mathcal{L} \equiv L$ of one year of taking data at the LHC assuming an efficiency of 30%.

Solution

The number of seconds (N_s) of data taking in one year, with 30% of efficiency is

$$N_s = \frac{30}{100} \cdot 365 \cdot 3600 \cdot 24 = 9460800\, s \backsim 10^7 s. \tag{15.8}$$

Then

$$\begin{aligned} L &= \mathcal{L} \cdot N_s = 10^{41}\mathrm{cm}^{-2} = 10^{41} \frac{1}{\mathrm{cm}^2} = 10^{41} \frac{1}{\mathrm{cm}^2} \left(\frac{10^{-24}\mathrm{cm}^2}{1\, b} \right) \\ &= 10^{17} \frac{1}{b} = \frac{10^{17}}{10^{15} fb} = 100\, fb^{-1}. \end{aligned} \tag{15.9}$$

15.6 Problem: $t\bar{t}$ production at the LHC

Knowing that $\sigma_{t\bar{t}} \simeq 7\,pb$ compute the number of $t\bar{t}$ pairs ($N_{t\bar{t}}$) produced each year at the LHC.

Solution

$$N_{t\bar{t}} = \sigma_{t\bar{t}} \cdot L = 7\,pb \cdot \frac{100}{fb} = 7 \cdot 100 \cdot \frac{1000 fb}{1 fb} = 7 \cdot 10^5. \tag{15.10}$$

15.7 Problem: e^- energy loss at LEP1

Calculate the energy loss of an electron following a circular orbit at LEP1 at energies near the peak of the Z_0 and compare the result with the energy loss of a proton.

Solution

For electrons one has

$$\Delta E = \frac{4\pi}{3} \frac{\alpha \hbar c \beta^3 \gamma^4}{R}, \tag{15.11}$$

namely

$$\Delta E[KeV] = 88.5\,\frac{E^4[GeV]}{R[m]}. \tag{15.12}$$

Since

$$E = 45 GeV, \qquad R = \frac{\mathcal{C}}{2\pi} = \frac{27000}{2\pi} = 4300\,m \tag{15.13}$$

one obtains

$$\Delta E = \frac{88.5 \cdot (45)^4}{4300}\,KeV = 84392\,KeV = 84\,MeV. \tag{15.14}$$

For protons, one would have, instead

$$\Delta E[KeV] = 88.5\,\frac{E^4[GeV]}{R([m])}\left(\frac{m_e}{m_p}\right)^4, \tag{15.15}$$

namely

$$\Delta E = 84\,MeV\,(10^{-13}) = 84 \cdot 10^6 eV \cdot 10^{-13} = 84 \cdot 10^{-7} eV \backsim 10^{-5} eV. \tag{15.16}$$

that is a negligible energy loss. That is the reason why it is much easier the acceleration of protons.

15.8 Problem∗: The SLHC

A project exists to upgrade the LHC to reach an instantaneous luminosity of

$$\mathcal{L} = 1 \cdot 10^{35}\,\mathrm{cm}^{-2} s^{-1}. \tag{15.17}$$

This upgraded LHC is called Super LHC (SLHC). Discuss the possible options to increase the Luminosity.

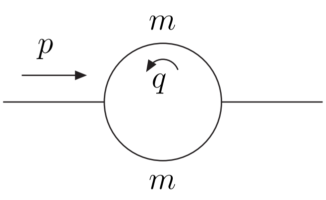
Chapter 16

Quantum Field Theory at one-loop

So far we have dealt with the problem of computing physical processes at the tree-level only. However, any Quantum Field Theory describing Nature should be internally consistent, meaning that, among other things, it should be possible to compute the so called *radiative corrections*, that is the contributions coming from Feynman diagrams with loops. In this chapter, we discuss the complications which arise in the one-loop case and show how they can be solved [4]. We do it both in the framework of the electroweak Standard Model (SM) and with the help of simple scalar $\lambda\phi^3$ and $g\phi^4$ theories. Finally, we present a tool to compute in a numerical way one-loop corrections for arbitrary processes.

16.1 UV divergent one-loop diagrams

When computing loop corrections in Quantum Field Theories divergences may appear due to the integration over large components of the momentum flowing in the loops. As an example, consider the interaction Lagrangian $\mathcal{L}_{\text{INT}} = -\lambda\phi^3/3!$. It gives rise to the following one-loop diagram



$$= \frac{\lambda^2}{2(2\pi)^4}\int d^4q \;\; \frac{1}{(q^2-m^2)[(q+p)^2-m^2]}, \tag{16.1}$$

which gives an infinite contribution. In fact, the $q \to \infty$ limit of the integral behaves as

$$\int^{\Lambda} dq\, \frac{q^3}{q^4} = \int^{\Lambda} \frac{dq}{q} = \ln(\Lambda), \tag{16.2}$$

which gives ∞ when $\Lambda \to \infty$. Divergences like that are dubbed *ultraviolet* (UV) divergences.

16.2 Problem: UV infinities in $g\phi^4$.

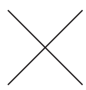
Classify the UV divergent one-loop diagrams and integrals appearing in the scalar $g\phi^4$ theory.

Solution

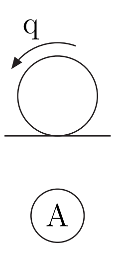
The Lagrangian is

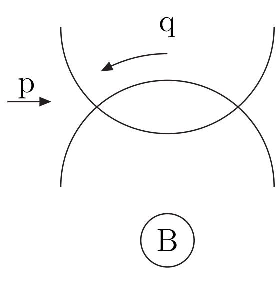
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 - \frac{g}{4!}\phi^4, \tag{16.3}$$

from which one derives the following Feynman rules

 $= \dfrac{i}{p^2 - m^2}$  $= -ig.$

They give rise to the following UV divergent one-loop diagrams





.

Diagram (A) is proportional to the integral

$$\int d^4q \frac{1}{q^2 - m^2} \sim \int^\Lambda dq\, \frac{q^3}{q^2} \sim \int^\Lambda dq\, q \sim \Lambda^2, \tag{16.4}$$

which diverges when $\Lambda \to \infty$.

Analogously, diagram (B) produces the same UV divergent integral of (16.1),

$$\int d^4q \frac{1}{(q^2 - m^2)[(q+p)^2 - m^2]} \sim \int^\Lambda dq\, \frac{q^3}{q^4} \sim \ln \Lambda. \tag{16.5}$$

16.3 Regularization

Prior to any attempt to manipulate loop integrals to get rid of the UV infinities, one needs a method to regularize them. This is necessary in order to deal with well defined mathematical objects. Such a procedure is called *regularization.* One example of regularization is the use of a cut-off Λ in the loop momentum, as shown in the previous examples. However, this is not suitable in the context of gauge theories, because it violates, in general, the Ward Identities dictated by the gauge invariance. [1] On the contrary, the *dimensional regularization* procedure described in the next section preserves gauge invariance. For this reason, it is nowadays the mostly common used method to regularize Quantum Field Theories.

16.4 Dimensional regularization

The basic observation is that the presence of UV divergences depends on the dimensionality of the space-time. For example, the diagram in (16.1) is UV convergent in 3 dimensions, but divergent in 4. Therefore, one computes the loop integrals in a generic n-dimensional space-time. In this way UV divergences appear as poles in $\epsilon = n - 4$ when taking the physical limit $n \to 4$. The advantage of this is that the Ward Identities at the base of the needed gauge cancellations do not depend upon the number of the space-time coordinates. This is the reason why dimensional regularization does not break gauge invariance.

The relevant formulas to be used are

[1] This means that important gauge cancellations among Feynman diagrams are broken by the presence of the regulator Λ.

- Feynman parameters:

$$\frac{1}{a_1^{\alpha_1}\cdots a_k^{\alpha_k}} = \frac{\Gamma(\alpha_1+\cdots\alpha_k)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_k)} \times \int_0^1 dx_1\cdots\int_0^1 dx_k \frac{x_1^{\alpha_1-1}\cdots x_k^{\alpha_k-1}\delta(1-x_1-\cdots-x_k)}{[a_1x_1+\cdots+a_kx_k]^{\alpha_1+\cdots+\alpha_k}}, \tag{16.6}$$

where the a_is have imaginary parts with the same sign.

- Wick rotation:

$$d^nq \to i d^nq \quad \text{and} \quad q^2\big|_M \to -q^2\big|_E,$$

where the subscripts M and E stand for Minkowskian and Euclidean space, respectively.

- Angular integration:

$$\int d\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

- Integration over the Euclidean norm of q:

$$\int_0^\infty dq \frac{q^\beta}{(q^2+\chi)^\alpha} = \frac{1}{2}\frac{\Gamma(\frac{\beta+1}{2})\Gamma(\alpha-\frac{\beta+1}{2})}{\Gamma(\alpha)\chi^{\alpha-\frac{\beta+1}{2}}}.$$

- Furthermore, the following properties of the Γ function are useful:

$$\Gamma(n) = (n-1)!\,,\quad z\Gamma(z) = \Gamma(z+1)\,,\quad \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon). \tag{16.7}$$

16.5 Problem: The one-loop scalar integrals

Compute the Pole Part ($P.P.$) of

$$A(m^2) := \int d^nq \frac{1}{q^2-m^2} \tag{16.8}$$

and

$$B(p^2,m^2,m^2) := \int d^nq \frac{1}{(q^2-m^2)\,[(q+p)^2-m^2]}. \tag{16.9}$$

Solution

Let us start with

$$A(m^2) = \int d^n q \frac{1}{q^2 - m^2} = -i \int_E d^n q \frac{1}{q^2 + m^2}, \tag{16.10}$$

where $\int_E$ means integration in the Euclidean space. Then

$$A(m^2) = -i \int d\Omega_n \int_0^\infty dq \frac{q^{n-1}}{q^2 + m^2} = -i \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \frac{1}{2} \frac{\Gamma(\frac{n}{2})\Gamma(1 - \frac{n}{2})}{(m^2)^{1-\frac{n}{2}}}. \tag{16.11}$$

By splitting $n = 4 + \epsilon$ gives

$$\begin{aligned} A(m^2) &= -i\pi^2 \pi^{\frac{\epsilon}{2}} \qquad \underbrace{\Gamma(-1 - \frac{\epsilon}{2})}_{\frac{\Gamma(-\frac{\epsilon}{2})}{-1-\frac{\epsilon}{2}} = -\frac{1}{1+\frac{\epsilon}{2}}\left(\frac{-2}{\epsilon} + \cdots\right)} \qquad (m^2)^{1+\frac{\epsilon}{2}} \\ &= -i\pi^2 \left(1 + \frac{\epsilon}{2}\ln\pi + \cdots\right)\left(\frac{2}{\epsilon} + \cdots\right) m^2 \left(1 + \frac{\epsilon}{2}\ln m^2\right) \\ &= -i\pi^2 m^2 \left(\frac{2}{\epsilon} + \text{finite parts}\right). \end{aligned} \tag{16.12}$$

Thus

$$P.P.\left[A(m^2)\right] = -i\pi^2 m^2 \left(\frac{2}{\epsilon}\right). \tag{16.13}$$

As for the second integral, one first puts together the two denominators

$$B(p^2, m^2, m^2) = \int d^n q \frac{1}{(q^2 - m^2)\left[(q+p)^2 - m^2\right]} = \int_0^1 dx \int d^n q \frac{1}{\left[D_0(1-x) + D_1 x\right]^2},$$

where $D_0 = (q^2 - m^2)$ and $D_1 = [(q+p)^2 - m^2]$, so that

$$[D_0(1-x) + D_1 x] = q^2 - m^2 + 2(q \cdot p)x + p^2 x. \tag{16.14}$$

The change of variables $q \to q - px$ gives

$$\begin{aligned} [D_0(1-x) + D_1 x] &\to q^2 + p^2x^2 - 2(q \cdot p)x - m^2 + p^2 x + 2px \cdot (q - px) \\ &= q^2 - \underbrace{(m^2 - p^2 x(1-x))}_{M^2(x)}. \end{aligned} \tag{16.15}$$

Thus

$$\begin{aligned}
B(p^2, m^2, m^2) &= \int_0^1 dx \int d^n q \frac{1}{[q^2 - M^2(x)]^2} = i \int_0^1 dx \int_E d^n q \frac{1}{[q^2 + M^2(x)]^2} \\
&= i \int_0^1 dx \int d\Omega_n \int_0^\infty dq \frac{q^{n-1}}{[q^2 + M^2(x)]^2} \\
&= i \int_0^1 dx \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \frac{1}{2} \frac{\Gamma(\frac{n}{2})\Gamma(2-\frac{n}{2})}{\Gamma(2)(M^2(x))^{2-\frac{n}{2}}}. \qquad (16.16)
\end{aligned}$$

Splitting $n = 4 + \epsilon$ gives

$$\begin{aligned}
B(p^2, m^2, m^2) &= i\pi^{2+\frac{\epsilon}{2}} \int_0^1 dx\, \Gamma(-\frac{\epsilon}{2})(M^2(x))^{\frac{\epsilon}{2}} \\
&= i\pi^2 \left(-\frac{2}{\epsilon} + \cdots\right)\left(1 + \frac{\epsilon}{2}\ln\pi + \cdots\right) \int_0^1 dx \left(1 + \frac{\epsilon}{2}\ln M^2(x)\right) \\
&= i\pi^2 \left(-\frac{2}{\epsilon}\right) + \text{finite parts}, \qquad (16.17)
\end{aligned}$$

so that

$$P.P.\left[B(p^2, m^2, m^2)\right] = -i\pi^2 \left(\frac{2}{\epsilon}\right). \qquad (16.18)$$

As promised, a meaning in $n = 4 + \epsilon$ dimensions is given to both UV divergent integrals at the price of having poles in $\epsilon = 0$.

16.6 Renormalization

After regularizing the loop integrals, the "hope" is that the regulator dependence only occur in the intermediate steps of the calculation, while disappear from physical predictions. If this happens, the theory under study is called *renormalizable* and can be used to describe Nature at a fundamental level. If it does not, it is *nonrenormalizable*, and cannot represent fundamental interactions. [2]

In renormalizable theories, the UV regulator leaves no trace in the physical S matrix elements because *the original parameters in the Lagrangian $\mathcal{L}$ (the so called bare*

[2]Nonrenormalizable Quantum Field Theories can still be used as *effective* models, valid up to energy scales of the order of the UV cut-off Λ, meaning that the theory is expected to change at higher energies.

parameters) can be made divergent in such a way that the two kinds of infinities compensate each other when bare parameters are fixed in terms of physical measurements. In other words, UV infinities are not observable because they can be reabsorbed in the free parameters of the theory.

Roughly speaking, a theory is renormalizable when the type of possible UV infinities is limited and does not increase with the number of loops. Thus, they can be accommodated by redefining a finite number of terms in $\mathcal{L}$. When computing physical observables, only couplings and masses need to be redefined. On the contrary, field redefinitions are also necessary to obtain UV finite Green's functions. The first approach is the one we use until section 16.13, while the second procedure is described in section 16.14.

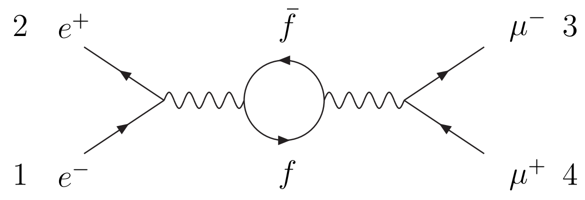
In summary, absorbing UV divergences in $\mathcal{L}$ is the essence of the renormalization procedure, which makes sense because *bare* parameters have no direct physical interpretation, unless linked to measurements. As a practical consequence, one computes divergent loop integrals in n dimensions and sets $n \to 4$ in physical quantities. If the theory is renormalizable all UV divergent terms cancel and the $n \to 4$ limit exists. The simplest realistic case is the renormalization of the electric charge discussed in the next section.

16.7 Problem: Charge renormalization in QED

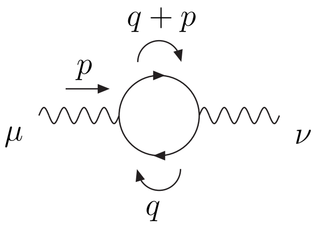
Renormalize the electric charge in QED.

Solution

Suppose we want to compute the QED one-loop corrections to the process $e^+e^- \to \mu^+\mu^-$. This means considering, among many others, the contribution



where f is a fermion. The relevant one-loop diagram is therefore



$\equiv \Pi_{\mu\nu}(p^2)$,

with

$$\Pi_{\mu\nu} = -\frac{(ie)^2}{(2\pi)^4} \ ii \int d^n q \frac{1}{D_1 D_2} Tr[\gamma_\mu \ (\not{q} + m) \ \gamma_\nu \ (\not{q} + \not{p} + m)], \tag{16.19}$$

where $D_1 = q^2 - m^2, \ D_2 = (q+p)^2 - m^2, \ m = m_f$ and the (-) sign is due to the fermion loop.

By using Feynman parametrization one obtains

$$\Pi_{\mu\nu} = -\frac{e^2}{16\pi^4} \int_0^1 \ dx \int \ d^n q \frac{Tr[\cdots]}{(D_1(1-x) \ + \ D_2 x)^2}. \tag{16.20}$$

But

$$D_1(1-x) + D_2 x = q^2 - m^2 + 2(q \cdot p)x + p^2 x, \tag{16.21}$$

so that, to get rid of the term $(q \cdot p)$, we change the integration variables by shifting $q \to q - px$, that gives

$$\Pi_{\mu\nu} = -\frac{e^2}{16\pi^4} \int_0^1 \ dx \int d^n q \frac{1}{[q^2-\chi]^2} Tr[\gamma_\mu(\not{q} - x\not{p} + m)\gamma_\nu(\not{q} + \not{p}(1-x) + m)],$$

with $\chi = m^2 - p^2 x(1-x)$. By computing the trace one obtains

$$\begin{aligned} Tr[\cdots] = 4\big\{&(q-xp)_\mu(q+p(1-x))_\nu + (q+p(1-x))_\mu(q-px)_\nu \\ &+ g_{\mu\nu}\left[m^2 - (q-xp)\cdot(q+p(1-x))\right]\big\}, \end{aligned} \tag{16.22}$$

and by virtue of the fact that $\int d^n q \frac{q_\mu}{[q^2-\chi]^2} = 0$ only a few terms survive, resulting in

$$\Pi_{\mu\nu} = -\frac{e^2}{4\pi^4} \int_0^1 dx \int d^n q \frac{1}{[q^2-\chi]^2} \big\{ 2 q_\mu q_\nu - g_{\mu\nu}\left(q^2 - m^2 - x(1-x)p^2\right) + A p_\mu p_\nu \big\}.$$

The $p_\mu p_\nu$ term <u>does not</u> contribute when inserted in the amplitude. [3] In fact, it would give a contribution proportional to

$$\begin{aligned} [\bar{v}_2 \not{p} u_1] \times [\bar{u}_3 \not{p} v_4] &= [\bar{v}_2 \not{p}_1 u_1 + \bar{v}_2 \not{p}_2 u_1] \times [\bar{u}_3 \not{p}_3 v_4 + \bar{u}_3 \not{p}_4 v_4] \\ &= (m_e - m_e)(m_\mu - m_\mu) \, \bar{v}_2 u_1 \bar{u}_3 v_4 = 0. \end{aligned} \tag{16.23}$$

[3] This is a consequence of the Ward Identities.

One is therefore left with the integral

$$\Pi_{\mu\nu} = -\frac{e^2}{4\pi^4}\int_0^1 dx \int d^n q \frac{1}{[q^2-\chi]^2}\big\{2q_\mu q_\nu - g_{\mu\nu}(q^2 - m^2 - x(1-x)p^2)\big\}. \quad (16.24)$$

The piece $\int d^n q \frac{1}{[\cdots]^2} q_\mu q_\nu$ must be proportional to $g_{\mu\nu}$,

$$\int d^n q \frac{1}{[\cdots]^2} q_\mu q_\nu = B g_{\mu\nu}. \quad (16.25)$$

By multiplying both sides by $g^{\mu\nu}$ and using $g_{\mu\nu}g^{\mu\nu} = n$, one obtains B,

$$\int d^n q \frac{q_\mu q_\nu}{[\cdots]^2} = \frac{1}{n}\int d^n q \frac{q^2}{[\cdots]^2} g_{\mu\nu}. \quad (16.26)$$

This gives

$$\Pi_{\mu\nu} = -\frac{e^2}{4\pi^4} g_{\mu\nu}\Pi_0(p^2), \quad (16.27)$$

with

$$\Pi_0(p^2) = \int_0^1 dx \left\{ \left(\frac{2}{n} - 1\right) J + \left(p^2 x(1-x) + m^2\right) I \right\}, \quad (16.28)$$

where

$$I = \int d^n q \frac{1}{[q^2-\chi]^2} \quad \text{and} \quad J = \int d^n q \frac{q^2}{[q^2-\chi]^2}. \quad (16.29)$$

The integral I can be computed as follows

$$\begin{aligned} I = i\int_E d^n q \frac{1}{[q^2+\chi]^2} &= i\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \underbrace{\int_0^\infty dq \frac{q^{n-1}}{[q^2+\chi]^2}}_{\frac{1}{2}\frac{\Gamma(\frac{n}{2})\Gamma(2-\frac{n}{2})}{\Gamma(2)}\chi^{(\frac{n}{2}-2)}} \\ &= i\pi^{\frac{n}{2}}\Gamma\left(2-\frac{n}{2}\right)\chi^{\frac{n}{2}-2}. \end{aligned} \quad (16.30)$$

Introducing $n = 4+\epsilon$ gives

$$I = i\pi^2\pi^{\frac{\epsilon}{2}}\Gamma\left(-\frac{\epsilon}{2}\right)\chi^{\frac{\epsilon}{2}}. \quad (16.31)$$

For J one obtains instead

$$\begin{aligned} J = -i \int_E d^n q \frac{q^2}{[q^2+\chi]^2} &= -i \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \underbrace{\int_0^\infty dq \frac{q^{n+1}}{[q^2+\chi]^2}}_{\frac{1}{2} \quad \frac{\Gamma(\frac{n+2}{2})\Gamma(2-\frac{n+2}{2})}{\Gamma(2)} \chi^{-(1-\frac{n}{2})}} \\ &= -i\pi^2 \pi^{\frac{\epsilon}{2}} \Gamma\left(-1-\frac{\epsilon}{2}\right) \chi^{1+\frac{\epsilon}{2}} \frac{1}{(2+\frac{\epsilon}{2})^{-1}}. \end{aligned} \tag{16.32}$$

Now we use

$$\Gamma\left(-1-\frac{\epsilon}{2}\right) = \frac{\Gamma(-\frac{\epsilon}{2})}{-(1+\frac{\epsilon}{2})} \tag{16.33}$$

so that

$$J = i\pi^2 \pi^{\frac{\epsilon}{2}} \Gamma\left(-\frac{\epsilon}{2}\right) \frac{4+\epsilon}{2+\epsilon} \chi \chi^{\frac{\epsilon}{2}}. \tag{16.34}$$

The part which diverges when $\epsilon \to 0$ is contained in $\Gamma(-\frac{\epsilon}{2})$, while the rest can be expanded in powers of ϵ. Putting everything together gives

$$\begin{aligned} \Pi_0(p^2) &= i\pi^2 \pi^{\frac{\epsilon}{2}} \Gamma\left(-\frac{\epsilon}{2}\right) \int_0^1 dx \left\{-\chi + (p^2 x(1-x) + m^2)\right\} \chi^{\frac{\epsilon}{2}} \\ &= i\pi^2 \pi^{\frac{\epsilon}{2}} \Gamma\left(-\frac{\epsilon}{2}\right) p^2 2 \int_0^1 dx x(1-x) \chi^{\frac{\epsilon}{2}}. \end{aligned} \tag{16.35}$$

But

$$\Gamma\left(-\frac{\epsilon}{2}\right) = -\frac{2}{\epsilon} - \gamma_E + O(\epsilon), \quad \chi^{\frac{\epsilon}{2}} = 1 + \frac{\epsilon}{2}\ln\chi + O(\epsilon^2), \quad \pi^{\frac{\epsilon}{2}} = 1 + \frac{\epsilon}{2}\ln\pi + O(\epsilon^2).$$

Hence

$$\Pi_0(p^2) = -i\pi^2 p^2 2 \int_o^1 dx x(1-x) \left[\Delta + \ln\chi\right] + \mathcal{O}(\epsilon),$$

where

$$\Delta = \frac{2}{\epsilon} + \gamma_E + \ln\pi$$

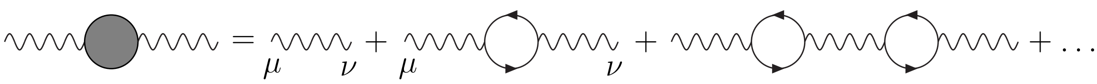
is the part which diverges when $\epsilon \to 0$. Finally

$$\Pi_0(p^2) = -i\pi^2 p^2 2 \left\{\frac{\Delta}{6} + \int_0^1 dx x(1-x) \ln\chi\right\}.$$

Thus [4]

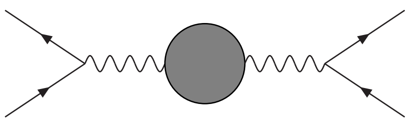
$$\Pi_{\mu\nu} = i g_{\mu\nu} \Pi_F, \quad \text{with} \quad \Pi_F = \frac{e^2}{2\pi^2} p^2 \left\{ \frac{\Delta}{6} + \int_0^1 dx x(1-x) \ln \chi \right\}. \tag{16.36}$$

This result serves to compute the so called *Dressed Photon Propagator*,



$$\begin{aligned} &= -i g_{\mu\nu} \frac{1}{p^2} - i g_{\mu\nu} \frac{1}{p^2} \Pi_F \frac{1}{p^2} + \cdots \\ &= -i g_{\mu\nu} \frac{1}{p^2} \left(1 + \frac{\Pi_F}{p^2} + \cdots \right) = -i g_{\mu\nu} \frac{1}{p^2 \left(1 - \frac{\Pi_F}{p^2} \right)} \\ &= -i g_{\mu\nu} \frac{1}{p^2 \left(1 - \frac{e^2}{2\pi^2} \left\{ \frac{\Delta}{6} + \int_0^1 dx x(1-x) \ln \chi \right\} \right)}. \end{aligned} \tag{16.37}$$

This dressed propagator has to be inserted in the amplitude we want to compute,

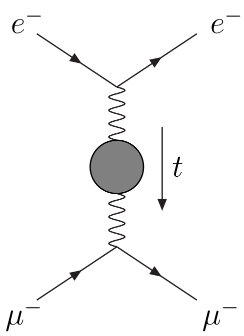


$$= (ie)^2 (-i) J_\beta(e^-) J^\beta(\mu^-) \frac{1}{p^2(\dots)}, \tag{16.38}$$

so that the following combination appears,

$$\frac{e^2}{\left(1 - \frac{e^2}{2\pi^2} \left\{ \frac{\Delta}{6} + \int_0^1 dx x(1-x) \ln \chi \right\} \right)}. \tag{16.39}$$

Before the theory can be used to predict observables, one has to measure the QED coupling e by using, for example, the low energy limit of the $e^-\mu^-$ scattering,



[4]Note that the fact that $\Pi_F \propto p^2$ means that the photon remains massless.

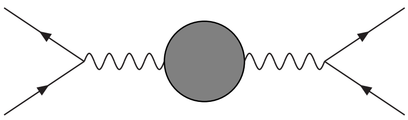
Due to the Ward Identities, only the propagator corrections contribute when $t \to 0$. Note the appearance of the dressed propagator we just computed. This fixes e in terms of the measured value of $\alpha \simeq 1/137.036$,

$$4\pi\alpha = \frac{e^2}{1 - \frac{e^2}{2\pi^2}\left\{\frac{\Delta}{6} + \int_0^1 dx x(1-x)\ln m^2\right\}}, \tag{16.40}$$

where we used $\lim_{t\to 0} \chi = m^2$. This condition determines $1/e^2$,

$$\frac{1}{e^2} = \frac{1}{4\pi\alpha} + \frac{1}{2\pi^2}\left\{\frac{\Delta}{6} + \int_0^1 dx x(1-x)\ln m^2\right\}. \tag{16.41}$$

Inserting this expression in (16.38) makes the theory UV finite and predictive,



$$= i\,\frac{4\pi\alpha\; J_\beta(e^-)J^\beta(\mu^-)}{1 - \frac{2\alpha}{\pi}\int_0^1 dx x(1-x)\ln\left[1 - \frac{p^2}{m^2}x(1-x)\right]}.$$

Note that the correction we just computed can be parametrized by introducing a *running* $\alpha(s)$,

$$\alpha(s) = \frac{\alpha(0)}{1 - \frac{2\alpha(0)}{\pi}\int_0^1 dx x(1-x)\ln\left[1 - \frac{s}{m^2}x(1-x)\right]}, \tag{16.42}$$

where $\alpha(0) \simeq 1/137.036$.

The asymptotic $s \gg m^2$ behaviour of real part of $\alpha(s)$ can be easily computed,

$$\Re e \int_0^1 dx x(1-x)\ln\left[1 - \frac{s}{m^2}x(1-x)\right] \overset{s\to\infty}{\to} \frac{1}{6}\ln\left|s/m^2\right|. \tag{16.43}$$

This gives

$$\Re e\left[\alpha(s)\right] \overset{s\to\infty}{\to} \frac{\alpha(0)}{1 - \frac{\alpha(0)}{3\pi}\ln|s/m^2|}, \tag{16.44}$$

so that $\alpha(s)$ increases with the energy.

The running of α is observed, for example, at LEP, where one measures

$$\Re e\left[\alpha(M_Z^2)\right] = \frac{1}{128.9}. \tag{16.45}$$

Finally, note that the dependence of $\alpha(s)$ with the energy can also be derived by using renormalization group arguments.

16.8 Large m_{top} corrections to M_W in the SM

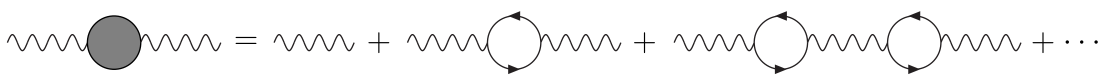
We are now ready to show the renormalization procedure at work in the full SM. For that we use a specific example, namely the computation of the corrections to M_W induced by top quark loops (in the limit $m_{top} \to \infty$). These corrections are fermionic ones, therefore gauge invariant. As already observed, *before the theory becomes predictive* we have to connect the Lagrangian's parameters to a specific set of measurements. The parameters in the Lagrangian are $\{g, s_\theta, M\}$ and we choose to relate them with the measured values of $\{\alpha, G_F, M_Z\}$. The latter set is the chosen *input parameter set*. Of course, we did not include there M_W, which is in fact what we want to predict! Again, renormalizing means finding the relations between the 2 sets,

$$\{g, s_\theta, M\} \longleftrightarrow \{\alpha, G_F, M_Z\}\,,$$

that can be achieved by considering the *dressed*, Dyson resummed propagators for the gauge bosons,

$$\bar{\Delta}_\gamma := i\frac{-g_{\mu\nu}}{p^2\left[1 - g^2 s_\theta^2 \Pi_\gamma(p^2)\right]},$$
$$\bar{\Delta}_W := i\frac{-g_{\mu\nu}}{p^2 - M^2 - \frac{g^2}{4}\Sigma_W(p^2)},$$
$$\bar{\Delta}_Z := i\frac{-g_{\mu\nu}}{p^2 - \frac{M^2}{c_\theta^2} - \frac{g^2}{4c_\theta^2}\Sigma_Z(p^2)}.$$

Those propagators are nothing but the sum of the series



and are divergent quantities (before renormalization). For example, in the previous section we have computed

$$\Pi_\gamma(p^2) = \frac{1}{2\pi^2}\left\{\frac{\Delta}{6} + \int_0^1 dx x(1-x)\ln\left[m^2 - p^2 x(1-x)\right]\right\}. \tag{16.46}$$

Analogously, one can calculate $\Sigma_Z(p^2)$ and $\Sigma_W(p^2)$ (see later). Π_γ, Σ_Z and Σ_W are called self-energies. By means of the dressed propagators we can find the relations between $\{g, s_\theta, M\}$ and $\{\alpha, G_F, M_Z\}$, namely the *Fitting Equations*,

$$\frac{g^2 s_\theta^2}{1 - g^2 s_\theta^2 \Pi_\gamma(0)} = 4\pi\alpha, \tag{16.47}$$

$$\frac{g^2}{8\left[M^2 + \frac{g^2}{4}\Sigma_W(0)\right]} = \frac{G_F}{\sqrt{2}} \equiv G, \tag{16.48}$$

$$\frac{M^2}{c_\theta^2} + \frac{g^2}{4c_\theta^2} Re\Sigma_Z(M_Z^2) = M_Z^2, \tag{16.49}$$

which link the experimental quantities on the r.h.s. to the combinations of Lagrangian's parameters and self energies appering in the l.h.s.

Equation (16.47) is the charge renormalization condition we have already discussed in section 16.7. Equation (16.48) defines G_F through the muon decay. Remember that we have computed it at the tree-level,

$$\frac{g^2}{8M_W^2} = \frac{G_F}{\sqrt{2}}, \tag{16.50}$$

where M_W^2 is nothing but low energy limit of the tree-level W propagator

$$\frac{1}{M_W^2} = -\lim_{p^2\to 0} \frac{1}{p^2 - M_W^2}.$$

Thus, when turning on loop corrections, the tree-level propagator must be replaced by the dressed one, whose low energy limit reads

$$-\lim_{p^2\to 0} \frac{1}{p^2 - M^2 - \frac{g^2}{4}\Sigma_W(p^2)},$$

which gives (16.48). Finally, (16.49) defines the Z mass to be the real part of the pole of the Z propagator. At the tree-level $M_Z^2 = \frac{M_W^2}{c_\theta^2}$, while at one-loop we have the condition [5]

$$M_Z^2 - \frac{M^2}{c_\theta^2} - \frac{g^2}{4c_\theta^2}\Sigma_Z^R = 0,$$

that is (16.49).

The next step is inverting (16.47), (16.48) and (16.49), namely determining g, s_θ, and M in terms of $\{\alpha, G_F, M_Z\}$. However, before doing so, let us explicitly note that the r.h.s. of the fitting equations is a finite quantity (it is a measured value!). So

[5] Here and in the following we define $\Sigma_{W,Z}^R := Re\,(\Sigma_{W,Z})$.

that also the l.h.s. must be finite. Since Π_γ and $\Sigma_{W,Z}$ contains divergences, that implies that also $\{g, s_\theta, M\}$ must be divergent in such way that the two divergences compensate each other. This is the essence of the renormalization procedure: one gives up with the idea of having finite parameters in the Lagrangian and accepts the fact that only physical, observable quantities must be finite. Indeed, in the following we will explicitly see that substituting $\{g, s_\theta, M\}$ in the real part of the W (dressed) propagator gives a finite prediction for the W mass.

16.9 Problem: Solving the Fitting Equations

Solve the Fitting Equations 16.47-16.49.

Solution

The first equation fixes $\frac{1}{g^2 s_\theta^2}$:

$$\frac{1}{g^2 s_\theta^2} = \Pi_\gamma(0) + \frac{1}{4\pi\alpha}. \tag{16.51}$$

The second equation fixes $\frac{M^2}{g^2}$:

$$4\frac{M^2}{g^2} = \frac{1}{2G} - \Sigma_W(0). \tag{16.52}$$

From the last equation we obtain $\frac{g^2}{c_\theta^2}$:

$$\begin{aligned} \frac{g^2}{4c_\theta^2} &= M_Z^2 \left\{\frac{4M^2}{g^2} + \Sigma_Z^R\right\}^{-1} \\ &= \frac{2GM_Z^2}{1+2G\Sigma_F}, \end{aligned} \tag{16.53}$$

where $\Sigma_F = \Sigma_Z^R - \Sigma_W(0)$. By multiplying the first and the third equation one derives

$$s_\theta^2 c_\theta^2 = \frac{\pi\alpha}{2GM_Z^2}\,\frac{1+2G\Sigma_F}{1+4\pi\alpha\Pi_\gamma(0)}. \tag{16.54}$$

Since we are interested in solutions at one-loop, we can look for a perturbative solution by setting $s_\theta^2 = \bar{s}^2 + \delta$ and $c_\theta^2 = \bar{c}^2 - \delta$ in (16.54):

$$\bar{s}^2\bar{c}^2 + \delta(\bar{c}^2 - \bar{s}^2) = \frac{\pi\alpha}{2GM_Z^2}\left\{1 + 2G\Sigma_F - 4\pi\alpha\Pi_\gamma(0)\right\}. \tag{16.55}$$

Then

$$\bar{s}^2\bar{c}^2 = \frac{\pi\alpha}{2GM_Z^2} \equiv a, \tag{16.56}$$

namely

$$\bar{s}^2 = \frac{1}{2}\left\{1 - \left(1 - \frac{2\pi\alpha}{GM_Z^2}\right)^{\frac{1}{2}}\right\}, \tag{16.57}$$

and

$$\delta = \frac{\bar{s}^2\bar{c}^2}{\bar{c}^2 - \bar{s}^2}\left[2G\Sigma_F - 4\pi\alpha\Pi_\gamma(0)\right].$$

Therefore our solution is:

$$\begin{aligned} s_\theta^2 &= \bar{s}^2\left\{1 + \frac{\bar{c}^2}{\bar{c}^2 - \bar{s}^2}\left[2G\Sigma_F - 4\pi\alpha\Pi_\gamma(0)\right]\right\} \\ &= \bar{s}^2 + \delta = \bar{s}^2(1 + \frac{\delta}{\bar{s}^2}). \end{aligned}$$

16.10 Computing M_W

Compute M_W by inserting the solution of the fitting equations into the dressed W propagator and extract the terms proportional to m_{top}^2.

Solution

M_W is defined to be the the zero of the Real part of the inverse W propagator. We then look for a $x \equiv M_W^2$ solution of

$$Re\left\{\frac{x}{g^2} - \frac{M^2}{g^2} - \frac{\Sigma_W(x)}{4}\right\} = 0. \tag{16.58}$$

Let us define $f(x) \equiv \frac{x}{g^2} - \frac{M^2}{g^2} - \frac{\Sigma_W(x)}{4}$. By inserting the fitting equations we just computed we derive

$$\begin{aligned}
f(x) &= (xs_\theta^2)\left(\frac{1}{g^2 s_\theta^2}\right) - \frac{1}{4}\left(\frac{4M^2}{g^2}\right) - \frac{\Sigma_W(x)}{4} \\
&= \frac{x\bar{s}^2}{4\pi\alpha}\left(1+\frac{\delta}{\bar{s}^2}\right)(1+4\pi\alpha\Pi_\gamma(0)) - \frac{1}{4}\left(\frac{1}{2G} - \Sigma_W(0)\right) - \frac{\Sigma_W(x)}{4} \\
&= -\frac{1}{8G} + \frac{\Sigma_W(0)-\Sigma_W(x)}{4} \\
&\quad + \frac{x\bar{s}^2}{4\pi\alpha}\left\{1+4\pi\alpha\Pi_\gamma(0) + \frac{\bar{c}}{\bar{c}^2-\bar{s}^2}\left[2G\Sigma_F - 4\pi\alpha\Pi_\gamma(0)\right]\right\}, \qquad (16.59)
\end{aligned}$$

and the solution is that value of x such that $Ref(x) = 0$. The above solution is the full fermionic contribution. Now we want to explicitly compute it in the leading approximation for large values of m_{top}. When $m_{top} \to \infty$ one obtains terms proportional to $\ln(m_t)$ and m_t^2. We will keep only the latter ones. $\Pi_\gamma(0)$ is logarithmic in m_t :

$$\Pi_\gamma(0) = \frac{1}{2\pi^2}\left\{\frac{\Delta}{6} + \frac{1}{6}\ln m_t^2\right\}, \qquad (16.60)$$

so that, at the leading order in m_t^2, $\Pi_\gamma(0) \sim 0$. In addition, when $m_t \to \infty$ there is just one scale left in the problem, namely the top mass, so that

$$\lim_{m_t\to\infty}\left(\Sigma_W(x) - \Sigma_W(0)\right) \sim 0. \qquad (16.61)$$

In this limit we then have

$$f(x) \sim -\frac{1}{8G} + \frac{x\bar{s}^2}{4\pi\alpha}\left\{1 + \frac{\bar{c}^2}{\bar{c}^2-\bar{s}^2}2G\Sigma_F\right\}, \qquad (16.62)$$

therefore

$$M_W^2 \equiv x \sim \frac{\pi\alpha}{2G\bar{s}^2}\left\{1 - \frac{2G\bar{c}^2\Sigma_F}{\bar{c}^2-\bar{s}^2}\right\}. \qquad (16.63)$$

Then, one has to compute the terms proportional to m_t^2 in the combination:

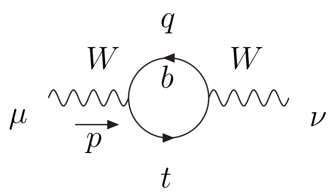
$$\Sigma_F = \Sigma_Z^R - \Sigma_W(0). \qquad (16.64)$$

16.11 Problem: Computation of the W self-energy

Compute the asymptotic behaviour of Σ_W when $m_{top} \to \infty$.

Solution

The diagram to be computed is



$$= -\left(\frac{-ig}{2\sqrt{2}}\right)^2 \frac{ii}{(2\pi)^4} I_{\mu\nu} \equiv \Sigma^W_{\mu\nu},$$

where

$$I_{\mu\nu} = \int d^n q \frac{N_{\mu\nu}}{D_1 D_2}, \tag{16.65}$$

with

$$D_1 = q^2 + i\epsilon, \quad D_2 = (q+p)^2 - m^2, \quad m \equiv m_{top}, \tag{16.66}$$

and

$$N_{\mu\nu} = 4Tr\left\{\gamma_\mu \not{q} \gamma_\nu (\not{q} + \not{p})\omega_+\right\}, \quad \omega_\pm = \frac{1}{2}(1 \pm \gamma_5). \tag{16.67}$$

The asymptotic behaviour of D_2 when $m \to \infty$ is

$$D_2 \sim q^2 - m^2, \tag{16.68}$$

therefore

$$I_{\mu\nu} \sim \int_0^1 dx \int d^n q \frac{N_{\mu\nu}}{(q^2 - m^2 x)^2}, \tag{16.69}$$

and since, by power counting, only terms quadratic in q can give a contribution $O(m^2)$:

$$N_{\mu\nu} \sim 4Tr\left\{\gamma_\mu \not{q} \gamma_\nu \not{q} \omega_+\right\} = 8\left\{2q_\mu q_\nu - q^2 g_{\mu\nu}\right\}, \tag{16.70}$$

so that

$$I_{\mu\nu} \sim 8\left(\frac{2}{n} - 1\right) \int_0^1 dx \underbrace{\int d^n q \frac{q^2}{(q^2 - m^2 x)^2}}_{J} g_{\mu\nu}. \tag{16.71}$$

We already computed a J-like integral in (16.34):

$$J = i\pi^2\pi^{\frac{\epsilon}{2}}\Gamma\left(-\frac{\epsilon}{2}\right)\frac{4+\epsilon}{2+\epsilon}\chi\chi^{\frac{\epsilon}{2}}, \quad \chi = m^2x, \quad n = 4+\epsilon, \tag{16.72}$$

furthermore

$$\left(\frac{2}{n}-1\right) = -\frac{2+\epsilon}{4+\epsilon}, \tag{16.73}$$

Then

$$\begin{aligned} I_{\mu\nu} &\sim -8i\pi^2\pi^{\frac{\epsilon}{2}}\Gamma\left(-\frac{\epsilon}{2}\right)\int_0^1 dx m^2x(m^2x)^{\frac{\epsilon}{2}}g_{\mu\nu} \\ &= 4i\pi^2 g_{\mu\nu}m^2\left\{\Delta + \ln m^2 - \frac{1}{2}\right\} + \mathcal{O}(\epsilon), \end{aligned} \tag{16.74}$$

where, as usual,

$$\Delta = \frac{2}{\epsilon} + \gamma_E + \ln\pi, \tag{16.75}$$

and where we used

$$\int_0^1 dx x \ln x = -\frac{1}{4}. \tag{16.76}$$

Therefore

$$\Sigma^W_{\mu\nu} = ig_{\mu\nu}\frac{g^2}{4}\Sigma_W(p^2), \tag{16.77}$$

with

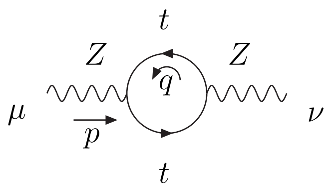
$$\Sigma_W(p^2) = -\frac{m^2}{8\pi^2}\left\{\Delta + \ln m^2 - \frac{1}{2}\right\}. \tag{16.78}$$

16.12 Problem: Computation of the Z self-energy

Compute the asymptotic behaviour of Σ_Z when $m_{top} \to \infty$.

Solution

The diagram to be computed is



$$= -\left(\frac{-ig}{2c_\theta}\right)^2 \frac{ii}{(2\pi)^4} \underbrace{\int d^n q \frac{N_{\mu\nu}}{D_1 D_2}}_{I_{\mu\nu}} \equiv \Sigma^Z_{\mu\nu}$$

where

$$\begin{aligned} D_1 &= q^2 - m^2, \quad D_2 \;=\; (q+p)^2 - m^2, \\ N_{\mu\nu} &= Tr\left[\gamma_\mu (v^+\omega^+ + v^-\omega^-)(\not{q} + m)\gamma_\nu (v^+\omega^+ + v^-\omega^-)(\not{q} + \not{p} + m)\right], \\ v_+ &\equiv v + a \;=\; -2s_\theta^2 Q_t, \quad v_- \;\equiv\; v - a \;=\; 1 - 2s_\theta^2 Q_t, \quad Q_t \;=\; 2/3. \end{aligned} \tag{16.79}$$

When $m \to \infty$

$$D_1 = D_2 \sim q^2 - m^2, \tag{16.80}$$

and

$$\begin{aligned} N_{\mu\nu} &\sim Tr[\gamma_\mu(\cdots)\not{q}\gamma_\nu(\cdots)\not{q}] + m^2[\gamma_\mu(\cdots)\gamma_\nu(\cdots)] \\ &= \alpha Tr[\gamma_\mu\not{q}\gamma_\nu\not{q}] + \beta m^2 Tr[\gamma_\mu\gamma_\nu], \end{aligned} \tag{16.81}$$

with

$$\alpha = \frac{v_+^2 + v_-^2}{2}, \qquad \beta = v_+ v_-. \tag{16.82}$$

Then

$$\begin{aligned} N_{\mu\nu} &\sim 4\left\{\alpha\left(2q_\mu q_\nu - q^2 g_{\mu\nu}\right) + m^2\beta g_{\mu\nu}\right\} \Longrightarrow \\ N_{\mu\nu} &\sim 4g_{\mu\nu}\left\{\alpha\left(\frac{2}{n} - 1\right)q^2 + m^2\beta\right\}, \end{aligned} \tag{16.83}$$

so that

$$I_{\mu\nu} \equiv \int d^n q \frac{N_{\mu\nu}}{D_1 D_2} \sim 4g_{\mu\nu}\left\{\alpha\left(\frac{2}{n} - 1\right)\underbrace{\int d^n q \frac{q^2}{(q^2 - m^2)^2}}_{J} + m^2\beta \underbrace{\int d^n q \frac{1}{(q^2 - m^2)^2}}_{I}\right\}. \tag{16.84}$$

We already computed (see eqs. (16.34) and (16.31))

$$\begin{aligned} J &= i\pi^2\pi^{\frac{\epsilon}{2}}\Gamma\left(-\frac{\epsilon}{2}\right)\frac{4+\epsilon}{2+\epsilon}m^2(m^2)^{\frac{\epsilon}{2}} \\ I &= i\pi^2\pi^{\frac{\epsilon}{2}}\Gamma\left(-\frac{\epsilon}{2}\right)(m^2)^{\frac{\epsilon}{2}}, \end{aligned} \qquad (16.85)$$

furthermore

$$\left(\frac{2}{n}-1\right) = -\frac{2+\epsilon}{4+\epsilon}. \qquad (16.86)$$

Therefore

$$\begin{aligned} I_{\mu\nu} &= 4g_{\mu\nu}m^2 i\pi^2(m^2\pi)^{\frac{\epsilon}{2}}\Gamma\left(-\frac{\epsilon}{2}\right)\underbrace{\{-\alpha+\beta\}}_{-\frac{1}{2}} \\ &= -2m^2 g_{\mu\nu} i\pi^2\left(1+\frac{\epsilon}{2}\ln(m^2\pi)\right)\left(-\frac{2}{\epsilon}-\gamma_E\right) \\ &= 2m^2 i\pi^2 g_{\mu\nu}\left(\underbrace{\frac{2}{\epsilon}+\gamma_E+\ln\pi}_{\Delta}+\ln m^2\right), \end{aligned} \qquad (16.87)$$

and

$$\Sigma^Z_{\mu\nu} = ig_{\mu\nu}\frac{g^2}{4c_\theta^2}\Sigma_Z(p^2), \qquad (16.88)$$

with

$$\Sigma_Z(p^2) = -\frac{m^2}{8\pi^2}\left(\Delta+\ln m^2\right). \qquad (16.89)$$

16.13 The leading m_{top} contribution to M_W

The combination appearing in the solution for the W mass (see (16.63)) is

$$\Sigma_F = Re\Sigma_Z(M_Z^2)-\Sigma_W(0) = -\frac{m_{top}^2}{16\pi^2}. \qquad (16.90)$$

Note that all divergences canceled out, together with $\ln m_{top}^2$ (this is necessary to have scale independent results) as expected. When considering a factor 3, due to the QCD color, one has to replace

$$\Sigma_F \to -\frac{3m_{top}^2}{16\pi^2}. \tag{16.91}$$

Therefore

$$M_W^2 = \frac{\pi\alpha}{2G\bar{s}^2}\left\{1 + \frac{3}{8\pi^2}\frac{G\bar{c}^2 m_{top}^2}{\bar{c}^2 - \bar{s}^2}\right\}. \tag{16.92}$$

Equation (16.92) can be now computed by using the experimental values

$$\begin{aligned} M_Z &= 91.1876 \text{ GeV } (LEP1), \\ m_{top} &= 173.5 \text{ GeV } (TEVATRON + LHC), \\ G_F &= 1.16637 \times 10^{-5} \ \text{ GeV}^{-2}, \\ \alpha &= \frac{1}{137.036}, \end{aligned} \tag{16.93}$$

to find the leading m_{top} contribution at one-loop

$$\begin{aligned} (M_W)_{tree} &= 80.939 \text{ GeV} \\ (M_W)_{1-loop} &= 81.459 \text{ GeV}, \end{aligned} \tag{16.94}$$

to be compared with the experimental value

$$(M_W)_{exp} = 80.385 \pm 0.015 \text{ GeV } (LEP2 + TEVATRON). \tag{16.95}$$

The corrections given by m_{top} seem then to go in the wrong direction. But there is one important ingredient missing, i.e. the vacuum polarization, namely the running of α_{EM} computed in (16.42). By using a different scheme, in which $\alpha(M_Z)$ is used to resum the large logs due to the light fermions:

$$\begin{aligned} M_Z &= 91.1876 \text{ GeV } (LEP1), \\ m_{top} &= 173.5 \text{ GeV } (TEVATRON + LHC), \\ G_F &= 1.16637 \times 10^{-5} \ \text{ GeV}^{-2}, \\ \alpha(M_Z) &= \frac{1}{128.89} \ (LEP1), \end{aligned} \tag{16.96}$$

one obtains a prediction which includes both leading m_{top} contributions and vacuum polarization effects

$$\begin{aligned} (M_W)'_{tree} &= 79.958 \text{ GeV} \\ (M_W)'_{1-loop} &= 80.495 \text{ GeV}, \end{aligned} \tag{16.97}$$

which is sow in very good agreement with the experimental value in (16.95). Of course sub-leading radiative corrections are also present (and can be computed!).

16.14 Problem: Multiplicative renormalization

By using multiplicative renormalization, renormalize the Lagrangian of the scalar theory $g\phi^4$.

Solution

We start writing the Lagrangian in terms of *bare* parameters and fields (denoted by the subscript $_0$) as follows

$$\mathcal{L}^0 = \frac{1}{2}\left[(\partial_\mu\phi_0)(\partial^\mu\phi_0) - m_0^2\phi_0^2\right] - \frac{g_0}{4!}\phi_0^4. \tag{16.98}$$

Then, we suppose that the *bare* (in general infinite in 4 dimensions) parameters are connected to the renormalized ones (finite in 4 dimension) by the following, multiplicative relations

$$\begin{aligned} \phi_0 &= Z_\phi^{\frac{1}{2}}\phi \\ m_0 &= Z_m m \\ g_0 &= Z_g g\mu^{-\epsilon} \equiv Z_g g_R \Leftrightarrow g_R = g\mu^{-\epsilon}, \end{aligned} \tag{16.99}$$

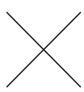
where $\mu^{-\epsilon}$ has been introduced in order to keep the coupling constant g dimensionless. Then one can split the original Lagrangian into a renormalized one ($\mathcal{L}$) plus a counterterm Lagrangian ($\mathcal{L}^c$) as follows

$$\mathcal{L}^0 = \mathcal{L} + \mathcal{L}^c. \tag{16.100}$$

Explicitly, the two parts read

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left[(\partial_\mu\phi)(\partial^\mu\phi) - m^2\phi^2)\right] - \frac{g_R}{4!}\phi^4, \\ \mathcal{L}^c &= \frac{1}{2}(Z_\varphi - 1)(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}(Z_m^2 Z_\varphi - 1)m^2\phi^2 - \frac{1}{4!}(Z_g Z_\varphi^2 - 1)g_R\phi^4. \end{aligned} \tag{16.101}$$

$\mathcal{L}$ gives the following Feynman rules in terms of finite parameters (in 4 dimensions)

 $= i\dfrac{1}{p^2 - m^2}$  $= -ig_R$

from which the counter terms in $\mathcal{L}^c$ can be fixed to compensate the infinities coming from the loop functions. All Z in $\mathcal{L}^c$ must therefore have the form

$$Z = \left(1 + \frac{\alpha_1}{\epsilon} + \frac{\alpha_2}{\epsilon^2} + \cdots\right). \tag{16.102}$$

16.15 Problem: The renormalization constants

Fix, at one-loop, the renormalization constants Z_ϕ, Z_m and Z_g in the theory $g\phi^4$.

Solution

In problem 16.5 we computed

$$P.P.(A(m^2)) = -i\pi^2 m^2 \left(\frac{2}{\epsilon}\right) \quad \text{and} \quad P.P.(B(p^2, m, m)) = -i\pi^2 \left(\frac{2}{\epsilon}\right). \tag{16.103}$$

Now we compute the $P.P$ of the corrections to the bare propagator ——— .

They are given by the following diagram

$$= \frac{1}{2}\frac{1}{(2\pi)^4}(-ig_R)(i)\int d^n q \frac{1}{(q^2 - m^2)}$$

$$= -\frac{g_R}{32\pi^4} i\pi^2 m^2 \left(\frac{2}{\epsilon} + \cdots\right)$$

$$= -\frac{ig_R}{32\pi^2} m^2 \left(\frac{2}{\epsilon} + \cdots\right).$$

Since this correction is $\propto m^2$ it only gives a contribution to the mass renormalization and no external field renormalization is necessary. Therefore

$$Z_\phi = 1. \tag{16.104}$$

From the term in $\mathcal{L}_c$

$$-\frac{m^2\phi^2}{2}(Z_m^2 - 1) \tag{16.105}$$

the following counter-term is generated

 $= -im^2(Z_m^2 - 1),$

and Z_m can then be fixed by requiring

$0 = P.P.\Big[$  $+$  $\Big] = -im^2(Z_m^2 - 1) - \frac{ig_R}{32\pi^2}m^2\left(\frac{2}{\epsilon}\right),$

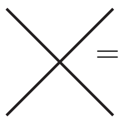
Namely

$$-im^2\left[Z_m^2 - 1 + \frac{g_R}{16\pi^2\epsilon}\right] = 0.$$

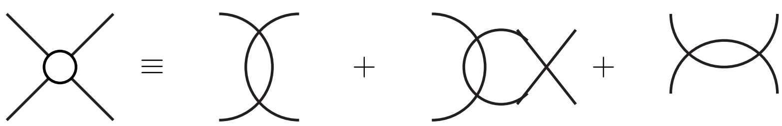
Therefore

$$Z_m = 1 - \frac{g_R}{32\pi^2\epsilon}. \tag{16.106}$$

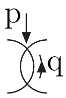
Now consider the vertex corrections to

 $= -ig_R.$

At one-loop one has



For the first diagram one computes



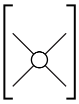
$$= \frac{1}{2}\frac{1}{(2\pi)^4}(-ig_R)^2(i)^2\int d^nq\frac{1}{(q^2-m^2)}\frac{1}{((q+p^2)-m^2)}$$

$$= \frac{1}{32\pi^4}g_R^2 B(p^2,m,m) = \frac{1}{34\pi^4}g_R^2(-i\pi^2\frac{2}{\epsilon}+\cdots)$$

$$= \frac{-i}{16\pi^2}g_R^2\frac{1}{\epsilon}+\cdots .$$

Therefore, since this result does not depend on p, also the other 2 diagrams give the same contribution, so that

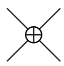
$$P.P.\left[\quad\right] = -\frac{3i}{16\pi^2}g_R^2\frac{1}{\epsilon}.$$



From the term in $\mathcal{L}_c$

$$-\frac{1}{4!}(Z_g-1)g_R\phi^4$$

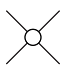
the following counter-term is generated

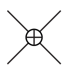


$$= -i(Z_g-1)g_R.$$

We then fix Z_g such that

$$P.P.\left[\quad + \quad\right] = 0, \qquad (16.107)$$





namely

$$\frac{-3i}{16\pi^2}g_R^2\frac{1}{\epsilon} - i(Z_g-1)g_R = 0, \qquad (16.108)$$

so that

$$Z_g = 1 - \frac{3}{16\pi^2}\frac{g_R}{\epsilon}. \qquad (16.109)$$

In summary

$$\begin{aligned} Z_\phi &= 1, \\ Z_m &= 1-\frac{g_R}{32\pi^2\epsilon}, \\ Z_g &= 1-\frac{3}{16\pi^2}\frac{g_R}{\epsilon}. \end{aligned} \qquad (16.110)$$

16.16 Tensor integrals

In practical calculations one-loop tensor integrals appear of the form

$$\int d^n q \, \frac{q^{\mu_1} \cdots q^{\mu_r}}{D_0 \cdots D_m}, \tag{16.111}$$

where $D_i = (q + p_i)^2 - m_i^2$ and $p_0 = 0$. Such integrals can always be reduced to scalar integrals (namely integrals with no q in the numerator) by means of the so called Passarino-Veltman reduction technique [6]. Because of that, one can write the following Master Equation for any one-loop amplitude $\mathcal{M}$

$$\mathcal{M} = \sum_i d_i \, \mathrm{Box}_i + \sum_i c_i \, \mathrm{Triangle}_i + \sum_i b_i \, \mathrm{Bubble}_i + \sum_i a_i \, \mathrm{Tadpole}_i + \mathrm{R}, \tag{16.112}$$

where d_i, c_i, b_i and a_i are the coefficients of the scalar 4-,3-,2-,1-point functions and R is a left over piece called Rational Part of the amplitude.

16.17 Problem: The rank-1 two point function

Express the rank-1 two point function

$$B^\mu(p_1^2, m_0^2, m_1^2) := \int d^n q \, \frac{q^\mu}{D_0 D_1} \tag{16.113}$$

in terms of one-loop scalar integrals.

Solution

Since p_1^μ is the only momentum at our disposal to obtain the desired tensor structure, one can write

$$B^\mu(p_1^2, m_0^2, m_1^2) = B_1 \, p_1^\mu. \tag{16.114}$$

The constant B_1 can be determined by multiplying (16.114) by $p_{1\mu}$,

$$\int d^n q \, \frac{(q \cdot p_1)}{D_0 D_1} = p_1^2 B_1. \tag{16.115}$$

Reconstructing denominators gives $(q \cdot p_1) = \frac{1}{2}\left(D_1 - D_0 + m_1^2 - m_0^2 - p_1^2\right)$, so that

$$
\begin{aligned}
B_1 &= \frac{1}{2\,p_1^2} \int d^n q \, \frac{D_1 - D_0 + m_1^2 - m_0^2 - p_1^2}{D_0 D_1} \\
&= \frac{1}{2\,p_1^2} \left[(m_1^2 - m_0^2 - p_1^2) \int d^n q \, \frac{1}{D_0 D_1} + \int d^n q \, \frac{1}{D_0} - \int d^n q \, \frac{1}{D_1} \right].
\end{aligned}
$$

Hence, the desired decomposition reads

$$
B^\mu(p_1^2, m_0^2, m_1^2) = \frac{p_1^\mu}{2\,p_1^2} \left[(m_1^2 - m_0^2 - p_1^2) \int d^n q \, \frac{1}{D_0 D_1} + \int d^n q \, \frac{1}{D_0} - \int d^n q \, \frac{1}{D_1} \right].
$$

16.18 Cuttools

The Passarino-Veltman technique can always be used, but it becomes very cumbersome for high point high rank tensor integrals. In addition, each tensor structure should be treated separately, with a lot of analytic work. Very recently, new numerical techniques appeared, where those problems have been solved by working at the *integrand level* of the loop function [7]. This techniques allow one to numerically compute the coefficients of the contributing scalar functions just by knowing *numerically* the numerator function $N(q)$ of the loop *integrand*. More in detail, rewriting the amplitude in equation 16.112 as follows

$$
\mathcal{M} = \int d^n q \, \frac{N(q)}{D_0 \cdots D_m}, \tag{16.116}
$$

all the coefficients d_i, c_i ,b_i and a_i can be determined by solving simple systems of linear equations involving the numerator function $N(q)$ computed at special values of q.

A program implementing such a strategy is CUTTOOLS [2] and can be downloaded in

http://www.ugr.es/local/pittau/CutTools/.

16.19 Problem*: The light-light scattering

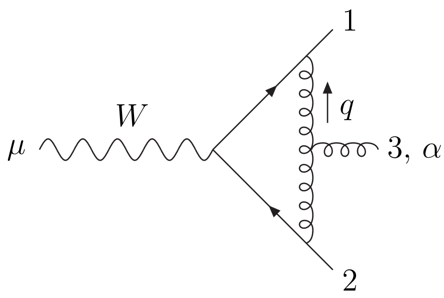
By using CUTTOOLS, prove, numerically, that the *P.P.* of the QED process

$$\gamma\gamma \to \gamma\gamma$$

is zero. Prove this both in the case of massless and massive fermion loop.

16.20 Problem*: $W \to 3$ jets

By using CUTTOOLS compute, numerically and in one phase space point, the following diagram contributing to $W \to 3$ jets



where particles 1 and 2 are massless quarks and the curly lines represent gluons.

Chapter 17

The β function

The ultraviolet divergent behaviour of a Quantum Field Theory describing Nature can be used to determine the running of its coupling constant. There are 2 possibilities

1. either the coupling constant *increases* with energy,
2. or the coupling constant *decreases* with energy.

This second possibility happens in QCD, that is the theory describing the so called *strong interactions*, and has the very important phenomenological consequence that, in the high energy regime, collisions of strong interacting particles, like protons, become *perturbatively* computable, as we have seen in Chapter 14. This Quantum Field Theory property is directly linked to a fundamental quantity called β function [4]. In this chapter, we introduce and explicitly compute the β function of the simple scalar $g\phi^4$ theory, and give a flavour of what happens in QED and QCD.

17.1 Problem: The dimension of the coupling constant in n dimensions

Calculate the dimensions of ϕ_0 and g_0 in the Lagrangian

$$\mathcal{L}^0 = \frac{1}{2}\left[(\partial_\mu\phi_0)(\partial^\mu\phi_0) - m_0^c ont2\phi_0^2\right] - \frac{g_0}{4!}\phi_0^4. \tag{17.1}$$

continued to $n = 4 + \epsilon$

Solution

In n dimensions the action $\int d^n\mathcal{L}$ should be dimensionless. Therefore

$$[\mathcal{L}] = M^n.$$

The dimension of ϕ_0 can be read from the kinetic term

$$M^n = \left[m^2\phi_0^2\right] = M^2\left[\phi_0\right]^2 \to \left[\phi_0\right] = M^{\frac{n-2}{2}}.$$

The dimension of g_0 can be read form the interaction term

$$M^n = [g_0]\left[\phi_0\right]^4 = [g_0]\, M^{2n-4}.$$

Therefore

$$[g_0] = M^{n-2n+4} = M^{4-n} = M^{-\epsilon}.$$

Note that M is an arbitrary scale put into the game Physics should not the depend on it. It is customary to call this arbitrary scale μ [1] and this has very important consequences, as we will see later.

17.2 Problem: The running of g

Show, heuristically, that the knowledge of the quantity

$$\beta \equiv \mu\frac{\partial g}{\partial \mu}, \tag{17.2}$$

where g is the renormalized coupling constant g (see equation 16.99 in the case of the scalar $g\phi^4$ theory) allows one to compute the running of g.

Solution

$$\beta = \mu\frac{\partial g}{\partial \mu}, \quad \mu \equiv e^t, \quad \frac{d\mu}{dt} = \mu, \quad \frac{\partial}{\partial \mu} = \frac{\partial t}{\partial \mu}\frac{\partial}{\partial t} = \frac{1}{\mu}\frac{\partial}{\partial Z}. \tag{17.3}$$

[1] This scale μ is the same appearing in equation 16.99.

Then

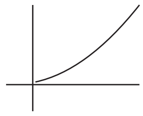
$$\beta = \mu \frac{1}{\mu} \frac{\partial}{\partial t} g(t) \tag{17.4}$$

namely

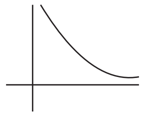
$$\beta = \frac{\partial g(t)}{\partial t}. \tag{17.5}$$

The situation can be depicted as follows

$$\beta = \frac{dg(t)}{dt} \Rightarrow \quad if \quad \beta > 0$$



$$if \quad \beta < 0$$



In the second case, the theory at hand is *asymptotically free*.

17.3 Problem: Computation of the β function

Given the pole structure of the bare coupling constant g_0, compute the β function.

Solution

We have seen, in the previous chapter, that one expects, in general, the following expression for the bare coupling constant g_0 of the theory under study

$$g_0 = \mu^{\alpha\epsilon} \left[g + \sum_{r=1}^{\infty} a_r \frac{1}{\epsilon^r} \right]. \tag{17.6}$$

On the other hand, as we have seen in problem 17.1, g_0 should not depend on the arbitrary mass scale μ. Therefore

$$
\begin{aligned}
0 &= \mu\frac{\partial g_0}{\partial \mu} = \mu\left\{(\epsilon\alpha)\mu^{(\epsilon\alpha-1)}\left[g+\sum_r \frac{a_r}{\epsilon^r}\right] + \mu^{\alpha\epsilon}\left(\frac{\partial g}{\partial \mu} + \sum_{r=1}^{\infty}\frac{1}{\epsilon^r}\frac{\partial a_r}{\partial \mu}\right)\right\} \\
&= \left\{\epsilon\alpha\left[g+\sum_r \frac{a_r}{\epsilon^r}\right] + \beta + \sum_{r=1}^{\infty}\frac{1}{\epsilon^r}\mu\frac{\partial a_r}{\mu}\right\} \\
&= \epsilon\alpha\left[g+\sum_r \frac{a_r}{\epsilon^r}\right] + \beta + \sum_{r=1}^{\infty}\left(\frac{1}{\epsilon^r}\mu\frac{\partial a_r}{\partial g}\frac{\partial g}{\partial \mu}\right) \\
&= \epsilon\alpha\left[g+\sum_r \frac{a_r}{\epsilon^r}\right] + \beta + \beta\sum_{r=1}^{\infty}\left(\frac{1}{\epsilon^r}\frac{\partial a_r}{\partial g}\right) \\
&= \beta\left(1+\sum_{r=1}^{\infty}\left(\frac{1}{\epsilon^r}\frac{\partial a_r}{\partial g}\right)\right) + \epsilon\alpha\left[g+\sum_r \frac{a_r}{\epsilon^r}\right].
\end{aligned}
\tag{17.7}
$$

Furthermore β should be analytic in ϵ, so that,

$$\beta = d_0 + \epsilon d_1 + \cdots$$

Then

$$(d_0 + \epsilon d_1 + \cdots)\left(1 + \frac{1}{\epsilon}\frac{\partial a_1}{\partial g} + \cdots\right) + \epsilon\alpha\left(g + \frac{a_1}{\epsilon} + \cdots\right) = 0.$$

From which one obtains

$$\begin{cases} d_0 + d_1\dfrac{\partial a_1}{\partial g} + \alpha a_1 = 0 \\ d_1 + \alpha g = 0 \end{cases} \Rightarrow \quad \begin{aligned} d_1 &= -\alpha g \\ d_0 &= -\alpha a_1 + \alpha g\frac{\partial a_1}{\partial g} \end{aligned}$$

Therefore

$$\beta = \alpha\left[-a_1 + g\frac{\partial a_1}{\partial g}\right]. \tag{17.8}$$

In summary, to compute the β function, one simply needs to know a_1, namely the simple pole of the renormalization constant Z_g.

17.4 Problem: The β function of $g\phi^4$

Compute the β function of the scalar theory $g\phi^4$.

Solution

In this case $\alpha = -1$ in equation 17.6, so that

$$g_0 = \mu^{-\epsilon} Z_g g = \mu^{-\epsilon}\left[g + \sum_{r=1}^{\infty} a_r \frac{1}{\epsilon^r}\right].$$

By comparing this equation with equation 16.109, one obtains

$$a_1 = -\frac{3}{16\pi^2} g^2, \tag{17.9}$$

and therefore

$$\beta = \frac{3g^2}{16\pi^2} > 0. \tag{17.10}$$

The coupling constant grows with energy.

17.5 Problem*: The β function of QED

Prove that the β function of QED is

$$\beta_{QED} = \frac{e^3}{12\pi^2}. \tag{17.11}$$

Is QED an asymptotically free theory?.

17.6 Problem*: The β function of QCD

Prove that the β function of QCD is

$$\beta_{QCD} = -\frac{g^3}{\pi^2}\left[\frac{11N_{col} - 2\,n_f}{48}\right], \tag{17.12}$$

where N_{col} and n_f are the number of colors and of active flavours, respectively. Is QCD an asymptotically free theory?.

Bibliography

[1] M. L. Mangano, M. Moretti, F. Piccinini, R. Pittau and A. D. Polosa, "ALPGEN, a generator for hard multiparton processes in hadronic collisions," JHEP **0307** (2003) 001 [arXiv:hep-ph/0206293].

[2] G. Ossola, C. G. Papadopoulos and R. Pittau, "CutTools: a program implementing the OPP reduction method to compute one-loop amplitudes," JHEP **0803** (2008) 042 [arXiv:0711.3596 [hep-ph]].

[3] G. 't Hooft and M. J. G. Veltman, NATO Sci. Ser. B **4**, 177 (1974).

[4] George Sterman, "An introduction to Quantum Field Theory", Cambrige University Press, 1993.

[5] Ta-Pei Cheng and Ling-Fong Li,, "Gauge theory of elementary particle physics", Oxford University Press, 1984.

[6] G. Passarino and M. J. G. Veltman, "One Loop Corrections For E+ E- Annihilation Into Mu+ Mu- In The Weinberg Model," Nucl. Phys. B **160**, 151 (1979).

[7] G. Ossola, C. G. Papadopoulos and R. Pittau, "Reducing full one-loop amplitudes to scalar integrals at the integrand level," Nucl. Phys. B **763** (2007) 147 [arXiv:hep-ph/0609007].