AN INTRODUCTION TO THE "DIWORLD"

JUANA SÁNCHEZ ORTEGA

Abstract. We present a generalization of associative algebras, called associative dialgebras. We give a simplified statement of the KP algorithm introduced by Kolesnikov and Pozhidaev for extending polynomial identities for algebras to corresponding identities for dialgebras. Applications to the KP algorithm are given.

Introduction

Dialgebras were introduced by Loday [10] to provide a natural setting for Leibniz algebras, a "noncommutative" version of Lie algebras. To be more precise, a **(right) Leibniz algebra** (see [6, 9] for details) is a vector space L, together with a bilinear map $L \times L \to L$, denoted $(a, b) \mapsto \langle a, b\rangle$, satisfying the **(right) Leibniz identity**, which says that right multiplications are derivations:

$$\langle\langle a, b\rangle, c\rangle \equiv \langle\langle a, c\rangle, b\rangle + \langle a, \langle b, c\rangle\rangle.$$

If $\langle a, a\rangle \equiv 0$ then the Leibniz identity is the Jacobi identity and L is a Lie algebra.

It is well known that every associative algebra becomes a Lie algebra if the product ab is replaced by the Lie bracket $ab - ba$. Loday's goal was to introduce a new structure which gives a Leibniz algebra by a similar procedure. His idea was to replace the product ab and its opposite ba by two distinct operations $a \dashv b$ and $b \vdash a$. As this way, the Leibniz bracket $a \dashv b - b \vdash a$ is not necessarily skew-symmetric, and we obtain the notion of an associative dialgebra.

An **associative dialgebra** is a vector space A with two bilinear maps $A \times A \to A$, denoted $\dashv$ and $\vdash$ and called the **left** and **right** products, satisfying the **left and right bar identities**, and **left, right and inner associativity**:

$$(a \dashv b) \vdash c \equiv (a \vdash b) \vdash c, \quad a \dashv (b \dashv c) \equiv a \dashv (b \vdash c),$$
$$(a \dashv b) \dashv c \equiv a \dashv (b \dashv c), \quad (a \vdash b) \vdash c \equiv a \vdash (b \vdash c), \quad (a \vdash b) \dashv c \equiv a \vdash (b \dashv c).$$

The Leibniz bracket in an associative dialgebra satisfies the Leibniz identity.

Dialgebras have become an active research area, attracting the attention of numerous authors who have considered other varieties of nonassociative dialgebras, which have been studied by Velásquez and Felipe [14, 15], Gubarev and Kolesnikov [8, 7], Pozhidaev [12, 13], Voronin [16], and Bremner, Felipe, Peresi and J.S.O [1, 2, 4, 3, 5], among many others.

The purpose of the talk is to present the distinct varieties of nonassociative dialgebras, and their related systems (here by a system, we will understand a pair of a triple).

Section 1 recalls basic definitions for free dialgebras. Section 2 presents a simplified statement of the general Kolesnikov-Pozhidaev (KP) algorithm for converting an arbitrary

Date: 20 February, 2012.

variety of multioperator algebras into a variety of dialgebras. Sections 3 and 4 are devoted to the application of the KP algorithm to the Jordan and Lie setting, respectively.

1. Free dialgebras

Loday has determined a basis for the free dialgebra.

Definition 1.1. A **dialgebra monomial** on a set X is a product $x = a_1 a_2 \cdots a_n$ where $a_1, \dots, a_n \in X$ with some placement of parentheses and some choice of operations. The **center** of x is defined inductively: if $n = 1$ then $c(x) = x$; if $n \geq 2$ then $x = y \dashv z$ or $x = y \vdash z$ and we set $c(y \dashv z) = c(y)$ or $c(y \vdash z) = c(z)$.

Lemma 1.2. (Loday [11]) *If $x = a_1 a_2 \cdots a_n$ is a monomial with $c(x) = a_i$ then x is determined by the order of its factors and the position of its center:*

$$x = (a_1 \vdash \cdots \vdash a_{i-1}) \vdash a_i \dashv (a_{i+1} \dashv \cdots \dashv a_n).$$

Definition 1.3. The right side of the last equation is the **normal form** of x and is abbreviated by the **hat notation** $a_1 \cdots a_{i-1} \widehat{a}_i a_{i+1} \cdots a_n$.

Lemma 1.4. (Loday [11]) *The set of monomials $a_1 \cdots a_{i-1} \widehat{a}_i a_{i+1} \cdots a_n$ in normal form with $1 \leq i \leq n$ and $a_1, \dots, a_n \in X$ forms a basis of the free dialgebra on X.*

2. The Kolesnikov-Pozhidaev algorithm

This algorithm, introduced by Kolesnikov [8] and Pozhidaev [13], converts a multilinear polynomial identity of degree d for an n-ary operation into d multilinear identities of degree d for n new n-ary operations.

Definition 2.1. KP Algorithm.

Part 1: We consider a multilinear n-ary operation, denoted by the symbol

$$\{-, -, \dots, -\} \qquad (n \text{ arguments}). \tag{1}$$

Given a multilinear polynomial identity of degree d in this operation, we describe the application of the algorithm to one monomial in the identity, and from this the application to the complete identity follows by linearity. Let $\overline{a_1 a_2 \dots a_d}$ be a multilinear monomial of degree d, where the bar denotes some placement of n-ary operation symbols. We introduce n new n-ary operations, denoted by the same symbol but distinguished by subscripts:

$$\{-, -, \dots, -\}_1, \quad \{-, -, \dots, -\}_2, \quad \dots, \quad \{-, -, \dots, -\}_n. \tag{2}$$

For each $i \in \{1, 2, \dots, d\}$ we convert the monomial $\overline{a_1 a_2 \dots a_d}$ in the original n-ary operation (1) into a new monomial of the same degree d in the n new n-ary operations (2), according to the following rule which is based on the position of a_i. For each occurrence of the original operation symbol in the monomial, either a_i occurs within one of the n arguments or not, and we have the following cases:

- If a_i occurs within the j-th argument then we convert the original operation symbol $\{\dots\}$ to the j-th new operation symbol $\{\dots\}_j$.
- If a_i does not occur within any of the n arguments, then either
  - a_i occurs to the left of the original operation symbol, in which case we convert $\{\dots\}$ to the first new operation symbol $\{\dots\}_1$, or

– a_i occurs to the right of the original operation symbol, in which case we convert $\{\dots\}$ to the last new operation symbol $\{\dots\}_n$.

In this process, we call a_i the central argument of the monomial.

Part 2: In addition to the identities constructed in Part 1, we also include the following identities for all $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$ and all $k, \ell \in \{1, 2, \dots, n\}$:

$$\{a_1, \dots, a_{i-1}, \{b_1, \cdots, b_n\}_k, a_{i+1}, \dots, a_n\}_j \equiv$$
$$\{a_1, \dots, a_{i-1}, \{b_1, \cdots, b_n\}_\ell, a_{i+1}, \dots, a_n\}_j.$$

This identity says that the n new operations are interchangeable in the i-th argument of the j-th new operation when $i \neq j$.

Example 2.2. The defining identities for associative dialgebras can be obtained by applying the KP algorithm to the associativity identity, which we write in the form $\{\{a, b\}, c\} \equiv \{a, \{b, c\}\}$. The original operation produces two new operations $\{-, -\}_1$ and $\{-, -\}_2$. Since associativity has degree 3, Part 1 produces three new identities of degree 3 by making a, b, c in turn the central argument:

$$\{\{a, b\}_1, c\}_1 \equiv \{a, \{b, c\}_1\}_1, \ \{\{a, b\}_2, c\}_1 \equiv \{a, \{b, c\}_1\}_2, \ \{\{a, b\}_2, c\}_2 \equiv \{a, \{b, c\}_2\}_2,$$

and Part 2 produces these two identities:

$$\{a, \{b, c\}_1\}_1 \equiv \{a, \{b, c\}_2\}_1, \quad \{\{a, b\}_1, c\}_2 \equiv \{\{a, b\}_2, c\}_2.$$

If we revert to the standard notation by writing $a \dashv b = \{a, b\}_1$ and $a \vdash b = \{a, b\}_2$, then these five identities are the defining identities for associative dialgebras.

3. Jordan dialgebras and Jordan triple disystems

We apply the KP algorithm to the defining identities for Jordan algebras to obtain the variety of **Jordan dialgebras**. The most recent results related to Jordan dialgebras will be stablished.

4. Leibniz triple systems

The KP algorithm will be applied to Lie triple systems to get a new variety of triple systems; we call these structures **Leibniz triple systems**. To conclude, we verify that Leibniz triple systems are the natural analogues of Lie triple systems in the context of dialgebras.

References

[1] M. R. Bremner: On the definition of quasi-Jordan algebra. *Communications in Algebra* 38 (2010) 4695–4704.

[2] M. R. Bremner, L. A. Peresi: Special identities for quasi-Jordan algebras. *Communications in Algebra* 39:7 (2011) 2313-2337.

[3] M. R. Bremner, R. Felipe and J. Sánchez-Ortega: Jordan triple disystems. *Computers and Mathematics with Applications* (to appear).

[4] M. R. Bremner, L. A. Peresi, and J. Sánchez-Ortega: Malcev dialgebras. *Linear and Multilinear Algebra* (to appear).

[5] M. R. Bremner and J. Sánchez-Ortega: Leibniz triple systems (Preprint)

[6] C. Cuvier: Algèbres de Leibnitz: définitions, propriétés. *Annales Scientifiques de l'École Normale Supérieure* (4) 27, 1 (1994) 1–45.

[7] V. Y. Gubarev, P. S. Kolesnikov: The Tits-Kantor-Koecher construction for Jordan dialgebras. *Communications in Algebra* 39 (2011) 497–520.
[8] P. S. Kolesnikov: Varieties of dialgebras and conformal algebras. *Sibirskiĭ Matematicheskiĭ Zhurnal* 49 (2008) 322–339; translation in *Siberian Mathematical Journal* 49 (2008) 257–272.
[9] J.-L. Loday: Une version non commutative des algèbres de Lie: les algèbres de Leibniz. *L'Enseignement Mathématique* 39, 3-4 (1993) 269–293.
[10] J.-L. Loday: Algèbres ayant deux opérations associatives: les digèbres. *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique* 321, 2 (1995) 141–146.
[11] J. L. Loday: Dialgebras. *Dialgebras and Related Operads*, pages 7–66. Lectures Notes in Mathematics, 1763. Springer-Verlag, 2001.
[12] A. P. Pozhidaev: Dialgebras and related triple systems. *Sibirskiĭ Matematicheskiĭ Zhurnal* 49 (2008) 870–885; translation in *Siberian Mathematical Journal* 49 (2008) 696Ð708.
[13] A. P. Pozhidaev: 0-dialgebras with bar-unity, Rota-Baxter and 3-Leibniz algebras. *Contemporary Mathematics* 499, 245–256. American Mathematical Society, 2009.
[14] R. Velásquez, R. Felipe: Quasi-Jordan algebras. *Communications in Algebra* 36 (2008) 1580–1602.
[15] R. Velásquez, R. Felipe: Split dialgebras, split quasi-Jordan algebras and regular elements. *Journal of Algebra and its Applications* 8 (2009) 191–218.
[16] V. Voronin: Special and exceptional Jordan dialgebras. `arXiv:1011.3683v1 [math.RA]`

Dpto. de Matemática Aplicada, E. T. S. I. de Telecomunicación, Campus de Teatinos, Universidad de Málaga

E-mail address: `jsanchezo@uma.es`