FAITHFUL ACTIONS ON DIFFERENTIAL GRADED ALGEBRAS DETERMINE THE ISOMORPHISM TYPE OF FINITE GROUPS

CRISTINA COSTOYA AND ANTONIO VIRUEL

ABSTRACT. It is well known that the isomorphism type of a finite group cannot be decided by enumerating the set of vector spaces (abelian groups) on which that group acts faithfully. In contrast with this situation, we show that the isomorphism type of a finite group is determined by the set of differential graded algebras on which that group acts faithfully.

1. INTRODUCTION

A classical problem in Mathematics is the Isomorphism Problem:

Question 1.1. Given a category $\mathcal{C}$, the Isomorphism Problem (IP in what follows) in $\mathcal{C}$ consists of providing an algorithm or procedure to determine whether or not two objects in $\mathcal{C}$ are isomorphic. If such a procedure exists and the techniques fit in some theory $\mathcal{T}$, we say that $\mathcal{T}$ tells objects in $\mathcal{C}$ apart.

Well known examples of the IP are Dehn's Finite Presentation Problem [2] and the Graph IP [8, 6]. Here we consider the Finite Group IP by means of Representation Theory:

Does $\mathcal{T}$ = "Representation Theory" tell finite groups apart?

The answer to the question above depends on what is meant by Representation Theory, and how it is used to compare finite groups.

If we consider Representation Theory just as Linear Representation of Groups, and we compare two finite groups by looking at their set of modules, the answer to Question 1.1 is negative: in [7] it is shown that there exist two non isomorphic finite groups G and H, both of size $2^{21}97^{28}$, such that $\mathbb{Z}[G] \cong \mathbb{Z}[H]$ as rings, thus every G-module admits an H-module structure.

Our approach to Representation Theory is broader, since we consider actions on rational Differential Graded Algebras (DGA's for short). In this setting the answer to Question 1.1 is positive:

Theorem 1.2. *Let G and H be finite groups, and (A, d) be a finitely generated rational DGA. Then the following statements are equivalent:*

- *G and H are isomorphic.*
- *$G \leq \mathrm{Aut}(A, d)$ if and only if $H \leq \mathrm{Aut}(A, d)$.*

2. DIFFERENTIAL GRADED ALGEBRAS

We follow the notation in [3].

Definition 2.1. A Graded Module V over $\mathbb{K}$, is a family $\{V_i\}_{i\in\mathbb{Z}}$ of $\mathbb{K}$-modules. By "abuse of language" we say that $v \in V_i$ is an element of V of degree i, and we write $|v| = i$.

Definition 2.2. A Graded Algebra over $\mathbb{K}$ is a $\mathbb{K}$-graded module R together with an associative linear map of degree zero, $R\otimes R \to R$, $x\otimes y \mapsto xy$, that has an identity $1 \in R$. A derivation of degree k is a linear map $\theta\colon R \to R$ of a degree k such that $\theta(xy) = \theta(x)y+(-l)^{k|x|}\theta(y)$.

Definition 2.3. A Differential Graded Algebra over $\mathbb{K}$ (DGA for short) is a $\mathbb{K}$-graded algebra R together with a differential in R that is a derivation. A morphism of differential graded algebras $f : (R, d) \to (E, d)$ is a morphism of graded algebras satisfying $fd = df$.

We are interested in the case $\mathbb{K} = \mathbb{Q}$, that is, in rational DGA's, and we consider only the commutative case. Here commutative means commutative in the graded sense, so $ab = (-1)^{|a|\cdot|b|}ba$.

Our interest for these DGA's comes from homotopy theory: for any topological space X Sullivan [9] defined a commutative DGA $A_{PL}(X)$, called the algebra of polynomial differential forms on X with rational coefficients. This DGA codifies the rational homotopy type of X.

3. Groups to Graphs to DGA's

A graph is a pair $\mathcal{G} = (V, E)$ of sets such that $E \subseteq V^2$. The elements v of V are the *vertices* of $\mathcal{G}$, the elements of E are its *edges* and we will write them as couples (v, w) of vertices. For $\mathcal{G} = (V, E)$ and $\mathcal{G}' = (V', E')$ we will say that $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic if there exists a bijection $\sigma : V \to V'$ with (v, w) in E if and only if $(\sigma(v), \sigma(w)) \in E'$ for every (v, w) in E. Such a map is called an isomorphism; if $\mathcal{G} = \mathcal{G}'$ it is called an automorphism. In this work we only consider simple graphs which implies they are non directed graphs, that is if (v, w) is in E then (w, v) is also in E, and they have no loops, that is $(v, v) \notin E$.

Groups actions on finite graphs tell finite groups apart. This follows by the result of Frucht in [4, 5]:

Theorem 3.1. *Given a finite group G, there exist infinitely many non-isomorphic connected (finite) graphs $\mathcal{G}$ whose automorphism group is isomorphic to G.*

Now, given a finite graph $\mathcal{G}$, one can construct a finitely generated rational DGA (A, d) whose automorphisms are closely related to those of $\mathcal{G}$. This follows by previous results of the authors in [1, Section 2]:

Theorem 3.2. *Let $\mathcal{G} = (V, E)$ be a finite graph without isolated vertices, and let*

$$\mathcal{M} = \big(\Lambda(x_1, x_2, y_1, y_2, y_3, z) \otimes \Lambda(x_v, z_v | v \in V), d\big)$$

be the rational DGA where dimensions and differential are

$$\begin{array}{ll} |x_1| = 8, & d(x_1) = 0 \\ |x_2| = 10, & d(x_2) = 0 \\ |y_1| = 33, & d(y_1) = x_1^3x_2 \\ |y_2| = 35, & d(y_2) = x_1^2x_2^2 \\ |y_3| = 37, & d(y_3) = x_1x_2^3 \\ |x_v| = 40, & d(x_v) = 0, \\ |z| = 119, & d(z) = y_1y_2x_1^4x_2^2 - y_1y_3x_1^5x_2 + y_2y_3x_1^6 + x_1^{15} + x_2^{12} \\ |z_v| = 119, & d(z_v) = x_v^3 + \sum_{(v,w)\in E} x_vx_wx_2^4. \end{array} \tag{1}$$

Then given $f \in \operatorname{Aut}(\mathcal{M})$, *there exists* $\sigma \in \operatorname{Aut}\mathcal{G}$ *such that*

$$\begin{aligned} f(x_1) &= x_1 \\ f(x_2) &= x_2 \\ f(y_1) &= y_1 \\ f(y_2) &= y_2 \\ f(y_3) &= y_3 \\ f(x_v) &= x_{\sigma(v)} \\ f(z) &= z + d(m_z) \\ f(z_v) &= z_{\sigma(v)} + d(m_{z_v}) \end{aligned} \tag{2}$$

with $|m_z| = |m_{z_v}| = 118$ *elements in* $\mathcal{M}$.

4. Proof of Theorem 1.2

Let G and H be finite groups, and (A, d) be a finitely generated rational DGA.

If $G \cong H$, then $G \leq \operatorname{Aut}(A, d)$ if and only if $H \leq \operatorname{Aut}(A, d)$.

Assume now that $G \ncong H$. Without loss of generality we may assume that H is not (isomorphic to) a subgroup of G, that is $H \nleq G$.

According to Theorem 3.1, there exists a connected finite graph $\mathcal{G}$ such that $\operatorname{Aut}(\mathcal{G}) = G$. Let $\mathcal{M}$ be the rational DGA associated to $\mathcal{G}$ by means of Theorem 3.2.

Projection over the module of indecomposable elements of $\mathcal{M}$ provides an split exact sequence of groups

$$K \longrightarrow \operatorname{Aut}(\mathcal{M}) \longrightarrow \operatorname{Aut}(\mathcal{G}) = G$$

where $f \in K$ if and only if

$$\begin{aligned} f(x_1) &= x_1 \\ f(x_2) &= x_2 \\ f(y_1) &= y_1 \\ f(y_2) &= y_2 \\ f(y_3) &= y_3 \\ f(x_v) &= x_v \\ f(z) &= z + d(m_z) \\ f(z_v) &= z_v + d(m_{z_v}) \end{aligned}$$

where $|m_z| = |m_{z_v}| = 118$ elements in $\mathcal{M}$, that is $f(m_z) = m_z$ and $f(m_{z_v}) = m_{z_v}$.

Therefore K is torsion free, and since $\operatorname{Aut}(\mathcal{M}) = K \rtimes G$, every a maximal finite subgroup of $\operatorname{Aut}(\mathcal{M})$ is (up to isomorphisms) a subgroup of G.

Recall that $H \nleq G$, then $H \nleq \operatorname{Aut}(\mathcal{M})$.

References

[1] C. Costoya, A. Viruel, *Every finite group is the group of self homotopy equivalences of an elliptic space*, arXiv:1106.1087v1.

[2] M. Dehn, *Über unendliche diskontinuierliche Gruppen*, Math. Ann. 71 (1911), 116–144.

[3] Y. Felix, S. Halperin, J.C. Thomas, Rational homotopy theory. Graduate Texts in Mathematics, 205. Springer-Verlag, New York, 2001.

[4] R. Frucht, *Herstellung von Graphen mit vorgegebener abstrakter Gruppe*, Compositio Math., 6 (1939), 239–250.

[5] R. Frucht, *Graphs of degree* 3 *with given abstract group*, Canad. J. Math., 1 (1949), 365–378.

[6] G. Gati, *Further annotated bibliography on the isomorphism disease*, J. of Graph Theory 3 (1979), 95–109.
[7] M. Hertweck, *A counterexample to the isomorphism problem for integral group rings*, Ann. of Math. 154 (2001), 115–138.
[8] R. Read and D. Corneil, *The graph isomorphism disease*, J. of Graph Theory 1 (1977), 339–363.
[9] D. Sullivan, *Infinitesimal computations in topology*, Publications Mathématiques de l'IHÉS 47 (1977), 269–331.

Dep. de Computación, Álxebra, Universidade da Coruña, Campus de Elviña, s/n, 15071 A Coruña, Spain.
E-mail address: cristina.costoya@udc.es

Dep. de Álgebra, Geometría y Topología, Campus de Teatinos, Universidad de Málaga, 29071 Málaga, Spain
E-mail address: viruel@agt.cie.uma.es