Weight modules over split Lie algebras

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Abstract

We study the structure of arbitrary weight modules V, (with no restrictions neither on the dimension nor on the base field), over split Lie algebras L. We show that if L is perfect and V satisfies $LV = V$ and $\mathcal{Z}(V) = 0$, then

$$L = \bigoplus_{i \in I} I_i \text{ and } V = \bigoplus_{j \in J} V_j$$

with any I_i an ideal of L satisfying $[I_i, I_k] = 0$ if $i \neq k$, and any V_j a (weight) submodule of V in such a way that for any $j \in J$ there exists a unique $i \in I$ such that $I_i V_j \neq 0$, being V_j a weight module over I_i. Under certain conditions, it is shown that the above decomposition of V is by means of the family of its minimal submodules, each one being a simple (weight) submodule.

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1 Introduction and previous definitions

Throughout this paper, weight modules V and split Lie algebras L are considered of arbitrary dimensions and over an arbitrary base field $\mathbb{K}$. It is worth to mention that, unless otherwise stated, there is not any restriction on $\dim V_\gamma$, $\dim L_\alpha$ or the products $L_\alpha V_\gamma$ where V_γ denotes the weight space associated to the weight γ of V and L_α the root space associated to the root α of L.

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Given an element x of a Lie algebra L, we denote by ad_x the adjoint mapping ad_x defined as $\mathrm{ad}_x(y) := [x, y]$ for any $y \in L$. A *splitting Cartan subalgebra* H of L is defined as a maximal abelian subalgebra of L, satisfying that the adjoint mappings ad_h, for $h \in H$, are simultaneously diagonalizable. If L contains a splitting Cartan subalgebra H, then L is called a *split Lie algebra*, (see for instance [5]). This means that we have a root spaces decomposition

$$L = H \oplus (\bigoplus_{\alpha \in \Lambda} L_\alpha)$$

where $L_\alpha = \{v_\alpha \in L : [v_\alpha, h] = \alpha(h)v_\alpha \text{ for any } h \in H\}$ for a linear functional $\alpha \in H^*$ and $\Lambda := \{\alpha \in H^* \backslash \{0\} : L_\alpha \neq 0\}$. The subspaces L_α for $\alpha \in H^*$ are called *root spaces* of L, (respect to H), and the elements $\alpha \in \Lambda \cup \{0\}$ are called *roots* of L, (respect to H). Clearly $L_0 = H$ and, as consequence of Jacobi identity, $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$ for any $\alpha, \beta \in \Lambda \cup \{0\}$. We also say that Λ is *symmetric* if for any $\alpha \in \Lambda$ we have that $-\alpha \in \Lambda$.

Definition 1.1. *Let V be a module over a Lie algebra L with splitting Cartan subalgebra H. For a linear functional $\gamma : H \longrightarrow \mathbb{K}$, the* weight space *of V, (respect to H), associated to γ is the subspace*

$$V_\gamma = \{v_\gamma \in V : hv_\gamma = \gamma(h)v_\gamma \ \ for \ \ any \ \ h \in H\}.$$

The elements $\gamma \in H^*$ satisfying $V_\gamma \neq 0$ are called weights *of V respect to H and we denote $\mathcal{P} := \{\gamma \in H^* \backslash \{0\} : V_\gamma \neq 0\}$. We say that V is a* weight module*, respect to H, if*

$$V = V_0 \oplus (\bigoplus_{\gamma \in \mathcal{P}} V_\gamma).$$

We also say that $\mathcal{P}$ is the weight system *of V.*

The weight system $\mathcal{P}$ is called *symmetric* if for any $\gamma \in \mathcal{P}$ we have that $-\gamma \in \mathcal{P}$.

Split Lie algebras are examples of weight modules over themselves, where $\mathcal{P} = \Lambda$ and $V_\gamma = L_\gamma$ for $\gamma \in \mathcal{P} \cup \{0\}$. Since the even part L^0 of the standard embedding of a split Lie triple system T and of a split twisted inner derivation triple system M is a split Lie algebra, the natural actions of L^0 over T and M make of T and M weight modules over the split Lie algebra L^0. So the present paper extend the results in [1, 2, 3].

2 Connections of weights. Decompositions of V

In the following, $V = V_0 \oplus (\bigoplus_{\gamma \in \mathcal{P}} V_\gamma)$ denotes a weight module with a symmetric weight system $\mathcal{P}$, respect to a split Lie algebra $L = H \oplus (\bigoplus_{\alpha \in \Lambda} L_\alpha)$ with a symmetric root system Λ. We begin by developing connections of weights techniques in this framework.

Definition 2.1. *Let γ and δ be two nonzero weights. We say that γ is* connected *to δ if there exist $\alpha_1, ..., \alpha_n \in \Lambda$ such that*

1. $\{\gamma + \alpha_1, \gamma + \alpha_1 + \alpha_2, ..., \gamma + \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}\} \subset \mathcal{P}$,
2. $\gamma + \alpha_1 + \alpha_2 + \cdots + \alpha_n \in \{\delta, -\delta\}$,

where the sums are considered in H^*.

We also say that $\{\gamma, \alpha_1, ..., \alpha_n\}$ is a connection *from γ to δ.*

Proposition 2.1. *The relation $\sim$ in $\mathcal{P}$ defined by $\gamma \sim \delta$ if and only if γ is connected to δ is of equivalence.*

Given $\gamma \in \mathcal{P}$, we denote by

$$\mathcal{P}_\gamma := \{\delta \in \mathcal{P} : \delta \sim \gamma\}.$$

Clearly if $\delta \in \mathcal{P}_\gamma$ then $-\delta \in \mathcal{P}_\gamma$ and, by Proposition 2.1, if $\eta \notin \mathcal{P}_\gamma$ then $\mathcal{P}_\gamma \cap \mathcal{P}_\eta = \emptyset$.

Our next goal is to associate an (adequate) weight submodule $V_{\mathcal{P}_\gamma}$ to any $\mathcal{P}_\gamma$. For $\mathcal{P}_\gamma, \gamma \in \mathcal{P}$, we define the following linear subspace of V:

$$V_{\mathcal{P}_\gamma} := (\sum_{\alpha \in \Lambda \cap \mathcal{P}_\gamma} L_{-\alpha} V_\alpha) \oplus (\bigoplus_{\delta \in \mathcal{P}_\gamma} V_\delta).$$

We also denote by $V_{0,\mathcal{P}_\gamma} := \sum_{\alpha \in \Lambda \cap \mathcal{P}_\gamma} L_{-\alpha} V_\alpha \subset V_0$.

Lemma 2.1. *The following assertions hold:*

1. *For any $\alpha \in \Lambda$ and $\gamma \in \mathcal{P}$ with $\alpha \neq -\gamma$, if $L_\alpha V_\gamma \neq 0$ then $\alpha + \gamma \sim \gamma$.*
2. *For any $\alpha, \beta \in \Lambda \cap \mathcal{P}$, if $L_\beta(L_{-\alpha} V_\alpha) \neq 0$ then $\alpha \sim \beta$.*

We recall that a Lie module V is said to be *simple* if its only submodules are $\{0\}$ and V.

Theorem 2.1. *Let $\gamma \in \mathcal{P}$. Then the following assertions hold.*

1. *$V_{\mathcal{P}_\gamma}$ is a weight submodule of V.*
2. *If V is simple, then there exists a connection from γ to δ for any $\gamma, \delta \in \mathcal{P}$ and $V_0 = \sum_{\alpha \in \Lambda \cap \mathcal{P}_\gamma} L_{-\alpha} V_\alpha$.*

Theorem 2.1-1 let us assert that for any $\gamma \in \mathcal{P}$, $V_{\mathcal{P}_\gamma}$ is a weight submodule of V that we call the submodule of V *associated* to $\mathcal{P}_\gamma$.

Proposition 2.2. *For a linear complement $\mathcal{U}$ of $span_{\mathbb{K}}\{L_{-\alpha} V_\alpha : \alpha \in \Lambda \cap \mathcal{P}\}$ in V_0, we have*

$$V = \mathcal{U} + (\sum_{[\gamma] \in \mathcal{P}/\sim} V_{[\gamma]}),$$

where any $V_{[\gamma]}$ is one of the weight submodules described in Theorem 2.1-1.

Recall that the *center* of V is defined as the set $\mathcal{Z}(V) = \{v \in V : Lv = 0\}$.

Theorem 2.2. *If $LV = V$ and $\mathcal{Z}(V) = 0$, then V is the direct sum of the ideals given in Proposition 2.2,*

$$V = \bigoplus_{[\gamma]\in\mathcal{P}/\sim} V_{[\gamma]}.$$

3 Connections of roots. Decompositions of L

We begin this section by introducing a concept of connections of roots for L in a slightly different way to the one of connection of weights for $\mathcal{P}$ developed in the previous section. To do that, we will connect the nonzero roots of L through nonzero roots of L and nonzero weights of $\mathcal{P}$ considered both as elements in H^*.

Definition 3.1. *Let α, β be two nonzero roots of L. We say that α is* connected *to β if there exist $\zeta_1, ..., \zeta_n \in \Lambda \cup \mathcal{P}$ such that*

1. $\alpha = \zeta_1$,
1. $\{\zeta_1, \zeta_1 + \zeta_2, ..., \zeta_1 + \cdots + \zeta_{n-1}\} \subset \Lambda \cup \mathcal{P}$,
2. $\zeta_1 + \cdots + \zeta_n \in \{\beta, -\beta\}$,

where the sums are considered in H^*.

We also say that $\{\zeta_1, ..., \zeta_n\}$ is a connection *from α to β.*

Proposition 3.1. *The relation $\approx$ in Λ defined by $\alpha \approx \beta$ if and only if α is connected to β is of equivalence.*

Given $\alpha \in \Lambda$, we denote by

$$\Lambda_\alpha := \{\beta \in \Lambda : \beta \approx \alpha\}.$$

We also have that if $\beta \in \Lambda_\alpha$ then $-\beta \in \Lambda_\alpha$ and, by Proposition 3.1, if $\mu \notin \Lambda_\alpha$ then $\Lambda_\alpha \cap \Lambda_\mu = \emptyset$.

Our next aim is to associate an adequate ideal of L to any Λ_α. For $\Lambda_\alpha, \alpha \in \Lambda$, we define

$$H_{\Lambda_\alpha} := span_{\mathbb{K}}\{[L_\beta, L_{-\beta}] : \beta \in \Lambda_\alpha\} \subset H,$$

and

$$V_{\Lambda_\alpha} := \bigoplus_{\beta\in\Lambda_\alpha} L_\beta.$$

We denote by L_{Λ_α} the following linear subspace of L,

$$L_{\Lambda_\alpha} := H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}.$$

Proposition 3.2. *Let $\alpha \in \Lambda$. Then the following assertions hold.*

1. $[L_{\Lambda_\alpha}, L_{\Lambda_\alpha}] \subset L_{\Lambda_\alpha}$.

2. *If* $\mu \notin \Lambda_\alpha$ *then* $[L_{\Lambda_\alpha}, L_{\Lambda_\mu}] = 0$.

By Proposition 3.2-1 we can assert that for any $\alpha \in \Lambda$, L_{Λ_α} is a subalgebra of L that we call the subalgebra of L *associated* to Λ_α.

Theorem 3.1. *The following assertions hold.*

1. *For any* $\alpha \in \Lambda$, *the subalgebra*

$$L_{\Lambda_\alpha} = H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}$$

of L *associated to* Λ_α *is an ideal of* L.

2. *If* L *is simple, then there exists a connection from* α *to* β *for any* $\alpha, \beta \in \Lambda$ *and* $H = \sum_{\alpha \in \Lambda} [L_\alpha, L_{-\alpha}]$.

Proposition 3.3. *For a linear complement* $\mathcal{U}$ *of* $span_{\mathbb{K}}\{[L_\alpha, L_{-\alpha}] : \alpha \in \Lambda\}$ *in* H, *we have*

$$L = \mathcal{U} + \sum_{[\alpha] \in \Lambda/\approx} I_{[\alpha]},$$

where any $I_{[\alpha]}$ *is one of the ideals* L_{Λ_α} *of* L *described in Theorem 3.1-1, satisfying* $[I_{[\alpha]}, I_{[\beta]}] = 0$ *if* $[\alpha] \neq [\beta]$.

Let us denote by $\mathcal{Z}(L) = \{e \in L : [e, L] = 0\}$ the *center* of L.

Theorem 3.2. *If* $\mathcal{Z}(L) = 0$ *and* $H = \sum_{\alpha \in \Lambda} [L_\alpha, L_{-\alpha}]$, *then* L *is the direct sum of the ideals given in Theorem 3.1,*

$$L = \bigoplus_{[\alpha] \in \Lambda/\approx} I_{[\alpha]}.$$

4 Relating the decompositions of V and L

Theorem 4.1. *Let* V *be weight module respect to a perfect split Lie algebra* L *such that* $LV = V$ *and* $\mathcal{Z}(V) = 0$. *Then*

$$L = \bigoplus_{i \in I} I_i \text{ and } V = \bigoplus_{j \in J} V_j$$

with any I_i *a nonzero ideal of* L *satisfying* $[I_i, I_k] = 0$ *if* $i \neq k$, *and any* V_j *a nonzero weight submodule of* V *in such a way that for any* $j \in J$ *there exists a unique* $i \in I$ *such that*

$$I_i V_j \neq 0.$$

Furthermore V_j *is a weight module over* I_i.

5 The simple components

In this section we are showing that, under certain conditions, the decomposition of V given in Theorem 4.1 can be given by means of the family of its minimal submodules, each one being a simple (weight) submodule.

Lemma 5.1. *Suppose $\mathcal{Z}(V) = 0$. If W is a submodule of V such that $W \subset V_0$, then $W = \{0\}$.*

Let us introduce the concepts of weight-multiplicativity and maximal length in the framework of weight modules over spit Lie algebras, in a similar way to the analogous ones for split Lie algebras, split Lie triple systems and split twisted inner derivation triple systems, (see [1, 2, 3] for these notions and examples).

Definition 5.1. *We say that a weight module V is of* maximal length *if* $\dim V_\gamma = 1$ *for any $\gamma \in \mathcal{P}$.*

Let us note that weight modules of maximal length appears in a natural way in several contexts. See for instance [4], [6] and [7] for the cases over Virasoro, generalized Virasolo and Witt algebras respectively.

Given any submodule W of V, it is well known that any submodule of a weight module is again a weight module. So we have

$$W = (W \cap V_0) \oplus (\bigoplus_{\gamma \in \mathcal{P}} (W \cap V_\gamma)).$$

Observe that if V is of maximal length then we can write

$$W = (W \cap V_0) \oplus (\bigoplus_{\gamma \in \mathcal{P}^W} V_\gamma),$$

where

$$\mathcal{P}^W := \{\gamma \in \mathcal{P} : W \cap V_\gamma \neq 0\}.$$

Definition 5.2. *We say that a weight module V over a split Lie algebra L is* weight-multiplicative *if given $\alpha \in \Lambda$ and $\gamma \in \mathcal{P}$ such that $\alpha + \gamma \in \mathcal{P}$, then $L_\alpha V_\gamma \neq 0$.*

Here we note that if V satisfies $V_0 = \sum_{\beta \in \Lambda \cap \mathcal{P}} L_{-\beta} V_\beta$ we will understand the weight-multiplicativity of V by supposing also that if $L_\beta(L_{-\beta}V_\beta) \neq 0$ then $L_{-\beta}(L_\beta V_{-\beta}) \neq 0$.

Theorem 5.1. *Let V be a weight module of maximal length, weight-multiplicative and with $\mathcal{Z}(V) = 0$ over a split Lie algebra L. If V has all its nonzero weights connected and $V_0 = \sum_{\alpha \in \Lambda \cap \mathcal{P}} L_{-\alpha} V_\alpha$ then either V is simple or $V = W \oplus W'$ with W and W' simple (weight) submodules of V.*

Theorem 5.2. *Let V be a weight module of maximal length, weight-multiplicative and with $LV = V$, $\mathcal{Z}(V) = 0$ over a split Lie algebra L. Then $L = \bigoplus_{i \in I} I_i$ with any I_i a nonzero ideal of L satisfying $[I_i, I_j] = 0$ if $i \neq j$, and $V = \bigoplus_{k \in K} V_k$ is the direct sum of the family of its minimal submodules, each one being a simple weight submodule of V in such a way that for any $k \in K$ there exists a unique $i \in I$ such that $I_i V_k \neq 0$. Furthermore V_k is a weight module over I_i.*

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