A LAZARD-LIKE THEOREM FOR QUASI-COHERENT SHEAVES

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Abstract. We study filtration of quasi-coherent sheaves. We prove a version of Kaplansky Theorem for quasi-coherent sheaves, by using Drinfeld's notion and the Hill Lemma. We also show a Lazard-like theorem for at quasi-coherent sheaves for quasi-compact and semi-separated schemes which satisfy the resolution property.

Introduction

Let $X = \operatorname{Spec} R$ be an affine scheme for a commutative ring R with unity. It is known that the category of all quasi-coherent sheaves on X is equivalent to R-Mod. In this equivalence, finite dimensional vector bundles on X correspond to finitely generated projective modules. Then a (classical) vector bundle on an arbitrary scheme X corresponds to a quasi–coherent sheaf $\mathcal{F}$ such that, for each affine open subset $U = \operatorname{Spec} R$, the corresponding R-module of sections $\Gamma(U, \mathcal{F})$ is finitely generated and free.

[Drinfeld, 2006] asks the following key problem: 'Is there a reasonable notion of not necessarily finite dimensional vector bundles on a scheme?'. In the same paper he proposes several possible answers to this question. Each one of these involves different classes of modules. In this work, we focus on the projective R-modules and flat Mittag-Leffler for a commutative ring R with unity. Then we center on the category of quasi-coherent sheaves over some special projective schemes and the several 'new' notions of (infinite dimensional) vector bundles attained to that class as proposed by Drinfeld. We prove structural results relative to the different generalization of vector bundles in terms of certain filtrations of locally countably generated quasi-coherent sheaves. In the case in which the vector bundles are built from the class of projective R-modules, our structural theorem yields a version of Kaplansky's Theorem for infinite dimensional vector bundles on these special projective schemes.

1. $\mathfrak{Q}co(X)$ As a Category of Representation

The aim of this chapter is to give a new and simpler category that is equivalent to $\mathfrak{Q}co(X)$. So, it allow us to work in $\mathfrak{Q}co(X)$ more easily.

Let Q be a quiver. A *representation* $\mathbf{R}$ of the quiver Q in the category of rings means that for each vertex $v \in V$ we have a ring $R(v)$ and a ring homomorphism

$$\mathbf{R}(a) : R(v) \longrightarrow R(w),$$

for each edge $a : v \to w$. An $\mathbf{R}$-*module* $\mathbf{M}$ is given by an $R(v)$-module $M(v)$, for each vertex $v \in V$, and an $R(v)$-linear morphism. The category of quasi-coherent $\mathbf{R}$-modules for a fixed quiver Q and a fixed representation $\mathbf{R}$ of the quiver Q is defined as the full subcategory of the category $\mathbf{R}$-Mod that contains all quasi-coherent $\mathbf{R}$-modules. We will denote it by $\mathbf{R}_{Qco}$-Mod.

Theorem 1.1. *Let X be a scheme. For a suitable quiver Q, the category of quasi-coherent $\mathbf{R}$-modules for this quiver Q is equivalent to $\mathfrak{Q}co(X)$.*

Proof. Consider the category of quasi-coherent sheaves on a scheme $(X, \mathcal{O}_X)$, denoted by $\mathfrak{Q}co(X)$. By the definition of a scheme, the scheme X has a family $\mathcal{B}$ of affine open subsets which is a base for X such that this family uniquely determines the scheme $(X, \mathcal{O}_X)$. Now, define a quiver Q having the family $\mathcal{B}$ as the set of vertices, and an edge between two affine open subsets $U, V \in \mathcal{B}$ as the only one arrow $U \to V$ provided that $V \subsetneq U$. Fix this quiver. Take the representation $\mathbf{R}$ as $R(U) = \mathcal{O}_X(U)$ for each $U \in \mathcal{B}$ and the restriction map $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ for the edge $U \to V$. Then the functor

$$\Phi : \mathfrak{Q}co(X) \longmapsto \mathbf{R}_{Qco}\text{-Mod},$$

which was defined by above argument, is well-defined. □

2. Filtration in $\mathfrak{Q}co(\mathbb{P}^n_R)$

It is known that there is a bijection between the class of the vector bundles in the sense of classical algebraic geometry and the class of the locally free coherent $\mathcal{O}_X$-modules of finite rank. So, in Sheaf Theory, the definition of vector bundle is taken as locally free coherent $\mathcal{O}_X$-module of finite rank. But in our study, we will drop the conditions finiteness and freeness. This leads to Drinfeld's definition of infinite-dimensional vector bundles.

Definition 2.1. [Drinfeld, 2006, Section 2] Let $(X, \mathcal{O}_X)$ be a scheme. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is said to be a *vector bundle* (in the sense [Drinfeld, 2006]) if $\mathcal{F}(U)$ is a projective $\mathcal{O}_X(U)$-module for every affine open subset U of X.

Definition 2.2. [Drinfeld, 2006, Section 2] Let $(X, \mathcal{O}_X)$ be a scheme. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is said to be a *locally flat Mittag-Leffler* if $\mathcal{F}(U)$ is a flat Mittag-Leffler $\mathcal{O}_X(U)$-module for every affine open subset U of X.

In Drinfeld's paper, it is stated that the notion in 2.2 is a local property. That is, conversely, if $\mathbf{M}$ is an $\mathbf{R}$-module such that $M(u)$ is a projective $R(u)$-module for each vertex u, then there exists a unique vector bundle $\mathcal{M}$ on the scheme X. As in the case of projective modules, the property of the flat Mittag-Leffler module is also local.

Theorem 2.3. *Every vector bundle on $\mathbb{P}^n_R$ is a filtration of locally countably generated vector bundles.*

Theorem 2.4. *Every locally flat Mittag-Lefer quasi-coherent sheaf on X is a direct union of locally countably generated vector bundles.*

Theorem 2.5. *Let X be a scheme having enough locally countably generated vector bundles. Let $\mathcal{F}$ be a flat quasi-coherent sheaf on X. Then $\mathcal{F} = \varinjlim \mathcal{F}_i$, where $\mathcal{F}_i$ is locally countably generated and flat with $\mathcal{V} \dim \mathcal{F}_i \leq 1$.*

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