DIFFERENTIAL RESULTANT FORMULAS FOR LINEAR OD-POLYNOMIALS

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Abstract. The sparse differential resultant $\partial\text{Res}(\mathfrak{P})$ of an overdetermined system $\mathfrak{P}$ of generic nonhomogeneous ordinary differential polynomials, was formally defined recently by Li, Gao and Yuan (2011). In this note, a differential resultant formula $\partial\text{FRes}(\mathfrak{P})$ is defined for linear systems. Under some conditions on the supports of the differential operators determining $\mathfrak{P}$, the formula is proved to be always nonzero and, up to a constant, equal to $\partial\text{Res}(\mathfrak{P})$.

Introduction

The implicitization problem of unirational algebraic varieties has been widely studied, and the results on the computation of the implicit equation of a system of algebraic rational parametric equations by algebraic resultants are well known. The generalization of these results to the differential (and more generally noncommutative) case is a field of research at an initial stage of development, where many interesting problems arise.

Let $\mathfrak{P}$ be an system of n generic sparse nonhomogeneous ordinary differential polynomials in $n-1$ differential variables. It would be useful to represent the sparse differential resultant $\partial\text{Res}(\mathfrak{P})$, defined in [4], as the quotient of two determinants, as done for the algebraic case in [3]. In the differential case, so called Macaulay style formulas do not exist, even in the simplest situation. The matrices used in the algebraic case to define Macaulay style formulas [3], are coefficient matrices of sets of polynomials obtained by multiplying the original ones by appropriate sets of monomials, [1]. In the differential case, in addition, derivatives of the original polynomials should be considered. The differential resultant formula defined by Carrà-Ferro in [2], is the algebraic resultant of Macaulay [5], of a set of derivatives of the ordinary differential polynomials in $\mathfrak{P}$. Already for linear differential polynomials these formulas vanish often, giving no information about the differential resultant $\partial\text{Res}(\mathfrak{P})$. The linear case can be seen as a previous stage to get ready to approach the nonlinear case, considering only the problem of taking the appropriate set of derivatives of the elements in $\mathfrak{P}$ for the moment.

Let us assume that $\mathfrak{P}$ is a system of linear differential polynomials. In [8], the linear complete differential resultant $\partial\text{CRes}(\mathfrak{P})$ was defined, as an improvement, in the linear case (non necessarily generic), of the differential resultant formula given by Carrà-Ferro. Still, $\partial\text{CRes}(\mathfrak{P})$ is the determinant of a matrix having zero columns in many cases. The linear differential polynomials in $\mathfrak{P}$ can be described via differential operators. We use appropriate bounds of the supports of those differential operators to decide on a convenient set $\text{ps}(\mathfrak{P})$ of derivatives of $\mathfrak{P}$, such that its coefficient matrix $\mathcal{M}(\mathfrak{P})$ is squared and has no zero columns.

Furthermore, we can guarantee that the linear sparse differential resultant $\partial\text{Res}(\mathfrak{P})$ can always be computed (up to a constant) as the determinant of a matrix $\mathcal{M}(\mathfrak{P}^*)$, for a super essential subsystem $\mathfrak{P}^*$ of $\mathfrak{P}$, as defined in Section 2. A key fact is that not every polynomial in $\mathfrak{P}$ is involved in the computation of $\partial\text{Res}(\mathfrak{P})$, only those in a super essential subsystem $\mathfrak{P}^*$ of $\mathfrak{P}$ are, and $\mathfrak{P}^*$ is proved to exist in all cases. An extended version of the results presented can be found in [7].

1. Sparse linear differential resultant

Let us suppose that the field $\mathbb{Q}$ of rational numbers is a field of constants of a derivation ∂. Let us consider the set $U = \{u_1, \ldots, u_{n-1}\}$ of differential indeterminates over $\mathbb{Q}$. By $\mathbb{N}_0$ we mean the natural numbers including 0. For $k \in \mathbb{N}_0$, we denote by $u_{j,k}$ the k-th derivative of u_j and for $u_{j,0}$ we simply write u_j. We denote by $\{U\}$ the set of derivatives of the elements of U.

For $i = 1, \ldots, n$ and $j = 1, \ldots, n-1$, let us consider subsets $\mathfrak{S}_{i,j}$ of $\mathbb{Z}$ to be the supports of generic differential operators

$$\mathcal{G}_{i,j} := \begin{cases} \sum_{k\in\mathfrak{S}_{i,j}} c_{i,j,k}\partial^k & , \mathfrak{S}_{i,j} \neq \emptyset, \\ 0 & , \mathfrak{S}_{i,j} = \emptyset. \end{cases}$$

Let us consider the set of differential indeterminates over $\mathbb{Q}$

$$C = \{c_1, \ldots, c_n\} \text{ and } \overline{C} := \cup_{i=1}^{n} \cup_{j=1}^{n-1} \{c_{i,j,k} \mid k \in \mathfrak{S}_{i,j}\}.$$

Let $\mathcal{K} = \mathbb{Q}\langle\overline{C}\rangle$, a differential field extension of $\mathbb{Q}$, and $\mathbb{D} = \mathcal{K}\{C\}$, a differential domain. Consider the set $\mathfrak{P} = \{\mathbb{F}_1, \ldots, \mathbb{F}_n\}$ of generic sparse linear differential polynomials in $\mathbb{D}\{U\}$ as follows

$$\mathbb{F}_i := c_i - \sum_{j=1}^{n-1} \mathcal{G}_{i,j}(u_j) = c_i - \sum_{j=1}^{n-1} \sum_{k\in\mathfrak{S}_{i,j}} c_{i,j,k}u_{j,k}, i = 1, \ldots, n.$$

Let $x_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, n-1$ be algebraic indeterminates over $\mathbb{Q}$. Let $X(\mathfrak{P}) = (X_{i,j})$ be the $n \times (n-1)$ matrix, such that

$$X_{i,j} := \begin{cases} x_{i,j} & , \mathcal{G}_{i,j} \neq 0, \\ 0 & , \mathcal{G}_{i,j} = 0. \end{cases}$$

The system $\mathfrak{P}$ is said to be differentially essential if $\text{rank}(X(\mathfrak{P})) = n - 1$.

Let $[\mathfrak{P}]$ be the differential ideal generated by $\mathfrak{P}$ in $\mathbb{D}\{U\}$. By [4], Corollary 3.4, the dimension of the elimination ideal

$$\text{ID}(\mathfrak{P}) = [\mathfrak{P}] \cap \mathbb{D}$$

is $n - 1$ if and only if $\mathfrak{P}$ is a differentially essential system. In such case, $\text{ID}(\mathfrak{P}) = \text{sat}(R)$, the saturation ideal of a differential polynomial R in $\mathbb{D}$, we can assume that $R \in \mathbb{Q}\{\overline{C}, C\}$ is irreducible. By [4], Definition 3.5, R is the sparse differential resultant of $\mathfrak{P}$, we will denote it by $\partial\text{Res}(\mathfrak{P})$.

2. Sparse differential resultant formula for supper essential systems

Let us assume that the order of $\mathbb{F}_i$ is $o_i \geq 0$, $i = 1, \ldots, n$. We define positive integers, to construct convenient intervals bounding the supports of the differential operators $\mathcal{G}_{i,j}$. Let $\mathrm{ldeg}(\mathcal{G}_{i,j}) := \min \mathfrak{S}_{i,j}$ and $\deg(\mathcal{G}_{i,j}) := \max \mathfrak{S}_{i,j}$. For $j = 1, \ldots, n-1$,

$$\overline{\gamma}_j(\mathfrak{P}) := \min\{o_i - \deg(\mathcal{G}_{i,j}) \mid \mathcal{G}_{i,j} \neq 0,\ i = 1, \ldots, n\},$$
$$\underline{\gamma}_j(\mathfrak{P}) := \min\{\mathrm{ldeg}(\mathcal{G}_{i,j}) \mid \mathcal{G}_{i,j} \neq 0,\ i = 1, \ldots, n\},$$

$$\gamma_j(\mathfrak{P}) := \underline{\gamma}_j(\mathfrak{P}) + \overline{\gamma}_j(\mathfrak{P}).$$

Therefore, for $\mathcal{G}_{i,j} \neq 0$ the next set of lattice points contains $\mathfrak{S}_{i,j}$,

$$I_{i,j}(\mathfrak{P}) := [\underline{\gamma}_j(\mathfrak{P}), o_i - \overline{\gamma}_j(\mathfrak{P})] \cap \mathbb{Z}.$$

Finally,

$$\gamma(\mathfrak{P}) := \sum_{j=1}^{n-1} \gamma_j(\mathfrak{P}).$$

We denote by $X_i(\mathfrak{P})$, $i = 1, \ldots, n$, the submatrix of $X(\mathfrak{P})$ obtained by removing its ith row. The system $\mathfrak{P}$ is said to be supper essential if $\det(X_i(\mathfrak{P})) \neq 0$, $i = 1, \ldots, n$. Given $N := \sum_{i=1}^{n} o_i$, let

$$L_i := N - o_i - \gamma(\mathfrak{P}), i = 1, \ldots, n.$$

If $\mathfrak{P}$ is super essential then $L_i \geq 0$, $i = 1, \ldots, n$ and we can construct the set

$$\mathrm{ps}(\mathfrak{P}) := \{\partial^k \mathbb{F}_i \mid k \in [0, L_i] \cap \mathbb{Z},\ i = 1, \ldots, n\},$$

containing $L := \sum_{i=1}^{n}(L_i + 1)$ differential polynomials, in the set $\mathcal{V}$ of $L - 1$ differential indeterminates

$$\mathcal{V} := \{u_{j,k} \mid k \in [\underline{\gamma}_j(\mathfrak{P}), N - \overline{\gamma}_j(\mathfrak{P}) - \gamma(\mathfrak{P})] \cap \mathbb{Z},\ j = 1, \ldots, n-1\}.$$

The coefficient matrix $\mathcal{M}(\mathfrak{P})$ of the differential polynomials in $\mathrm{ps}(\mathfrak{P})$ as polynomials in $\mathbb{D}[\mathcal{V}]$ is an $L \times L$ matrix. We define a linear differential resultant formula for $\mathfrak{P}$, denoted by $\partial\mathrm{FRes}(\mathfrak{P})$, and equal to:

$$\partial\mathrm{FRes}(\mathfrak{P}) := \det(\mathcal{M}(\mathfrak{P})).$$

3. Main results

The implicitization of linear DPPEs (differential polynomial parametric equations) by differential resultant formulas was studied in [8] and [6]. In [7], some of the results in [6] are extended and used to obtain the next conclusions.

Given a differentially essential system $\mathfrak{P}$, $\mathrm{ID}(\mathfrak{P}) = [\partial\mathrm{Res}(\mathfrak{P})]_{\mathbb{D}}$, the differential ideal generated by $\partial\mathrm{Res}(\mathfrak{P})$ in $\mathbb{D}$. Furthermore, $R = \partial\mathrm{Res}(\mathfrak{P})$ is a linear differential polynomial verifying:

(1) $R = \sum_{i=1}^{n} \mathcal{L}_i(c_i)$, $\mathcal{L}_i \in \mathcal{K}[\partial]$ and a greatest common left divisor of $\mathcal{L}_1, \ldots, \mathcal{L}_n$ belongs to $\mathcal{K}$, that is R is ID-primitive.

(2) R belongs to $(\mathrm{ps}(\mathfrak{P})) \cap \mathbb{D}$, where $(\mathrm{ps}(\mathfrak{P}))$ is the algebraic ideal generated by $\mathrm{ps}(\mathfrak{P})$ in $\mathbb{D}[\mathcal{V}]$.

(3) The highest positive integer c such that $\partial^c R \in (\mathrm{ps}(\mathfrak{P})$ is

$$c = |\mathrm{ps}(\mathfrak{P})| - 1 - \mathrm{rank}(\mathcal{M}(\mathcal{V})),$$

where $\mathcal{M}(\mathcal{V})$ is the submatrix of $\mathcal{M}(\mathfrak{P})$ whose $L-1$ columns are indexed by the elements in $\mathcal{V}$.

Using these properties we can prove the next result.

Theorem 3.1. *Let $\mathfrak{P}$ be a system of generic sparse linear differential polynomials. If $\mathfrak{P}$ is super essential then $\partial\mathrm{FRes}(\mathfrak{P}) \neq 0$.*

Furthermore, if $\mathfrak{P}$ is not super essential, we can prove the existence of a super essential subsystem $\mathfrak{P}^*$ of $\mathfrak{P}$ and provide a computation method, [7], Section 4. Furthermore, $\mathfrak{P}$ is differentially essential if and only it has a unique super essential subsytem.

Theorem 3.2. *Let us consider a differentially essential system $\mathfrak{P}$, of generic sparse linear differential polynomials, and the super essential subsystem $\mathfrak{P}^*$ of $\mathfrak{P}$. There exists a nonzero constant $\alpha \in \mathcal{K}$ such that $\partial\mathrm{Res}(\mathfrak{P}) = \alpha\partial\mathrm{FRes}(\mathfrak{P}^*)$.*

The previous results, allow us to give a bound of the order of $\partial\mathrm{Res}(\mathfrak{P})$ in the differential indeterminates C. Namely, given $I^* := \{i \mid \mathbb{F}_i \in \mathfrak{P}^*\}$ and $i \in \{1, \ldots, n\}$

$$\mathrm{ord}(\partial\mathrm{Res}(\mathfrak{P}), c_i) = -1 \text{ if } i \notin I^*,$$
$$\mathrm{ord}(\partial\mathrm{Res}(\mathfrak{P}), c_i) = N^* - o_i - \gamma(\mathfrak{P}^*) \text{ if } i \in I^*,$$

with $N^* = \sum_{i \in I^*} o_i$ and equality holds for some $i \in I^*$.

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