GRÖBNER FINITE PATH ALGEBRAS

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Let k be a field and Γ a finite directed multi-graph. In this talk we will give a complete description of all path algebras $k\Gamma$ and admissible orders with the property that all of their finitely generated ideals have finite Gröbner bases and of those which contain a finitely generated ideal whose Gröbner bases are all infinite.

We will define Gröbner finite path algebras to be the path algebras such that all of their finitely generated ideals have finite Gröbner bases. We go on to classify which path algebras are Gröbner finite. Furthermore we offer a simple description of the orders in which Gröbner finite path algebras have finite Gröbner bases. These results may be found in [2].

A graph Γ may have arrows from a vertex to itself or multiple arrows between the same set of vertices. Let β denote the set of paths of finite length, β_0 denote the set of vertices and β_1 denote the set of arrows. Arbitrary vertices and arrows will be denoted by v_i and α_i respectively. We define functions $o : \beta \to \beta_0$ and $t : \beta \to \beta_0$, such that for any path $p \in \beta$, $o(p)$ is the origin or first vertex of a path p and $t(p)$ is the terminus or last vertex of a path p. When writing paths as a product of arrows the convention will be to write a path, $\alpha_1\alpha_2\cdots\alpha_n$, from left to right, such that $t(\alpha_i) = o(\alpha_{i+1})$. For a path p, define the *length* function $l(p)$ to be the number of arrows that occur in a path p, counting multiplicities. Two paths will be said to intersect if they share a common vertex. A cycle is a path that begins and ends at the same vertex. A trivial cycle is a path of zero length beginning and ending at the same vertex.

We define multiplication of paths, such that for p, q in β, if $t(p) = o(q)$, then their product pq is the path adjoining p and q by concatenation. Otherwise, if $t(p) \neq o(q)$ then $pq = 0$. The path algebra $k\Gamma$ is defined to be the set of all finite linear combinations of paths in β with coefficients in k. Addition in $k\Gamma$ is the usual $k-$vector space addition, where β is a k-basis for $k\Gamma$. Multiplication in the path algebra, $k\Gamma$, extends k-linearly from the definition for multiplication of paths. Note that the identity element is always of the form $\sum_{v\in\beta_0} v$.

A Gröbner basis for an ideal in a path algebra is dependent upon choosing an ordering for the paths in β. A path order $<$ is considered to be an *admissible* order if it satisfies the following four conditions, for all $p, q, r, s \in \beta$:

(1) If $p \neq q$, then $p < q$ or $q < p$;
(2) Every nonempty set of paths has a least element;
(3) If $p < q$ and $rps, rqs \neq 0$, then $rps < rqs$ (r and s may be trivial); and
(4) If $p = qr \neq 0$, then $p \geq q$ and $p \geq r$.

Definition 1. *Let $x = \sum_{i=1}^{n} \gamma_i p_i \in K\Gamma$ with $\gamma_i \in K - \{0\}$. Then the support of x is the set $Supp(x) = \{p_1, \ldots, p_n\}$.*

Definition 2. *Given an admissible order $<$, for any nonzero $x \in K\Gamma$, the tip of x, denoted $Tip(x)$, is the largest path in $Supp(x)$. That is, $Tip(x) \in Supp(x)$ and for all $p \in Supp(x)$, $p \leq Tip(x)$.*

Definition 3. *Given $X \subset K\Gamma$ the set $\{p \in \beta|\ p = Tip(x)$ for some $x \in X\}$ is denoted as $Tip(X)$.*

Definition 4. *Let I be an ideal in the path algebra $K\Gamma$ with admissible order $<$. We say a set $\mathcal{G} \subset I$ is a Gröbner basis for I when for all $x \in I - \{0\}$, there exists $g \in \mathcal{G}$ such that $Tip(g)$ divides $Tip(x)$.*

Let X be a subset of $K\Gamma$ and $y \in K\Gamma$. We say y can be *reduced* by X if for some p in $Supp(y)$, there exists $x \in X$ such that $Tip(x)$ divides p. A *reduction* of y by X is given by $y - kpxq$, where $x \in X$, $p, q \in \beta$, and $k \in K - \{0\}$ such that $kp(Tip(x))q$ is a term in y. A *total reduction* of y by X is an element resulting from a sequence of reductions that cannot be further reduced by X. In general, two total reductions of an element y need not be the same element. However, when X is a Gröbner basis for an ideal all total reductions y produce the same element. We say an element y reduces to 0 by X if there is a total reduction of y by X which is 0. A set $X \subset K\Gamma$ is said to be *reduced* if for all $x \in X$, x cannot be reduced by $X - \{x\}$.

Definition 5. *The unique reduced monic Gröbner basis is called the reduced Gröbner basis.*

Given an ideal I in the path algebra $K\Gamma$ with admissible order if the reduced Gröbner basis is not finite then no other Gröbner basis will be finite.

Definition 6. *Let $f, g \in K\Gamma$, with admissible order $<$ on $K\Gamma$. Suppose there are paths p and q, of positive length, such that $Tip(f)p = qTip(g)$, with $l(p) < l(Tip(g))$. Then f and g have an overlap relation, denoted $o(f, g, p, q)$, given by*

$$o(f, g, p, q) = c_{Tip(f)}^{-1} fp - c_{Tip(g)}^{-1} qg \quad .$$

Given elements f and g whose tips overlap, the p and q will not necessarily be unique and consequently the same two elements f and g may have multiple overlap relations. Additionally an element may have an overlap relation with itself i.e. $o(f, f, p, q)$ is possibly an overlap relation.

Lemma 7. [1] *Let $\mathcal{G}$ be a set of uniform elements that form a generating set for the ideal $I \subset K\Gamma$, such that for all $g, g' \in \mathcal{G}, Tip(g)$ does not divide $Tip(g')$. Suppose that for each $f, g \in \mathcal{G}$ every overlap relation $o(f, g, p, q)$ reduces to 0 by $\mathcal{G}$. Then, $\mathcal{G}$ is a Gröbner basis for the ideal I.*

The following iterative construction produces a Gröbner basis in the limit:

- Let G_0 be generating set for an ideal I;
- $G_i' = G_i \cup \{o(f, g, p, q)|\ f, g \in G_i$ and p, q are overlap relations $\}$;
- G_{i+1} is any reduced set produced by repeatedly replacing elements $x \in G_i'$ with a reduction of x by $G_i' \setminus \{x\}$.

Then $G = \{g|\ g \in G_i$ for $i \gg 0\}$ is a reduced Gröbner basis for I. Also for all d there exists N such that for all $n \geq N$ we have

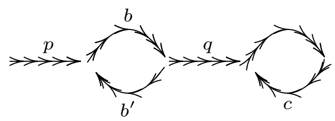
$$\{g \in G|\ l(g) \leq d\} = \{g \in G_n|\ l(g) \leq d\}.$$

However, for an arbirary ideal over arbitrary path algebra the value of N for a given d is indeterminable. In fact the ability to determine such an N in general would imply a solution to the word problem.

Definition 8. *Define a path algebra to be Gröbner finite if there is an admissible order $<$ such that all of its finitely generated ideals have finite reduced Gröbner bases.*

The following examples are key to determining when a path algebra contains a finitely generated ideal with an infinite Gröbner basis under a specific admissible order.

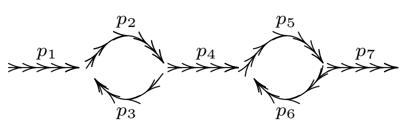
Example 9. *Let Γ be the graph given below.*



Let $<$ be an admissible order on the paths of Γ such that for some $i \in \mathbb{N}$ we have $qc^i > b'bq$. Let i be fixed for this example. In this case the 2-generated ideal $\langle pbq, qc^i - b'bq\rangle$ of $k\Gamma$ has an infinite reduced Gröbner basis $\{pb(b'b)^j q, qc^i - b'bq |\ j \in \mathbb{N}_0\}$.

However, for any admissible order $<$ such that $qc^i < b'bq$ the ideal $\langle pbq, b'bq - qc^i\rangle$ has a finite Gröbner basis $\{pbq, b'bq - qc^i\}$. This is readily apparent since there are no overlap relations between the elements of the set $\{pbq, b'bq - qc^i\}$.

Example 10. *Let Γ be the graph given below.*



Let $<$ be an admissible order on the paths of Γ. If $p_4p_5p_6 > p_3p_2p_4$, then the ideal $\langle p_1p_2p_4, p_4p_5p_6 - p_3p_2p_4\rangle$ has reduced Gröbner basis

$$\{p_1p_2(p_3p_2)^i p_4,\ p_4p_5p_6 - p_3p_2p_4|\ \in \mathbb{N}_0\}.$$

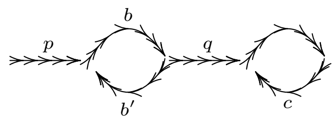
Else if $p_3p_2p_4 > p_4p_5p_6$, then the ideal $\langle p_4p_5p_7, p_3p_2p_4 - p_4p_5p_6\rangle$ of $k\Gamma$ has infinite reduced Gröbner basis

$$\{p_4(p_5p_6)^i p_5p_7,\ p_3p_2p_4 - p_4p_5p_6|\ i \in \mathbb{N}_0\}.$$

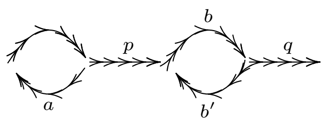
It follows that $k\Gamma$ has a finitely generated ideal with an infinite reduced Gröbner basis, under any admissible order.

Example 11. *Now let Γ be a graph which contains two nontrivial cycles P and Q, which intersect at a vertex v. Let p be the path from v to itself that goes around P once and let q be the path from v to itself that goes around Q once. If $pq^2 > p^2q$, then the ideal $\langle pqpq, pq^2 - p^2q\rangle$ has reduced Gröbner basis $\{pqp^i q, pq^2 - p^2q|\ i \in \mathbb{N}\}$. Else if $p^2q > pq^2$, then the ideal $\langle pqpq, p^2q - pq^2\rangle$ of $K\Gamma$ has reduced Gröbner basis $\{pq^i pq, p^2q - pq^2|\ i \in \mathbb{N}\}$. Therefore, under any admissable order a path algebra which contains two intersecting cycles, also contains a finitely generated ideal with an infinite reduced Gröbner basis.*

Theorem 12. *A path algebra $k\Gamma$ with admissible ordering $<$ contains a finitely generated ideal with infinite reduced Gröbner basis if and only if the graph Γ and the order $<$ satisfy one of the following conditions.*
(1)Γ *contains 2 intersecting cycles.*
(2)Γ *contains a subgraph of the form*



where $l(b)$ may be zero, with $b'bq < qc^i$ for some i.
(3)Γ *contains a subgraph of the form*



where $l(b)$ may be zero, with $pbb' < a^i p$ for some i.

Remark 13. *A path algebra $k\Gamma$ is Noetherian, if and only if, Γ both enters and exits a non-trivial cycle.*

Corollary 14. *A path algebra $k\Gamma$ is Gröbner finite if and only if no path on Γ enters and exits two distinct non-trivial cycles.*

By the unsolvability of the word problem it follows that whenever Γ contains two distinct non-trivial intersecting cycles then $k\Gamma$ contains a finitely generated ideal for which it is impossible to determine a Gröbner basis under any admissible order.

Question 15. *Which path algebras have the property that every finitely generated ideal has a determinable Gröbner basis under some admissible order?*

References

[1] G. Bergman: The diamond lemma for ring theory. Adv. Math. **29**, 178-218 (1978).
[2] M. Leamer, *Gröbner finite path algebras*, J. Symb. Comp., **41-1** (2006), 98-111.ract