

NUMERICAL SEMIGROUPS MINI-COURSE

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Lecture 1. Three ways for representing a numerical semigroup. Generalities

A numerical semigroup is a subset of the set of nonnegative integers (denoted here by \mathbb{N}) closed under addition, containing the zero element and with finite complement in \mathbb{N} . Observe that a numerical semigroup is a commutative monoid, thus is somehow surprising that it is required that zero belongs to the set under consideration. Some authors use the term numerical monoid to stress out this property. Note also that up to isomorphism the set of numerical semigroups classify the set of all submonoids of $(\mathbb{N}, +)$. Let S be submonoid of \mathbb{N} , the condition of having finite complement in \mathbb{N} is equivalent to saying that the greatest common divisor (gcd for short) of its elements is one.

Numerical semigroups probably where first considered while studying the set of nonnegative solutions of Diophantine equations. Given positive integers a_1, \dots, a_n with greatest common divisor one, the set of elements $b \in \mathbb{N}$ such that $a_1x_1 + \dots + a_nx_n = b$ has a nonnegative integer solution is a numerical semigroup. Actually, one of the first known problems related to numerical semigroups was to determine in terms of a_1, \dots, a_n , which is the largest integer for which there is no nonnegative integer solution. This is known as the Frobenius problem, since it seems that Frobenius it problem in one of his lectures.

1. GENERATORS (FIRST CHOICE)

The set of integers for which there is a nonnegative integer solution of $a_1x_1 + \dots + a_nx_n = b$ can be expressed as $\{a_1x_1 + \dots + a_nx_n \mid x_1, \dots, x_n \in \mathbb{N}\}$, or in a more abbreviated notation, as $\langle a_1, \dots, a_n \rangle$. We say that $\{a_1, \dots, a_n\}$ is a system of generators of S , or simply, that $\{a_1, \dots, a_n\}$ generates S . If no proper subset of $\{a_1, \dots, a_n\}$ generates S , then we, as expected, say that $\{a_1, \dots, a_n\}$ is a minimal system of generators of S . As S is cancellative ($a + b = a + c$ implies $b = c$), S admits a unique minimal system of generators, say $(S \setminus \{0\}) \setminus ((S \setminus \{0\}) + (S \setminus \{0\}))$. The cardinality of the minimal system of generators of S is known as the embedding dimension of S (we will see later why this weird name).

Observe that if S is generated by $\{a_1, \dots, a_n\}$, then $\text{gcd}(\{a_1, \dots, a_n\}) = 1$ (and that if $\text{gcd}(\{a_1, \dots, a_n\}) = 1$, then the submonoid of \mathbb{N} spanned by $\{a_1, \dots, a_n\}$ is a numerical semigroup).

2. MULTIPLICITY, FROBENIUS NUMBER, GAPS, (COHEN-MACAULAY) TYPE

As we mentioned above, Frobenius proposed the problem of finding a formula for the largest integer for which there is no (nonnegative integer) solution to $a_1x_1 + \dots + a_nx_n = b$. This in our notation is equivalent to say which is the largest integer not in the numerical semigroup $S = \langle a_1, \dots, a_n \rangle$. This is why $\max(\mathbb{Z} \setminus S)$ is known as the Frobenius number of S (here we are using \mathbb{Z} to denote the set of integers). If g is the Frobenius number of S , then $g + (\mathbb{N} \setminus \{0\}) \subset S$, and in particular $g + (S \setminus \{0\}) \subseteq S$. The integers fulfilling this condition are known as pseudo-Frobenius numbers, and its cardinality is the (Cohen-Macaulay) type of S .

Given a numerical semigroup S , we can define on \mathbb{Z} the following order relation: $a \leq_S b$ if $b - a \in S$. It turns out that the set of pseudo-Frobenius numbers is the set of maximal elements of $\mathbb{Z} \setminus S$ with respect to this ordering.

The positive integers not belonging to S are the gaps of S , and its cardinality is sometimes known as the gender of S . If g is the Frobenius number of S , some authors reserve the word hole for those integers x verifying that $x \notin S$ and $g - x \notin S$. Thus every hole is a gap, but the converse needs not to be true.

The least positive integer belonging to a numerical semigroup is its multiplicity. The multiplicity of a numerical semigroup is an upper bound for the embedding dimension of S . This is because there cannot be two different minimal generators congruent modulo the multiplicity.

3. APÉRY SETS, THE TOOL

Recall that two minimal generators of a numerical semigroup cannot be congruent modulo the multiplicity, in fact, they cannot be congruent modulo any nonzero element of the numerical semigroup. In this sense, minimal generators are minimal also with respect to be congruent to any nonzero element of the numerical semigroup. Following this idea for a given numerical semigroup S and a nonzero element n of S , we can consider the set $\{w_0, \dots, w_{n-1}\}$ where w_i is the least element in S congruent to i modulo n . One can easily see that this set corresponds to the set $\{s \in S \mid s - n \notin S\}$. Apéry was the first exploiting this idea and this is why this set is known as the Apéry set of n in S . If n is chosen as the multiplicity of S , then sometimes this set is called a standard basis of S . Since this set appears everywhere in our approach to the study of numerical semigroups, we introduce the notation $\text{Ap}(S, n)$ to refer to it.

The Apéry set of n in S has some wonderful properties. We enumerate some of them here, but later more will arise.

- Every integer x can be expressed uniquely as $x = kn + w$ for some $k \in \mathbb{Z}$ and $w \in \text{Ap}(S, n)$. And $x \in S$ if and only if $k \geq 0$.
- Thus if we want to know if x belongs to S , we find $w \in \text{Ap}(S, n)$ such that $x \equiv w \pmod{n}$; then $x \in S$ if and only if $w \leq x$.
- The Frobenius number of S is $\max(\text{Ap}(S, n)) - n$.
- More generally, an integer g is a pseudo-Frobenius number of S if and only if $g + n$ is maximal in $\text{Ap}(S, n)$ with respect to \leq_S . It follows that the type of S is the cardinality of $\text{Maximals}_{\leq_S}(\text{Ap}(S, n))$.
- By Selmer's formula, the number of gaps equals $\frac{1}{n} \sum_{w \in \text{Ap}(S, n)} w + \frac{n-1}{2}$.

Hence the knowledge of the Apéry set of a numerical semigroup S with respect to any of its nonzero elements, solves the membership problem, allows us to know the Frobenius number of S , its pseudo-Frobenius numbers (and thus its type) and its gender.

4. FUNDAMENTAL GAPS (SECOND CHOICE)

The Frobenius number of \mathbb{N} is -1 . If S is a numerical semigroup other than \mathbb{N} , then the Frobenius number of S is a positive integer, and the same holds for

its pseudo-Frobenius numbers, that is, they are gaps of S . There are 1,156,012 numerical semigroups with Frobenius number 39. Thus the Frobenius number is not suitable to describe uniquely a numerical semigroup (it can be shown that a numerical semigroup is uniquely determined by its Frobenius number g if and only if $g \in \{-1, 1, 2, 3, 4, 6\}$). Among these 1,156,012 numerical semigroups, 227 of them have $\{39\}$ as set of pseudo-Frobenius numbers. Hence pseudo-Frobenius numbers are also a bad choice to uniquely describe a numerical semigroup.

Clearly, the set of gaps of S uniquely determines S . But in this set a lot of information is redundant, since if $x|y$ (read x divides y) and y is a gap of S , then x must also be a gap of S . Hence among the gaps of S we only need those that are maximal with respect to $|$. These are known as fundamental gaps of S , and they uniquely determine S . Clearly an integer x is a fundamental gap of S if and only if $x \notin S$ and $\{2x, 3x\} \subset S$.

Let X be a subset of $\mathbb{N} \setminus \{0\}$. Denote by $D(X)$ the set of positive integers dividing some $x \in X$. If X is the set of fundamental gaps of S , then $S = \mathbb{N} \setminus D(X)$. If g is the Frobenius number of S (observe that this means that $g = \max X$), then

$$\left\lceil \frac{g}{6} \right\rceil \leq \#X \leq \left\lceil \frac{g}{2} \right\rceil.$$

There are positive integers g for which there is no numerical semigroup not reaching the lower bound, whilst the upper bound is always reached by $\{0, g+1, \rightarrow\}$.

5. THE SET OF OVERSEMIGROUPS OF A NUMERICAL SEMIGROUP

Minimal generators of a numerical semigroup S can be characterized as those elements $n \in S$ for which $S \setminus \{n\}$ is a numerical semigroup. We could then consider the dual of this property, that is, which are the integers $x \notin S$ such that $S \cup \{x\}$ is a numerical semigroup? If $S \cup \{x\}$ is a numerical semigroup, then

- $kx \in S$ for every integer k greater than one, or in other words, $\{2x, 3x\} \subset S$, and
- $x + (S \setminus \{0\}) \subseteq S$.

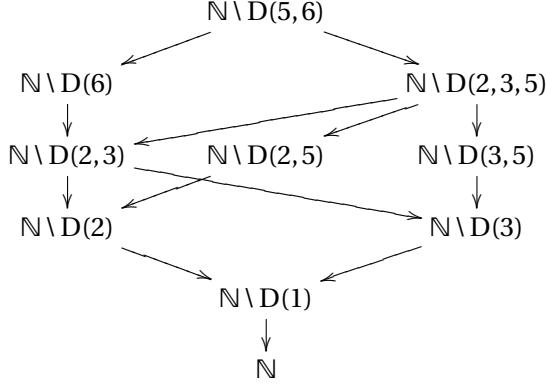
Hence x is both a pseudo-Frobenius number and a fundamental gap of S . These gaps are known as special gaps of S . Thus these are those fundamental gaps that are maximal with respect to \leq_S .

By using this idea it is easy to construct the set of all numerical semigroups containing S by adjoining to S each of its fundamental gaps, and then repeat the process for each resulting semigroup until we reach \mathbb{N} (this will happen after a finite number of steps). The key to perform this in an easy way is the following: if X is the set of fundamental gaps of S and Y is the set of fundamental gaps of $S \cup \{x\}$ for some $x \in X$, then

$$Y = (X \setminus \{x\}) \cup \left\{ \frac{x}{p} : p \text{ a prime dividing } x \text{ and } \frac{x}{p} \notin D(X \setminus \{x\}) \right\}.$$

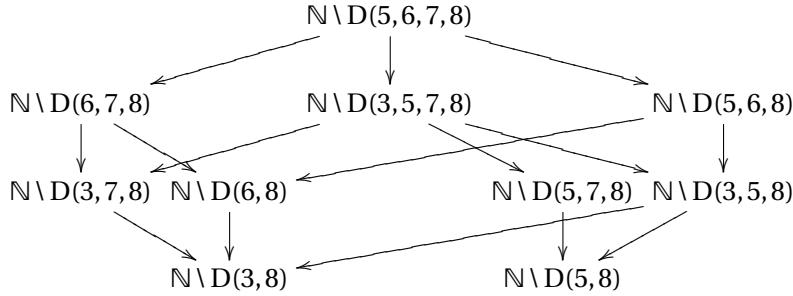
Example 1. Let us consider the semigroup $S = \mathbb{N} \setminus D(5, 6)$. We have that 5 is prime and $6 = 2 \cdot 3$, and both are maximals of $\{5, 6\}$ with respect to \leq_S . Thus our semigroup has two “children”: $\mathbb{N} \setminus D(6)$ (by removing 5) and $\mathbb{N} \setminus D(2, 3, 5)$ (from the

decomposition of 6). Proceeding in this way, we obtain the following graph of inclusions



□

Example 2. We construct the set of numerical semigroups with Frobenius number eight. They all contain the semigroup $\{0, 9, 10, \rightarrow\} = \mathbb{N} \setminus D(5, 6, 7, 8)$. When adjoining a special gap, we will never use 8, so that the Frobenius number is preserved.



Hence there are ten semigroups with Frobenius number eight.

6. PRESENTATIONS (THIRD CHOICE)

Let S be the numerical semigroup generated by $\{2, 3\}$. We could think of S as the commutative monoid generated by two elements x and y such that $3x = 2y$. This is the idea of a presentation. Let us formalize it. Assume that S is minimally generated by $\{n_1, \dots, n_p\}$. The map

$$\varphi : \mathbb{N}^p \rightarrow S, \varphi(a_1, \dots, a_p) = a_1 n_1 + \dots + a_p n_p$$

is a monoid epimorphism, and thus S is isomorphic to $\mathbb{N}^p / \text{Ker}(\varphi)$, where $\text{Ker}(\varphi) = \{(a, b) \in \mathbb{N}^p \times \mathbb{N}^p \mid \varphi(a) = \varphi(b)\}$. A presentation of S is just a system of generators of $\text{Ker}(\varphi)$ (as a congruence).

Rédei proved that every finitely generated commutative monoid is finitely presented, and thus every numerical semigroup is finitely presented. Moreover, for numerical semigroups the concepts of minimality with respect to cardinality and set inclusion of a presentation coincide.

Rosales gave a procedure to construct a minimal presentation of a numerical semigroup from its minimal system of generators. We describe this method briefly. Assume that S is minimally generated by $\{n_1, \dots, n_p\}$. Let $n \in S$. Associated to n we define a graph G_n whose vertices are

$$V_n = \{n_i \mid n - n_i \in \mathbb{N}\}$$

and with edges

$$E_n = \{n_i n_j \mid n - (n_i + n_j) \in \mathbb{N}\}.$$

If G_n is connected, then set $\rho_n = \emptyset$. Otherwise, assume that C_1, \dots, C_k are its connected components. For every $i \in \{1, \dots, k\}$ there exists a factorization (expression) of n in which only vertices of C_i appear, or in other words, there exists $\gamma_i \in \varphi^{-1}(n)$ such that the j th coordinate of γ_i is zero whenever n_j is not a vertex of C_i . Set $\rho_n = \{(\gamma_1, \gamma_2), (\gamma_1, \gamma_3), \dots, (\gamma_1, \gamma_k)\}$. Then $\rho = \bigcup_{n \in S} \rho_n$ is a minimal presentation of S (moreover, every minimal presentation can be obtained in this way if we allow in the definition of ρ_n other pairs so that there is a path linking every two different connected components of G_n). There are finitely many $n \in S$ for which G_n is not connected. Rosales proved that if G_n is not connected, then n is of the form $n = n_i + w$ with $i \in \{2, \dots, p\}$ and $0 \neq w \in \text{Ap}(S, n_1)$.

7. SOME (NUMERICAL) SEMIGROUP RINGS

Let K be a field and S be a numerical semigroup. We choose t to be a symbol. Define $K[S] = \bigoplus_{s \in S} Kt^s$ and $K[[S]] = \prod_{s \in S} Kt^s$. We will represent the elements h of $K[[S]]$ as $h = \sum_{s \in S} a_s t^s$, with $a_s \in \mathbb{N}$ for all s . The element h is in $K[S]$ if and only if $a_s = 0$ for almost all $s \in S$ (all but a finite number of them). We can add two elements of $K[[S]]$ (and of $K[S]$) by adding the coefficients componentwise, and we can multiply them by using the distributive law and the rule $t^s t^{s'} = t^{s+s'}$. In this way, both $K[[S]]$ and $K[S]$ are rings. Moreover, $K[[S]]$ is a local ring whose maximal ideal is $m = (t^{n_1}, \dots, t^{n_p})$, with $\{n_1, \dots, n_p\}$ the minimal system of generators of S (this is why p is called the embedding dimension of S). Some properties of $K[[S]]$ and of $K[S]$ can be determined from properties of S . This study caused some concepts in numerical semigroups to be named after their already existing counterpart in ring theory.

The integral closure of $K[[S]]$ is $K[[t]]$, and if g is the Frobenius number of S , then $t^{g+1} K[[t]] \subseteq K[[S]]$. This is why sometimes the Frobenius number plus one is called the conductor of S .

We can extend the semigroup morphism φ described in the preceding section as follows:

$$\psi : K[x_1, \dots, x_p] \rightarrow K[S], \quad \psi(x_i) = t^{n_i} \quad (i \in \{1, \dots, p\}).$$

The kernel of the ring morphism ψ is known as the defining ideal of S .

In order to simplify the notation, we write $X^a = x_1^{a_1} \cdots x_p^{a_p}$ for $a = (a_1, \dots, a_p) \in \mathbb{N}^p$.

Herzog proved that $(a, b) \in \text{Ker}(\varphi)$ if and only if $X^a - X^b \in \text{Ker}(\psi)$. Moreover, if ρ is a minimal presentation of S , then the set $\{X^a - X^b \mid (a, b) \in \rho\}$ is a minimal system of generators of $\text{Ker}(\psi)$.

On $K[[S]]$ one can define the map $\nu : K[[S]] \rightarrow S$, $\nu(\sum_{s \in S} a_s t^s)$ to be least element in S such that $a_s \neq 0$. This defines a valuation on $K[[S]]$. Several authors have exploited this map. If I is a fractional ideal of $K[[S]]$, then $\nu(I)$ is a relative ideal of S , that is, a subset of \mathbb{Z} (the quotient group of S) such that $I + S \subseteq I$ and $I + s \subseteq S$ for some $s \in S$. If I and J are two fractional ideals with $J \subseteq I$, then the length of I/J equals the cardinality of the set $\nu(I) \setminus \nu(J)$. In particular $I = J$ if and only if $\nu(I) = \nu(J)$.

Lecture 2. Big families

There are several families that have been of interest in the literature due to their “extreme” properties, and for their applications in ring theory.

8. SYMMETRIC AND PSEUDO-SYMMETRIC NUMERICAL SEMIGROUPS

A symmetric numerical semigroup is a numerical semigroup without holes. Or in other words, a numerical semigroup S with Frobenius number g is symmetric if for every integer x , $x \notin S$, implies $g - x \in S$. Fröberg, Gottlieb and Häggkvist proved that symmetric numerical semigroups are those numerical semigroups with odd Frobenius number with the least possible number of gaps, and that this last condition is equivalent to say that they are maximal (with respect to set inclusion) in the set of numerical semigroups with the same Frobenius number. One sees in these conditions, that indeed, symmetric numerical semigroups verify some extreme properties. There are still some other characterizations. For instance, it can be shown that S is symmetric if and only if the cardinality of $\text{Maximals}_{\leq_S}(\text{Ap}(S, n))$ is one (with n any, or all, nonzero element of S). Recall that in particular this means that symmetric numerical semigroups are those numerical semigroups with Cohen-Macaulay type one. Kunz proved that $K[[S]]$ is Gorenstein (Cohen-Macaulay ring with type equal to one) if and only if S is symmetric. Thus, when looking for Gorenstein rings, one can look in the big bag of symmetric numerical semigroups, where, as Rosales showed, you can choose with given embedding dimension and multiplicity.

One might wonder if we can impose analogous restrictions to a numerical semigroup with even Frobenius number. If g is the Frobenius number of S and g is even, then $\frac{g}{2}$ cannot be in S , and $g - \frac{g}{2}$ yields $\frac{g}{2}$. Thus the condition could be: if x is an integer not in S and $x \neq \frac{g}{2}$, then $g - x \in S$. A numerical semigroup fulfilling this condition is called pseudo-symmetric. The (Cohen-Macaulay) type of a pseudo-symmetric numerical semigroup is two (the pseudo-Frobenius numbers are $\frac{g}{2}$ and g ; not every numerical semigroup with type two is pseudo-symmetric). These semigroups still share some nice properties with symmetric numerical semigroups: they are maximal among those numerical semigroups with the same Frobenius number, and are numerical semigroups with the least possible number of gaps.

Observe that from the point of view of fundamental gaps, both concepts can be unified by saying that S is symmetric or pseudo-symmetric if and only if the

set of fundamental gaps has a maximum with respect to \leq_S (observe that $\frac{g}{2}$ cannot be a fundamental gap, for g the Frobenius number of S). This, actually, has to do with the way we described to produce the set of all oversemigroups of a numerical semigroup, since these semigroups are maximal with respect to set inclusion in the set of all numerical semigroups with Frobenius number g .

9. IRREDUCIBLE NUMERICAL SEMIGROUPS. DECOMPOSITIONS INTO IRREDUCIBLES

There is still another idea that unifies both families of numerical semigroups in one. A numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups containing it properly. It turns out that irreducible equals maximality with respect to set inclusion in the set of all numerical semigroups with given Frobenius number. Thus, a numerical semigroup is irreducible if and only if it is symmetric (and thus has odd Frobenius number) or pseudo-symmetric (with even Frobenius number).

Every numerical semigroup can be expressed as the intersection of finitely many irreducible numerical semigroups. Branco and Rosales did a nice work characterizing these factorizations, and studying which semigroups could be expressed in terms of just symmetric or pseudo-symmetric numerical semigroups. We know how to produce minimal decompositions, but still do not have any idea of the number of numerical semigroups involved in such a decomposition, in contrast of which is known for ideals of polynomial rings.

10. COMPLETE INTERSECTIONS AND TELESCOPIC NUMERICAL SEMIGROUPS

A numerical semigroup is a complete intersection if the cardinality of any of its minimal presentations equals its embedding dimension minus one (recall that the embedding dimension is the cardinality of the minimal system of generators of the numerical semigroup). This, in terms of numerical semigroups, means that the semigroup can be described with the least possible number relations. Hence again we have an “extreme” condition characterizing a family of numerical semigroups. It turns out that complete intersections are always symmetric. Thus many authors while looking for examples of Gorenstein rings chose semigroup rings over a complete intersection numerical semigroup (which are as rings a complete intersection, as expected). Delorme proved that a numerical semigroup is a complete intersection if it is the gluing of two complete intersection numerical semigroups, where gluing roughly speaking means that the presentation of the resulting semigroup is obtained by the presentations of each of the semigroups glued plus one relation connecting the generators of the first semigroup with the generators of the second. Numerical semigroups generated by two elements are the easiest example of complete intersections. If we are able to glue a numerical semigroup generated by two elements with a submonoid of \mathbb{N} generated by one element (and thus with no relators), the resulting semigroup is again a complete intersection. We can repeat this procedure and obtain complete intersections with more than three generators. The semigroups obtained

are called telescopic, and their presentations have, from the way they are constructed, a stair shape. Since they are relatively easy to construct, these semigroups have been extensively used in the literature.

11. MAXIMAL EMBEDDING DIMENSION NUMERICAL SEMIGROUPS. ARF PROPERTY

The multiplicity (least element) of a numerical semigroup is an upper bound for the embedding dimension (number of minimal generators). When this bound is reached, we obtain a maximal embedding dimension numerical semigroup. These semigroups not only fulfill this maximal condition, but also they are those with the maximum possible number of relators. This was proved by Sally and the equivalence by Rosales. Observe also that for these semigroups, every nonzero element of the Apéry set of the numerical semigroup with respect to its multiplicity is a minimal generator (the converse is always true, every minimal generator other than the multiplicity is in any Apéry set).

If S is a numerical semigroup, m is its multiplicity, and as usual we write $\text{Ap}(S, m) = \{0, w_1, \dots, w_{m-1}\}$, then $\langle m, m + w_1, \dots, m + w_{m-1} \rangle$ is a maximal embedding dimension numerical semigroup. This exhibits the abundance of semigroups of this kind. Observe that $m + w_i > 2m$ for all i . If we take a maximal embedding dimension numerical semigroup $\langle m, x_1, \dots, x_{m-1} \rangle$ such that $x_i > 2m$ for all i , then $S = \langle m, x_1 - m, \dots, x_{m-1} - m \rangle$ is a numerical semigroup with multiplicity m and $\text{Ap}(S, m) = \{0, x_1 - m, \dots, x_{m-1} - m\}$. Thus there is a one to one correspondence between the set of numerical semigroups with multiplicity m and the set of maximal embedding dimension with multiplicity m verifying that every generator other than the multiplicity is greater than twice the multiplicity.

There are several ways to characterize the maximal embedding dimension property, but we will focus on the following one, which will allow us to introduce some restrictions and will be the inspiration for what comes later in this session. A numerical semigroup S is of maximal embedding dimension if and only if for every x and y nonzero elements of S , $x + y - m \in S$, where m is the multiplicity of S . From this it easily follows that the intersection of two maximal embedding dimension numerical semigroups sharing the same multiplicity is again a maximal embedding dimension with the same multiplicity. Moreover if S is a maximal embedding dimension numerical semigroup with multiplicity m and Frobenius number g , then so is $S \cup \{g\}$.

Note that in the condition $x + y - m \in S$, we are choosing $x, y \in S \setminus \{0\}$, or in other words, $x, y \in S$ with $x, y \geq m$. Thus we can slightly modify this condition by: for every x, y and z in S , with $x, y \geq z$, we have that $x + y - z \in S$. A numerical semigroup fulfilling this condition is (trivially) a maximal embedding dimension numerical semigroup, but not every maximal embedding dimension numerical semigroup has this property. This condition is known as the Arf property. If two numerical semigroups have the Arf property, so does the intersection. This class is also closed under adjoin of the Frobenius number (observe that no restriction on the multiplicity is required here).

12. FAMILIES CLOSED UNDER INTERSECTION AND ADJOIN OF THE FROBENIUS NUMBER

As we have seen above some families of numerical semigroups \mathcal{F} fulfill that

- (C1) if $S_1, S_2 \in \mathcal{F}$, then $S_1 \cap S_2 \in \mathcal{F}$,
- (C2) if $S \in \mathcal{F}$ and g is the Frobenius number of S , then $S \cup \{g\} \in \mathcal{F}$.

We can think on the numerical semigroup S generated in \mathcal{F} by a subset A of \mathbb{N} with greatest common divisor equal to one. This semigroup can be defined as the intersection of all $T \in \mathcal{F}$ such that $A \subseteq T$. We then say that A is an \mathcal{F} -system of generators of S , or that S is the \mathcal{F} -closure of S . Of course, we say that A is minimal if no proper subset of A \mathcal{F} -generates S . We observed that for the families of maximal embedding dimension numerical semigroups with fixed multiplicity, numerical semigroups having the Arf property (and also those having the saturated property, a generalization of the Arf property) and system proportionally modular numerical semigroups (of which we will talk in our next lecture), minimal \mathcal{F} -systems of generators were unique. All these families had in common that (C1) and (C2) hold for them. Moreover, an element is in a minimal \mathcal{F} -system of generators of S if and only if $S \setminus \{m\}$ is again in \mathcal{F} , just as happens with “classical” minimal generators. This enabled us to obtain recurrently the tree of all numerical semigroups in these families.

Rosales proved that these two conditions suffice to show that \mathcal{F} -minimal systems of generators are unique.

Lecture 3. Numerical semigroups and Diophantine inequalities

In our first lecture, the connection between numerical semigroups and Diophantine equalities was made explicit. In this last lecture we will show how the set of all numerical semigroups with fixed multiplicity can be described in terms of a finite system of Diophantine inequalities. We will also give some families of numerical semigroups that can be defined as the set of solutions of a Diophantine inequality.

13. THE SET OF ALL NUMERICAL SEMIGROUPS WITH FIXED MULTIPLICITY IS A MONOID

We say that a set $X \subseteq \mathbb{N}$ is a complete system modulo a positive integer m if the cardinality of X is m and for every $i \in \{1, \dots, m\}$ there exists $x_i \in X$ congruent with i modulo m . This property might sound familiar to the reader, since Apéry sets with respect to an element in a numerical semigroup fulfilled this condition. Not every complete system modulo m is the Apéry set of a numerical semigroup containing m , there is still another restrictions required: first that $x_0 = 0$, and second that $x_i + x_j = x_{(i+j) \bmod m} + km$ for some nonnegative integer k . Observe also that if X is the Apéry set of a numerical semigroup S in m , then $X \cup \{m\}$ generates S and completely determines it (just recall the nice properties of Apéry sets). If one wants to use Apéry sets to describe a numerical semigroup, the

cheapest choice is to take the Apéry set with respect to the multiplicity, since this has the least possible number of elements.

Let S be a numerical semigroup and m be its multiplicity. If $\text{Ap}(S, m) = \{w_0 = 0, w_1, \dots, w_{m-1}\}$ with w_i congruent to i modulo m , then $w_i = k_i m + i$ for some nonnegative integer k_i . Since m is the multiplicity and $w_i \in S$, if $i \neq 0$, then $w_i \geq m$ and thus $k_i \geq 1$. The condition $w_i + w_j = w_{(i+j) \bmod m} + km$ translates to $(k_i + k_j)m + i + j = k_{(i+j) \bmod m}m + (i + j) \bmod m + km$. As $i + j = \lfloor \frac{i+j}{m} \rfloor m + (i + j) \bmod m$, we obtain that (k_1, \dots, k_{m-1}) ($k_0 = 0 = w_0$ gives no information) is a nonnegative integer solution to the system of inequalities

$$(1) \quad \begin{cases} x_i \geq 1, & 1 \leq i \leq m-1, \\ x_i + x_j + \lfloor \frac{i+j}{m} \rfloor \geq x_{(i+j) \bmod m}, & 1 \leq i \leq j \leq m-1, i \neq j. \end{cases}$$

Moreover, if (k_1, \dots, k_{m-1}) is a nonnegative integer solution to (1), then

$$S = \langle m, k_1 m + 1, k_2 m + 2, \dots, k_{m-1} m + m - 1 \rangle$$

is a numerical semigroup with multiplicity m and $\text{Ap}(S, m) = \{0, k_1 m + 1, k_2 m + 2, \dots, k_{m-1} m + m - 1\}$. Let $\mathcal{T}(m)$ be the set of elements of \mathbb{N}^{m-1} that are solutions of (1). Then $\mathcal{T}(m)$ is the ideal of a finitely generated commutative monoid (the monoid of solutions of the associated homogeneous system of inequalities, which belongs to a class of affine semigroups that has been widely studied in the literature). Thus this set can be described by a finite set of elements in \mathbb{N}^{m-1} , and it is bijective with the set of all numerical semigroups with multiplicity m .

If one changes the inequalities $x_i + x_j + \lfloor \frac{i+j}{m} \rfloor \geq x_{(i+j) \bmod m}$ in (1) with $x_i + x_j + \lfloor \frac{i+j}{m} \rfloor > x_{(i+j) \bmod m}$ (or equivalently with $x_i + x_j + \lfloor \frac{i+j}{m} \rfloor \geq x_{(i+j) \bmod m} + 1$), then the set of nonnegative integer solutions to this new system corresponds to the set of maximal embedding dimension numerical semigroups with multiplicity m . This set of solutions is a submonoid of \mathbb{N}^{m-1} , and using the correspondence described in our last lecture between numerical semigroups with multiplicity m and maximal embedding dimension numerical semigroups with multiplicity m and the rest of minimal generators greater than $2m$, one obtains that the set of numerical semigroups with fixed multiplicity is a monoid (isomorphic to the one obtained by replacing $x_i \geq 1$ with $x_i \geq 2$).

14. MODULAR DIOPHANTINE NUMERICAL SEMIGROUPS

Fixed m a nonzero element in a numerical semigroup, recall that a nonnegative integer x belongs to S if and only if $w_{x \bmod m} \leq x$, where w_i stands for the least element in S congruent with i modulo m (these are precisely the elements in $\text{Ap}(S, m)$). If we define $f_S : \mathbb{N} \rightarrow \mathbb{Q}$ (needless to say that we use \mathbb{Q} to denote the set of rational numbers) as $f_S(x) = w_{x \bmod m}$, then, in view of the properties of the Apéry set studied in the preceding section, $f_S(x+y) \leq f_S(x) + f_S(y)$. Observe also that $f_S(x+m) = f_S(x)$, that is, f_S is subadditive, $f_S(0) = 0$ and it is periodic with period m . Moreover,

$$S = \{x \in \mathbb{N} \mid f_S(x) \leq x\}.$$

The converse is also true, every subadditive function f such that $f(0) = 0$ and $f(x + m) = f(x)$ defines a numerical semigroup

$$S_f = \{x \in \mathbb{N} \mid f(x) \leq x\}.$$

Let a and b be positive integers, and set $f(x) = ax \bmod b$. Then f is subadditive, $f(0) = 0$ and $f(x + b) = f(x)$. Thus

$$S(a, b) = \{x \in \mathbb{N} \mid ax \bmod b \leq x\}$$

is a numerical semigroup. We say that a numerical semigroup S is modular if there exist a and b such that $S = S(a, b)$. This was the beginning of a research that yielded Urbano-Blanco's thesis and in which M. Delgado was also engaged (together with our team in Granada).

Even though the membership problem for these semigroups is trivial, surprisingly we know relatively few things about these semigroups. We still do not know a formula for the Frobenius number in terms of a and b , neither for the multiplicity. However the cardinality of the set of gaps of $S(a, b)$ is

$$\frac{b+1-\gcd\{a,b\}-\gcd\{a-1,b\}}{2}.$$

This formula was not obtained using Selmer's formula (presented the first day), since we do not have a nice way to describe the set of elements in the Apéry sets of $S(a, b)$. Recall that symmetric and pseudo-symmetric numerical semigroups were numerical semigroups with the least possible number of gaps with odd and even Frobenius number, respectively. Thus if S is symmetric (respectively pseudo-symmetric) with Frobenius number g , then the cardinality of the set of gaps of S is $\frac{g+1}{2}$ (respectively $\frac{g+2}{2}$). In this way it is easy to derive when a modular numerical semigroup is symmetric or pseudo-symmetric.

We were able to describe, for some subfamilies of modular numerical semigroups, some of the invariants mentioned above. We found also an algorithm procedure to recognize modular numerical semigroups, which has been recently improved by Urbano-Blanco and Rosales, giving all possible pairs a and b for which a numerical semigroup $S = S(a, b)$. These improvements were achieved by using Bézout sequences, which are the topic of our next section.

15. PROPORTIONALLY MODULAR DIOPHANTINE NUMERICAL SEMIGROUPS, OR FINDING THE NON-NEGATIVE INTEGER SOLUTIONS TO $ax \bmod b \leq cx$

If we choose now $f(x) = \frac{1}{c}(ax \bmod b)$, with a, b and c positive integers, then this map is also subadditive, $f(0) = 0$ and $f(x + b) = f(x)$ for all $x \in \mathbb{N}$. Thus

$$S(a, b, c) = \{x \in \mathbb{N} \mid ax \bmod b \leq cx\}$$

is a numerical semigroup. These semigroups are known as proportionally modular numerical semigroups, and clearly, every modular numerical semigroup belongs to this class. The point is that we do not have in general, as we had for modular numerical semigroups, a formula of the number of gaps of $S(a, b, c)$. However while generalizing from modular to proportionally modular, we learned a lot more than we knew about modular numerical semigroups, due mainly to the

tools we developed in this more general setting. We explain briefly some of them here.

Assume that I is a non-empty interval of \mathbb{Q}^+ . The submonoid of \mathbb{Q}^+ generated by I is $\bigcup_{k \in \mathbb{N}} kI$. If we cut this monoid with \mathbb{N} , we obtain a numerical semigroup, which amazingly is always a proportionally modular numerical semigroup. We will denote this numerical semigroup by $S(I)$. Moreover, every proportionally numerical semigroup can be obtained in this way:

$$S(a, b, c) = S\left([\frac{b}{a}, \frac{b}{a-c}]\right)$$

(since we are performing computations modulo b , we can assume that $a < b$, and if $c \geq a$, then trivially $S(a, b, c) = \mathbb{N}$; thus we assume that $c < a < b$). This in particular means that for every nonnegative integer x , we have that $x \in S(a, b, c)$ if and only if there exists $k \in \mathbb{N} \setminus \{0\}$ such that $\frac{b}{a} \leq \frac{x}{k} \leq \frac{b}{a-c}$ (from the restrictions assumed for a , b and c , this implies that $k \in \{1, \dots, x-1\}$). This allows us to decide if a numerical semigroup is proportionally modular or not, since we only have to find α and β in \mathbb{Q} such that for every minimal generator n of the semigroup there is a $k \in \{1, \dots, x-1\}$ for which $\alpha \leq \frac{n}{k} \leq \beta$, and such that for every fundamental gap h there is not such k .

If a, b, c and d are positive integers such that $\frac{a}{c} < \frac{b}{d}$ and $bc - ad = 1$, then it can be shown that $S([\frac{a}{c}, \frac{b}{d}]) = \langle a, b \rangle$. Thus this enables us to compute a system of generators of $S([\frac{a}{c}, \frac{b}{d}])$, when $bc - ad = 1$. But, what happens if $ad - bc > 1$? How can we obtain a generating system for $S([\frac{a}{c}, \frac{b}{d}])$? If we were able to solve this, for any $a, b, c \in \mathbb{N}$, we would be able to determine those integers that are solution to $ax \bmod b \leq cx$, or in other words, find a system of generators of $S(a, b, c)$.

Thus let us go back once more to the problem of finding a system of generators of $S([\frac{a}{c}, \frac{b}{d}])$, and assume that there are a_1, \dots, a_n and b_1, \dots, b_1 positive integers such that

- $\frac{a_1}{b_1} < \dots < \frac{a_n}{b_n}$,
- for all $i \in \{1, \dots, n-1\}$, $a_{i+1}b_i - a_i b_{i+1} = 1$,
- $\frac{a_1}{b_1} = \frac{a}{c}$ and $\frac{a_n}{b_n} = \frac{b}{d}$.

Then $\frac{a}{c} < \frac{x}{k} < \frac{b}{d}$ if and only if for some $i \in \{1, \dots, n-1\}$, $\frac{a_i}{b_i} < \frac{x}{k} < \frac{a_{i+1}}{b_{i+1}}$, or equivalently, $x \in \langle a_i, a_{i+1} \rangle$. Thus if such a sequence exists, $S([\frac{a}{c}, \frac{b}{d}]) = \langle a_1, \dots, a_n \rangle$ (moreover, $S([\frac{a}{c}, \frac{b}{d}]) = \langle a_1, a_2 \rangle \cup \dots \cup \langle a_{n-1}, a_n \rangle$).

We say that the sequence $\frac{a_1}{b_1} < \dots < \frac{a_n}{b_n}$ is a Bézout sequence joining $\frac{a}{c}$ and $\frac{b}{d}$. Bézout sequences connecting a rational number with another larger rational number always exist and are easy to compute. Their properties have shed some light in the world of proportionally modular numerical semigroups. For instance, we know that the numerators of a Bézout sequence form a convex sequence, whence the first two minimal generators of a proportionally modular numerical semigroup are always adjacent in the sequence. This in particular means that they are coprime. Hence $\langle 4, 6, 7 \rangle$ is not a proportionally modular numerical semigroup.

16. TOMS' RESULT ON THE POSITIVE CONE OF THE K_0 -GROUP FOR SOME C^* -ALGEBRAS

We say that a numerical semigroup S is system proportionally modular if it is the intersection of finitely many proportionally modular numerical semigroups. That is, S is the set of integer solutions to a system of equations of the form:

$$\left\{ \begin{array}{l} a_1 x \bmod b_1 \leq c_1 x, \\ a_2 x \bmod b_2 \leq c_2 x, \\ \vdots \\ a_k x \bmod b_k \leq c_k x, \end{array} \right.$$

with a_i, b_i, c_i positive integers. Trivially, system proportionally modular numerical semigroups are closed under intersections, and it can be shown that they are closed also under the operation of adjoining the Frobenius number. Thus, as seen in the last lecture, it makes sense to talk about minimal systems of generators for numerical semigroups in this family, and one can recurrently construct the set of all system proportionally numerical semigroups and arrange it in a tree. We already knew that some irreducible numerical semigroups were not proportionally modular, and thus not every numerical semigroup is system proportionally modular.

Urbano-Blanco and Rosales proved that every proportionally modular numerical semigroup is of the form $\frac{\langle m, n \rangle}{d}$, where in general for a numerical semigroup S and a positive integer d ,

$$\frac{S}{d} = \{x \in \mathbb{N} \mid dx \in S\},$$

which is also a numerical semigroup. Unfortunately even if we have a formula for the Frobenius number of S (as we have for $\langle n_1, n_2 \rangle$), we do not know how is the Frobenius number of $\frac{S}{d}$. The same stands for the minimal generators and other invariants of the semigroup.

It follows that every system proportionally numerical semigroup can be expressed as $\frac{\langle m_1, n_1 \rangle}{d_1} \cap \dots \cap \frac{\langle m_l, n_l \rangle}{d_l}$. We were able to show that modifying conveniently m_i and n_i , we could choose $d_1 = \dots = d_l$, and so that m_i, n_i, d_i are pairwise coprime. We were looking for this since recently Toms proved that for numerical semigroups S of this form, there is always a C^* -algebra for which its K_0 -ordered group is isomorphic to (\mathbb{Z}, S) .

We already know that not every numerical semigroup is system proportionally modular, but the question of determining a C^* -algebra fulfilling the above condition for this semigroup still remains open.

17. REFERENCES

It is difficult to collect a complete list of references related to numerical semigroups. Our friend Jorge Ramírez Alfonsín already did a wonderful work in his book. Barucci, Dobbs and Fontana wrote a wonderful monograph highlighting the connections between numerical semigroups and analytically unramified

one-dimensional local domains, which in addition can be used as a dictionary for the names we use for the invariants of a numerical semigroup. Barucci recently wrote a nice survey on numerical semigroup rings that will be included in a book in honour to Gilmer (to whom all the people working in commutative semigroups are in debt). This survey is available in her web page. Of course, I must mention Rosales' production in the field of numerical semigroups, since he was who introduced me in this fascinating world. And indeed I am happy he did...