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BEYOND SUBJECT MATTER: A PSYCHOLOGICAL TOPOLOGY OF TEACHERS' PROFESSIONAL KNOWLEDGE

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1. INTRODUCTION

In both educational psychology and mathematical education, the professional knowledge of teachers is increasingly becoming an object of research. In recent years, it has become clear that innovations in the curriculum and in teaching methods are successful only when what the teacher does with these innovations is taken into account (Steiner, 1987). However, this depends on which conceptual tools teachers possess in order to deal with their work situation. The professional knowledge of teachers is, in part, the content they discuss during the lesson, but it is also evident that they must possess additional knowledge in order to be able to teach mathematics in an appropriate way to their students. However, what belongs to the professional knowledge of teachers, and how does it relate to their practical abilities?

There is a rather recent research tradition in the field of educational psychology that studies teachers as experts. The notion of "experts" expresses the programmatic reference to questions, research methods, and views of expert research in cognitive psychology. This approach analyzes the connection between the professional knowledge and professional activity of good performers within a certain field of activity. The expert approach provides a good starting position to approach such questions with empirical methods. When applying this approach to the study of teachers' cognitions, one is faced with the question of what shall be counted as professional knowledge. The concept of professional knowledge must be decomposed analytically. This is what this contribution is about.

2. A TOPOLOGY OF TEACHERS' PROFESSIONAL KNOWLEDGE

At first glance, professional knowledge seems to be sufficiently described by "subject matter," "pedagogy," and "specific didactics." These fields, however, have to be decomposed further if the intention is to understand the special characteristics of professional knowledge.

R. Biehler, R. W. Scholz, R. Sträßer, B. Winkelmann (Eds.),
Didactics of Mathematics as a Scientific Discipline, 73-88.

Shulman (1986) has presented a classification of teachers' knowledge. It comprises: "content knowledge," "curricular knowledge," "pedagogical knowledge," and "pedagogical content knowledge." These suggestions have proved to be very stimulating for research into teacher cognitions (Grossmann, 1990). In order to be able to describe qualitative features of professional knowledge, Shulman's categories must be differentiated further. This is why I take up his suggestion, but extended by both the concept of "philosophy of content knowledge" and a clear distinction between the knowledge of the academic discipline and that of the subject in school.

This section will provide a brief sketch of my topology of areas of teachers' professional knowledge. The following sections shall consider some areas of this topology in greater depth in order to cast light on the complex nature of professional knowledge.

2.1 Content Knowledge About Mathematics as a Discipline

This is what the teacher learns during his or her studies, and it contains, among other things, mathematical propositions, rules, mathematical modes of thinking, and methods.

2.2 School Mathematical Knowledge

The contents of teaching are not simply the propaedeutical basics of the respective science. Just as the contents to be learned in German lessons are not simplified German studies, but represent a canon of knowledge of their own, the contents of learning mathematics are not just simplifications of mathematics as it is taught in universities. The school subjects have a "life of their own" with their own logic; that is, the meaning of the concepts taught cannot be explained simply from the logic of the respective scientific disciplines. Or, in student terms: Mathematics and "math," theology and "religious studies" are not the same. Rather, goals about school (e.g., concepts of general education) are integrated into the meanings of the subject-specific concepts. For the psychological analysis of professional knowledge, this is important, as these aspects of meaning are, in part, implicit knowledge.

2.3 Philosophy of School Mathematics

These are ideas about the epistemological foundations of mathematics and mathematics learning and about the relationship between mathematics and other fields of human life and knowledge. The philosophy of the school subject is an implicit content of teaching as well, and it includes normative elements. Students, for instance, will learn whether the teacher adheres to the view that the "essential thing" in mathematics is operating with a clear, completely defined language, no emphasis being set on what the things used refer to, or whether the view is that mathematics is a tool to describe a reality, however it might be understood.

2.4 Pedagogical Knowledge

This means that part of knowledge that has a relatively independent validity separate from the school subjects. This includes how to introduce the behavior patterns necessary for handling a class (Kounin, 1970). It also concerns coping with parents in order to explain and influence student behavior. The pedagogical ethics of teachers with regard to treating their students justly is neatly interwoven with their pedagogical knowledge (Oser, in press). Pedagogical knowledge, of course, is very important for the teacher's professional activity; however, it shall not be treated extensively here, as I shall focus on those areas that are related to the subject matter.

2.5 Subject-Matter-Specific Pedagogical Knowledge

On the basis of the logical structure of the subject matter taken alone, no teaching decision can yet be made. Lesson observation shows still large interindividual differences in the didactical approach chosen, even if the subject matter and the textbook are the same (Leinhardt & Smith, 1985). To find suitable forms of presenting the subject matter, to determine the temporal order of treating the topics, and to assess which matters have to be treated more intensely requires subject-matter-specific pedagogical knowledge (Chevallard, 1985, chaps. 5, 6). This field of knowledge has a special character. It is integrated knowledge cross-referring both pedagogical knowledge and the teacher's own experience to the subject-matter knowledge. This integration is exhibited, for instance, when the logical structure of the subject matter is reshaped into a temporal sequence. Further, it consists in changing the structuring and relative weight of concepts and rules; something that is of central importance from the viewpoint of mathematical theory may be accorded less weight from the perspective of teaching.

2.6 The Cognitive Integration of Knowledge From Different Disciplines

The professional knowledge of teachers is not simply a conglomerate of various fields. Rather, an integration takes place during the course of practical training and professional experience, and the various fields of knowledge are related to practical experience. The fusing of knowledge coming from different origins is the particular feature of the professional knowledge of teachers as compared to the codified knowledge of the disciplines in which they have been educated.

In mathematics teachers, the subject-matter-specific pedagogical knowledge is to a large part tied to mathematical problems. In a way, it is "crystallized" in these problems, as research into everyday lesson planning has shown. In their lesson preparation, experienced mathematics teachers concentrate widely on the selection and sequence of mathematical problems. Both "thinking aloud" protocols (Bromme, 1981) and interviews with mathematics teachers, have provided hardly any indications of pedagogical considerations prior to the selection of problems. Nevertheless, pedagogical questions of shaping the lessons are also considered by teachers in their lesson planning, as these questions codetermine the decision about tasks. By choosing tasks with regard to their difficulty, their value for motivating students, or to illustrate difficult facts, and so forth, the logic of the subject matter is linked to teachers' assumptions about the logic of how the lesson will run and how the students will learn (for similar results, see, also, Tietze, 1986). Thus, the mathematical problems already contain the subject-matter core of the scenarios of activity that structure the teachers' categorical perception of the teaching process.

Teachers often do not even realize the integration they effect by linking subject-matter knowledge to pedagogical knowledge. One example of this is their (factually incorrect) assumption that the subject matter (mathematics) already determines the sequence, the order, and the emphasis given to teaching topics. The pedagogical knowledge that flows in remains, in a way, unobserved. To teachers who see themselves more as mathematicians than as pedagogues, their teaching decisions appear to be founded "in the subject matter," as Sträßer (1985) found in his interviews with teachers in vocational schools. In case studies with American teachers, Godmundsdottir and Shulman (1986) have reported an implicit integration of methodological and subject-matter ideas in teachers.

3. SUBJECT-MATTER KNOWLEDGE AND INSTRUCTIONAL OUTCOME

The subject-matter knowledge is not only an object of the professional activity of teachers but also, as a prerequisite of this activity, a major and extensive content of their professional training. But, how much knowledge of this type is necessary to be a successful teacher?

In the 1970s, some surprising empiricial studies were published. According to these, there was no measurable connection between the extent of teachers' subject-matter knowledge and instructional outcomes (Gage & Berliner, 1977, pp. 646-647). It seems to be immediately evident that teachers must have the subject-matter knowledge they are supposed to teach. This, however, does not permit the conclusion that there is a direct linkage between the extent of subject-matter knowledge and students' instructional outcomes measured by means of standardized tests.

Eisenberg (1977) tested the knowledge of 28 teachers in algebra, looking for connections to the growth of knowledge in their students. While student variables such as verbal competence and previous knowledge prior to the teaching unit contributed to the variance of the performance measured, this proved not to be true for teachers' amount of knowledge, confirming similar results obtained by Begle (1972). Both authors conclude that a relatively low stock of knowledge is sufficient to teach students. In a meta-analysis of 65 studies of teaching in the natural sciences, Druva and Anderson (1983) summarized the empirically established relationships between teacher variables (age, extent of education in the natural sciences) and both teacher behavior and student behavior as well as performance in class. The number of courses the teachers had taken in the natural sciences (as a measure of their knowledge) explained about 10% of the variance in student performance. Similar explanatory power was found for instructional quality variables, for instance, the posing of complex questions. The small (in absolute terms) share of variance explained by these variables is stressed by several authors and considered serious (Romberg, 1988). In contrast to this conclusion, it must be stated, however, that this indirect indicator of academic knowledge is even a good predictor of student performance, for individual variables in research on teaching, be they variables of teaching or so-called background variables in teachers or students, will always be able to explain only a relatively small percentage of variance, except for the variable of "pretest scores" (Brophy & Good, 1986; Dunkin & Biddle, 1974).

Nevertheless, a correlative connection between the extent of a teacher's training in the subject matter and student learning outcomes does not lend itself to causal interpretation as long as the process of mediation between these two variables is no topic. There are a few studies shedding light on some steps of these mediating processes. To give one example concerning the variable of clarity, a teacher's subject-matter knowledge contributes to his or her being able to stress important facts and ideas within the curriculum. This knowledge influences the quality of explanations given (Roehler et al., 1987) and the ability to integrate into their teaching student contributions that do not lie precisely on the teacher's intended level of meaning (Hashweh, 1986).

The effects of limited subject-matter knowledge were analyzed in a case study by Stein, Baxter, and Leinhardt (1990). They questioned a mathematics teacher extensively on his mathematical knowledge and educational ideas concerning the concept of function. Afterwards, they observed his teaching, looking for episodes in the videotape recordings in which a connection between subject-matter knowledge and teaching was recognizable. The teacher's ideas were limited to interpreting function as a calculating rule. He made no allowance for interpreting functions as mappings of quantities upon one another, nor for the possibility of one element being assigned to several corresponding elements. This limited idea of the function concept did not lead to classroom statements that were strictly false, but to the following three weaknesses in developing the subject matter in class: (a) Too much emphasis on special cases: The explanation of function given by the teacher was correct only for cases of one-to-one relations between the elements of the two quantities. (b) Too little profiting from teaching opportunities: Drawing function graphs was not referred back to defining functions, and hence appeared to the students as something entirely new. (c) Omission of preparation for an extended understanding of the concept:

While the examples had been chosen to solve the problems of this very class level, a more general understanding of the concept of function was more impeded than promoted.

Carlsen (1987) studied the connection between subject-matter knowledge and teachers' questioning in science teaching. He used interviews and sorting procedures to inquire into the knowledge of four student teachers. Classroom observations (9th to 12th grade) and analyses of lesson transcripts showed linkages between intraindividual differences in the extent of subject-matter knowledge and the teachers' questioning within their lessons. In teaching units on topics on which the teachers knew relatively little, they asked more direct questions, the questions having a low cognitive level. In topics on which the teachers knew their way better, the students talked more, offered more spontaneous contributions, and their contributions were longer; the teachers implicitly communicating how they expected the students to behave both by the manner of their questions and by the interest they showed in the subject matter (the variable of "enthusiasm"). Only teachers who possess good subject-matter knowledge are sufficiently sure of themselves to be able to direct classroom activities even in cases when the students take new paths of work (Dobey & Schafer, 1984).

Leinhard and Smith (1985) questioned teachers about their subject-matter knowledge on division (using interviews and sorting procedures) and subsequently observed their lessons. The teachers had different levels of knowledge about the properties of fractions. By strict confinement to algorithmic aspects of fractions, even those teachers with less conceptual knowledge were able to give lessons on this topic. In the classrooms, interindividual differences in the availability of various forms of representing fractions (e.g., as area sections, on the number line) were observed as well. The teachers who showed conceptual gaps in their knowledge also belonged to the expert group, having obtained good learning performance with their classes over years. The authors supposed that there is some kind of compensation between lack of subject-matter knowledge and more knowhow about techniques of organizing the teaching in class (but only within definite limits).

The partly disappointing results of the studies on the correlations between subject-matter knowledge and teaching success are rather more suited to point out the complexity of what belongs to a teacher's professional knowledge than to put in question the basic idea of investigating the relation between professional knowledge and successful teaching. The connection between a teacher's subject-matter knowledge and the students' learning performance is very complex. A large number of variables "interfere" with the effect the teacher's amount of subject-matter knowledge has on student performance. There is an interesting parallel to this in the history of educational psychology. With their Pygmalion effect, Rosenthal and Jacobson (1971) also described a connection between a cognitive teacher variable

(anticipated student performance) and a product variable (actual student scores in tests). Only later studies (Brophy & Good, 1974; Cooper, 1979) were able to show how teacher expectations are communicated and how they are connected to student behavior, student cognitions, and, finally, student performance.

4. THE "PHILOSOPHY OF SCHOOL MATHEMATICS" IN TEACHERS

Structuring the problems to be worked on and evaluating goals and subgoals is a typical abilty for effective professionals in several professional fields (Schön, 1983). It requires normative components within the professional knowledge. Those professions that legitimize their daily activities by referring to a so-called scientific base often gloss over these normative elements in silence. Hence, such normative ideas will be treated here somewhat more extensively.

Only recently, normative ideas of teachers related to the subject matter and their effect on teaching (mostly called teachers' beliefs) have come under closer scrutiny (For the teaching of English: Grossmann, 1990; the natural sciences: Hollon & Anderson, 1987; mathematics: Cooney, 1985, this volume; Heymann, 1982; Kesler, 1985; McGalliard, 1983; Pfeiffer, 1981; Thompson, 1984; Tietze 1986; comparison of school subjects: Yaacobit & Sharan, 1985).

The concept of "philosophy" for this part of teachers' knowledge is intended to stress that this means an evaluating perspective on the content of teaching. It is not a matter of subjectively preferring this or that part of the curriculum. Therefore I prefer the notion of philosophy instead of the notion of belief in order to emphasize that it is a part of metaknowledge, soaked with implicit epistemology and ontology (see, also, Ernest, this volume) .

The effect of teachers' philosophy of school mathematics on their teaching is much more strongly verified empirically than the influence of the amount of subject-matter knowledge discussed above. A good example for studies on the philosophy of school mathematics is that of Thompson (1984). The author compared ideas about mathematics teaching in three woman teachers. Teacher J considered mathematics to be a logical system existing independent of whether it is acquired or not. She took her task to be clear and consistent presentation of the subject matter. She expected her students to learn, first of all, the connection between what they had already learned and what was new. In contrast, Teacher K had a more process-oriented conception of mathematics. Accordingly, her teaching was aligned to encourage students to discover for themselves. A third principle found was to listen attentively to and to take up and understand the ideas that students advanced. Thompson (1984) also found discrepancies between teachers' normative ideas and their teaching behavior. Thus, while Teacher J stressed how important mathematics is for solving practical problems, she had difficulties in introducing practical examples of this into her teaching. In two case studies, Cooney (1985) and Marks (1987) each examined a teacher's conception of problem-solving. Both teachers named "mathematical problem-solving" as their most important goal. They showed, however, rather different conceptions of what can be termed problem-solving in mathematics and can be encouraged by a teacher.

We compared the mathematics instruction on the topic of "stochastics" given by two teachers whose teaching obviously did not have the same degree of "smoothness" (Bromme & Steinbring, 1990). A group of teachers was observed across several lessons, and their behavior was judged according to scales listing their quality of teaching (providing guidance to the class, clearness in presenting the subject matter, etc.). This served to identify the two teachers. The next step was to investigate their difference in instructional quality. For this purpose, lesson transcripts were coded for two subsequent lessons for each teacher. The coding focused on the question of which aspects of mathematical meaning had been thematized by the teachers in class: the symbolic-formal side, the applications of formal calculus, or the relationship between formal calculus and the object to which it is applied. Both teachers were confronted with student contributions alternately thematizing these two aspects of mathematical meaning in an inconsistent way. The two teachers differed markedly in how they treated student contributions and in how they used what had been offered to develop the subject matter. The teacher whose teaching went more "smoothly" showed a more appropriate switching between the aspects of mathematical meaning and the establishment of explicit relationships between the levels of meaning. This suggests the assumption that normative views about school mathematical knowledge (i.e., about what is really worth knowing in a mathematical object) influence teacher behavior.

In the present empirical studies concerning the subject-matter knowledge of teachers, there is a partial overlapping of the above-mentioned conceptual distinction between "subject-matter-specific pedagogical knowledge" and "philosophy of school mathematics." A strict distinction may not be appropriate. Certain variants of the philosophy of school mathematics also require a more profound mathematical understanding as well as more and different subject-matter-specific pedagogical knowledge. The philosophy of school mathematics contains certain judgments about what are the central concepts and procedures that should be taught, and what characterizes mathematical thought. These values, however, are tied closely to the subject matter-specific pedagogical knowledge and to disciplinary knowledge of facts, and they are often implicit. It may well be possible for a teacher to belong to a certain school of thought without being aware of the fact that subject-matter knowledge also contains a set of values. A psychological theory of teachers' professional knowledge must take into account that normative elements are interwoven with all areas of knowledge (Bromme, 1992, chap. 8.2).

5. FORMING PROFESSIONAL KNOWLEDGE BY PRACTICAL EXPERIENCE: EVERYONE MUST LEARN BY EXPERIENCE

Teachers do not have to effect the integration of pedagogical knowledge and subject-matter knowledge alone. The education of teachers in most countries contains practical elements aiming at such a linkage. Nevertheless, the teacher is still obliged to adapt his or her general knowledge to the conditions of teaching with which he or she is confronted. In the following, some empirical results will be described supporting the hypothesis that teachers' professional knowledge is a quite particular mixture of the above-mentioned areas of knowledge (especially subject-matter knowledge, philosophy, and pedagogical knowledge), and that this mixture is structured by teachers' practical experience with their own classrooms.

The requirements of teaching compel teachers to modify their previously learned theories about the content and the ways of teaching it. This, however, must not be seen as a mere simplification of previously differentiated knowledge, but rather as an enrichment by information referring to situations. Empirical evidence can be found in studies examining whether teachers rely on psychological theories or make allowance for facts that have been proven to be relevant for learning processes in psychological studies. The question thus is not whether these teachers had explicitly heard about such results; this can be left aside. What matters is only whether they think and act in a way that seems reasonable to the interviewers according to psychological facts about student learning. Thus, some of the empirical studies inspired by Shulman's (1986) concept of "pedagogic content knowledge" examine the question whether teachers consider recent concepts of their subject's didactics and developmental psychological concepts of strategies of learning (Clift, Ghatala, & Naus, 1987; Shefelbine & Shiel, 1987). To the disappointment of their authors, these studies showed that the teachers studied did not rely on psychological theories, but used other knowledge referring to experience. These results must sometimes be read at odds with their authors' interpretations in order to note that the teachers studied do not simply show a deficit in subject-matter-specific pedagogical knowledge.

The following study provides an example of this: Carpenter, Fennema, Peterson, and Carey (1988) have analyzed teachers' concepts about student errors in arithmetic. The psychological basis of this analysis was developmental findings on 1st-grade children's addition strategies. According to how the task is formulated and to age group, several techniques of counting visible elements (fingers) can be observed (Carpenter & Moser, 1984). The task (5 + ? = 13): "How many marbles do you still need if you already have 5 marbles and want 13?" for instance, is solved in three steps: counting 5 objects, continuing to count from 5 to 13, and then counting the fingers that have been added. Later, the first of these steps is left out. The authors interviewed 40 experienced elementary school teachers (with an average of 11 years of experience) regarding what they knew of such strategies, then studied the connection between knowledge and both teaching behavior and teaching performance. For this, they used a collection of tasks containing the various task types. Subjects had to compare tasks as to their difficulty for 1st-grade students (in general, not for their own students). The degree of difficulty assumed was then compared to empirically found solution rates (Carpenter & Moser, 1984). For most of the task types, the majority of assessments were correct. The teachers, however, had difficulties in stating reasons for their assessments. Above all, they did not name the students' solving strategies, such as counting the concrete objects. Only eight of the teachers referred to student strategies at all in assessing the difficulty of the task. In the case of the above subtraction task, 18 teachers mentioned the difficulty that what is sought is at the beginning of the task description, but did not relate this to the counting strategy. Instead, the subjects gave the formulation of the problem or the occurrence of key terms as reasons for the task's difficulty, for example: "If the task says 'how many more marbles has . . . ' the children will at once think of a problem of addition." The teachers presumed that the students seek to establish whether it is a problem of addition or one of subtraction. They grouped the tasks according to whether the problem formulation in the text facilitates this search or makes it more difficult.

The next step of the study concerned the students' solving strategies. The teachers were shown videotapes of children using various strategies while working on tasks. Then the teachers were presented with tasks of the same kind and asked to predict whether the student observed would be able to solve this task, and how he or she would proceed. Using this method, the researchers intended to find out whether teachers recognize that the above subtraction and addition task differs for the students in the very fact that a direct representation by fingers is possible in one case and impossible in the other. The result was that, while teachers were able to describe the students' strategy, they obviously had no concept of it, and hence had difficulties in predicting the solution behavior in tasks in which they could not observe the student's actual work on them.

Subsequently, subjects were asked to predict solving strategies and success for students from their own class chosen at random, and to describe the strategy they expected. The students were tested independent of the teachers. On average, teachers were able to predict success correctly in 27 of 36 cases, and to predict the solving strategy correctly in almost half of the cases. In the strategy prediction, however, the differences between teachers were much larger than in their predictions about success. There was, however, no significant connection between general knowledge about strategies (which was measured in the second step) and the quality of the prediction with regard to their students, nor between this knowledge and student performance on the tasks themselves.

Carpenter, Fennema, Peterson, and Carey (1988) were disappointed at this lack of "pedagogical content knowledge." In the teachers, the authors missed the knowledge about individual solving strategies of the students working on the tasks. They said that the teachers looked to superficial task features to assess the difficulty, instead of at the strategies the students used in solving.

The teachers' way of proceeding, however, indicates rich knowledge from experience. Thus, it is a basic difficulty for students to find out which type of task they have to work on. In the classroom context, tasks are connected with the previous tasks. The student is called to recognize whether he or she may maintain his or her former strategy (i.e., adding, because adding problems were on), or whether a new strategy is required. Nesher and Teubal (1975) found that students use key terms in a problem text in order to identify the required operations. Establishing which part of mathematical knowledge is asked for at the moment is an important element of mathematical competence (Greeno, Riley, & Gelman, 1984). The teachers' assessments are thus very much an indication of experience-based professional knowledge about these facts. This knowledge is more realistic than the observations of research on strategies of adding, as the real student performance in class does not just depend on the individually available strategy of learning. Their certitude in this judgment, on the one hand, and their difficulties in giving reasons for it, on the other, are an indication that this is a case of intuitive knowledge from experience (Hoge & Coladarci, 1989; Leinhart & Smith, 1985; more evidence about expert teachers' abilities to assess the difficulty of mathematical tasks can be found in Schrader & Helmke, 1989).

6. ACCUMULATING PROFESSIONAL EXPERIENCE: THE EXAMPLE OF TEACHERS' KNOWLEDGE ABOUT THEIR STUDENTS' UNDERSTANDING

The previous sections described the professional knowledge that is acquired in teacher training and then changed by experience. The following will consider the collecting of experience more closely. Teachers' observations on their students during lessons shall serve as examples.

In educational psychology, there is a widespread normative idea that teaching should be adapted as individually as possible to the knowledge and abilities of individual students (Corno & Snow, 1986), and that, hence, the difficulties encountered by students during lessons should be perceived as accurately as possible. The categorical perception of student understanding is a good example for the application of professional knowledge. Studies presented up to now show a rather negative picture. They reveal that teachers notice very little of the understanding of their students (Jecker, Macoby, & Breitrose, 1965; Putnam, 1987). Shroyer (1981) interviewed teachers while they jointly viewed videotape recordings after lessons. The teachers were asked to recall instances in which students had experienced particular difficulties or in which they had shown unexpected progress. Shroyer carried out parallel observations of these lessons and found that only 3% of the difficulties and advances observed were actually perceived by the teachers.

The above studies, however, are based on an implicitly unrealistic idea of the requirements asked of a teacher during a lesson, which, again, has resulted in an underestimation of teachers' professional knowledge. The following study on mathematics teachers has yielded indications of this (Bromme, 1987).

Our question was which problems of, and which progress in, understanding do mathematics teachers perceive. Interviews were based on a brief list of mathematical tasks in the lesson. Interviews of nineteen 5th- to 7th-grade mathematics teachers, which referred to one lesson each, were analyzed with regard to their content. We intended to establish whether the teachers remembered advances of learning or problems of understanding, and who played the active part in an episode: the entire class, individual students known by name, or subgroups of the class. Per lesson, the teachers named only an average of two students, with a maximum of six by two teachers. Eight of the 19 teachers did not remember a student known by name having problems of understanding in the lesson just given. In the case of the advances in learning, an average of three students was named.

Hence, there was little perception as to the way the subject matter was understood individually. Instead, the teachers interviewed had observed the class as a whole. For "the class" as actor, observations could be found in all the teachers, whereas almost half of the teachers were unable to name a student having problems of understanding, as has been said. The number of student problems and learning advances remembered was thus, on the whole, surprisingly small. The result is – at first glance – just as negative as that obtained in Shroyer's study mentioned above (1981). Only few episodes in the teaching process containing problems and progress of understanding were remembered. These, however, were precisely those episodes in which new steps in working through the curriculum were initiated. From the teachers' view, these were the key episodes. Student contributions were remembered if they had been of strategical value for the flow of dialogue about the subject matter, for example: "Nobody was able to give an answer to my question, then Alexander came up with a good idea." The term "strategical value" means that these contributions occurred in situations during the lesson in which there was, according to the teachers' view, "a hitch" (as one of the woman teachers said), or in which the transition proper from the old to the new knowledge was intended.

The teachers' memory and, as may be assumed, their categorical perception as well, did not concentrate on the diagnosis of individual student errors, but rather on the Gestalt of the entire lesson's flow. The active subject of learning activities was not the individual learner, but rather an abstract,

but psychologically real unit that I have labeled the "collective student" (Bromme, 1987; see, also, Putnam, 1987, for similar results obtained in a laboratory setting). These results show that teachers judge their students' problems and advances of understanding against the background of an intended activity structure. The way of talking most teachers use in saying that "the class" did good work today, or had more difficulties with fractional calculus than others, is not only a verbal simplification but also an indication that entire classes are categorical units of perception for teachers (see, also, the similar result in Rutter, Manghan, Mortimore, & Queston, 1980). The categorical unit "whole class" is rather neglected in theories on mathematical education, the focus being more on the "individual student" as a categorical unit of perceiving and thinking. Therefore teachers have to develop their own concepts about the class as a unit, and it is not by chance that the notion of "the class" as an indvidual unit is an important element of teachers' professional slang.

7. SUMMARY AND CONCLUSIONS

In the 1970s, there were a number of studies according to which teachers with better curricular expertise did not perform better in their teaching. These studies, however, had two deficits: They compared subject-matter knowledge of facts (as measured by tests or by the number of university courses taken) directly with the learning performance of students, omitting to analyze the connection between subject-matter knowledge and teaching activity of teachers. Subsequent studies in which lessons were observed as well showed, among other things, an influence of the amount of subject-matter knowledge and of the philosophy of school mathematics on the flexibility of teachers in coping with unexpected student suggestions. In addition, there was, within certain limits, the possibility of mutual substitution between the richness of subject-matter knowledge and more pedagogical knowledge. A second deficit of these studies was their poor theoretical conception of subject-matter knowledge. The mere familiarity with the contents of teaching constitutes only a part of the conceptual tools necessary for teachers' daily work. For the mathematics teacher, we can distinguish between five such fields of knowledge that are needed for teaching: (a) knowledge about mathematics as a discipline; (b) knowledge about school mathematics; (c) the philosophy of school mathematics; (d) general pedagogical (and, by the way, psychological) knowledge; and (e) subject-matter-specific pedagogical knowledge. Two of these areas have been treated more extensively, as they are significant for further empirical research on the structure of teachers' professional knowledge.

One of these fields comprises evaluative views about school mathematics, for instance, about the value of certain concepts and techniques for what makes mathematics a content of education. Several empirical studies have shown a strong impact of the values and goals about the school subject matter on the teaching process. These have been termed "philosophy of school mathematics" here in order to emphasize that the normative elements are closely tied in with the subject's facts and procedures. Hence, this is not a case of purely subjective beliefs. While it would seem to be undisputed that professional activity also follows normative principles and requires value decisions, it is less self-evident that such value systems are in a way interwoven into the subject-matter knowledge about mathematics. The close linkage between normative and factual elements, however, must be taken into account in a psychological theory of professional knowledge.

The second field of professional knowledge that has been treated more extensively is that of subject-matter-specific pedagogical knowledge. The concern here is with the relationship between curricular content and teaching-learning process, and it must be developed by one's own experience. In mathematics teachers, it crystallizes predominantly in their ideas about mathematical tasks and their uses in the classroom. The teacher categories about scenarios of activity are another example of this. These are categories within which knowledge of different origins (here: mathematics and pedagogy) and personal experience have been fused. The integration of knowledge originating from various fields of knowledge, discussions with colleagues, and experience is an important feature of the professional knowledge of teachers, that has to be taken into account when thinking about any educational innovation that requires the teachers' cooperation.

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DIALOGUE BETWEEN THEORY AND PRACTICE IN MATHEMATICS EDUCATION

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1. NEW PERSPECTIVES ON THE RELATION BETWEEN THEORY AND PRACTICE

Traditionally, the central task of mathematics education has been to contribute in a more or less direct manner to improving the practice of teaching mathematics and to solve teaching problems. Accordingly, the didactics of mathematics is mainly conceived of as an auxiliary science, which has to transform the scientific mathematical knowledge into a suitable form of knowledge for teachers and students and which has to provide well-tested methodological procedures to teach this knowledge effectively. Mathematics education often is taken as a methodology for elementarizing, simplifying, and adapting scientific subject matter to the abilities of students.

Additionally, the role of the referential sciences, such as pedagogics, psychology, or the social sciences, is mostly understood as a further support for this central task of didactics: to improve everyday teaching practice. In particular, these sciences should help solve those educational, psychological, and social problems that go beyond the actual field of teaching mathematics.

Also with regard to the mathematics teacher and his or her pre- and in-service training, the didactics of mathematics primarily has the role of a servant: Didactics should prepare teacher students methodically for their future teaching practice and endow them with useful teaching strategies. And, in in-service seminars, experienced teachers expect more or less direct support for their everyday teaching practice from confirmed research results and reliable teaching materials.

Such an expectation toward didactics of mathematics seems to be dominant in the beliefs of many mathematics teachers and researchers: Useful research in mathematics education is characterised by a straightforward applicability of research findings to the problems of teaching practice. This ought to bring about direct improvements of practice. But, contrary to this widespread opinion about didactics of mathematics, there is agreement that

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