STRONGLY PRESERVER PROBLEMS

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Let A be an algebra. An element $b \in A$ is called the *Drazin inverse* of $a \in A$, if

$$ab = ba, \quad bab = b, \quad \text{and} \quad a^k ba = a^k \text{ for some positive integer k.}$$

The least of such k is the *index* of a, denoted by ind(a), and when ind(a) = 1, b is called the *group inverse* of a.

The Drazin inverse, and hence the group inverse, is unique whenever it exists. Let A^D and A^G denote the sets of all Drazin and group invertible elements in A, respectively. Similarly a^D and a^G denote the Drazin inverse and group inverse of a, respectively.

Every unital Jordan homomorphism between Banach algebras *strongly preserves invertibility*, that is, $T(a^{-1}) = T(a)^{-1}$, for every invertible element $a \in A$. Moreover, Hua's theorem (see [6]) states that every unital additive map between fields that strongly preserves invertibility is either an isomorphism or an anti-isomorphism. In [1, 11] the authors started the study of Hua's type characterizations for Banach algebras: if $T : A \to B$ is a Jordan homomorphism between Banach algebras, then it strongly preserves group invertibility ($T(a^G) = T(a)^G$ for every $a \in A^G$) and Drazin invertibility ($T(a^D) = T(a)^D$ for every $a \in A^D$); see [11, Theorem 2.1]. Conversely, if $T : A \to B$ is a linear map and $T(\mathbf{1}) = \mathbf{1}$ (respectively, $T(\mathbf{1})$ is invertible or $\mathbf{1} \in T(A)$), strongly preserving invertibility, group invertibility or Drazin invertibility, then T (respectively $T(\mathbf{1})T$) is a unital Jordan homomorphism (and $T(\mathbf{1})$ commutes with the image of T), [1, Theorem 4.2].

Recall that an element $a \in A$ is *regular* if there is $b \in A$ such that $aba = a$ and $b = bab$. When A is a C^*-algebra, we say that b is a *Moore-Penrose inverse* of a if $a = aba$, $b = bab$ and ab and ba are selfadjoint. It is known that every regular element a in A has a unique Moore-Penrose inverse that will be denoted by $a^\dagger$.

There exists a wider class of complex Banach spaces containing all C*-algebras in which the notion of regularity makes sense and extends the concept given for C*-algebras. We refer to the class of JB*-triples introduced by W. Kaup in [8]. Every C*-algebra is a JB*-triple via the triple product given by

$$\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x),$$

and every JB*-algebra is a JB*-triple under the triple product

$$\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*.$$

For each x in a JB*-triple E, $Q(x)$ will stand for the conjugate linear operator on E defined by $Q(x)(y) = \{x, y, x\}$.

An element a in a JB*-triple E is called *von Neumann regular* if there exists (a unique) $b \in E$ such that $Q(a)(b) = a$, $Q(b)(a) = b$ and $Q(a)Q(b) = Q(b)Q(a)$, or equivalently $Q(a)(b) = a$ and $Q(a)(b^{[3]}) = b$ (see [5, 9, 10]). The element b is called the *generalized inverse* of a. Denote by $E^\wedge$ the set of regular elements in the JB*-triple E, and for an element $a \in E^\wedge$ let $a^\wedge$ denotes its generalized inverse.

For a C*-algebra A, an element a in A has Moore-Penrose inverse b, if and only if a has generalized inverse b^* in the JB*-triple structure. That is $a^\wedge = (a^\dagger)^* = (a^*)^\dagger$.

Let A and B be C*-algebras. We say that a linear map $T : A \to B$ *strongly preserves regularity* if $T(a^\wedge) = T(a)^\wedge$, for all $a \in A^\wedge$. Notice that if T is selfadjoint, then T strongly preserves Moore-Penrose invertibility if and only if T strongly preserves regularity.

We are concerned with the study of additive and linear maps strongly preserving Drazin, group and generalized invertibility.

Let A and B be Banach algebras. Assume that A is unital. We prove that an additive map $T : A \to B$ strongly preserves Drazin (or equivalently group) invertibility, if and only if T is a Jordan triple homomorphism. When A and B are C*-algebras, we characterize the linear maps strongly preserving generalized invertibility (in the Jordan systems' sense), and as consequence we determine the structure of selfadjoint linear maps strongly preserving Moore-Penrose invertibility.

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