GENERALIZED TILTING THEORY

PEDRO NICOLÁS AND MANUEL SAORÍN

Abstract. We study the relationship between (generalized) tilting modules and triangle equivalences between (quotients of) derived categories.

Introduction

The fundamental theorem of Morita theory gives a list of necessary and sufficient conditions for an equivalence between two module categories to exist (see for example [6]). Equivalences between derived categories of rings generalize Morita theory, and in this sense, derived Morita theory is a generalization of Morita theory. The fundamental theorem of derived Morita theory gives a list of necessary and suffcient conditions for a triangle equivalence between two derived categories of rings to exist (see [4, 5]). This is at the basis of classical tilting theory. Indeed, if A is a ring, a right A-module T is *classical tilting* if and only if the adjoint pair of derived functors

$$\begin{array}{c} \mathcal{D}A \\ {\scriptstyle ?\otimes^{\mathbf{L}}_{B}T}\uparrow \;\downarrow{\scriptstyle \mathrm{RHom}_A(T,?)} \\ \mathcal{D}B \end{array}$$

induces a triangle equivalence between the derived category $\mathcal{D}A$ of A and the derived category $\mathcal{D}B$ of its endomorphisms ring $B = \mathrm{End}_A(T)$. In particular, a classical tilting module is finitely generated. The notion of classical tilting module has evolved to the notion of generalized tilting module, which includes other interesting modules which are not necessarily finitely generated. Recently, Bazzoni-Mantese-Tonolo [2] proved that a *good (generalized) tilting* A-module T induces a triangle equivalence

$$\begin{array}{c} \mathcal{D}A \\ {\scriptstyle ?\otimes^{\mathbf{L}}_{B}T}\uparrow \;\downarrow{\scriptstyle \mathrm{RHom}_A(T,?)} \\ \mathcal{D}B/\ker(?\otimes^{\mathbf{L}}_{\mathrm{B}}\mathrm{T}) \end{array}$$

between the derived category of A and the quotient of the derived category of $B = \mathrm{End}_A(T)$ by the kernel of the derived tensor functor $? \otimes^{\mathbf{L}}_{B} T$. This equivalence exists if and only if (see [3]) the derived Hom-functor

$$\mathrm{RHom}_A(T, ?) : \mathcal{D}A \to \mathcal{D}B$$

is fully faithful. A detailed inspection reveals that if T is a good tilting module, this functor is not only fully faithful, but it also preserves compact objects.

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We summarize the different notions of tilting modules we have been refering:

Definition 0.1. Let A be an ordinary algebra, and let T be a right A-module. Consider the following conditions:

a) There is an exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to T \to 0$ in $\mathrm{Mod}\, A$ where the modules P_i are projective.
a') There is an exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to T \to 0$ in $\mathrm{Mod}\, A$ where the modules P_i are finitely generated projective.
b) There is an exact sequence $0 \to A \to T_0 \to T_1 \to \cdots \to T_m \to 0$ in $\mathrm{Mod}\, A$ where the modules T_i are direct summands of coproducts of copies of T.
b') There is an exact sequence $0 \to A \to T_0 \to T_1 \to \cdots \to T_m \to 0$ in $\mathrm{Mod}\, A$ where the modules T_i are direct summands of finite coproducts of copies of T.
c) $\mathrm{Ext}^i_A(T, T^{(\alpha)}) = 0$ for each $i > 0$ and each cardinal α.

We say that T is a *generalized n-tilting* module if it satisfies a), b) and c). We say that it is a *classical n-tilting* module if it satisfies a'), b') and c) (for $\alpha = 1$). We say that T is a *good n-tilting* module if it satisfies a), b') and c). We say that a module is generalized tilting (resp. classical tilting, good tilting) if it is generalized n-tilting (resp. classical n-tilting, good n-tilting) for some $n \geq 1$.

However, it is was still unclear which is the precise relationship between generalized tilting modules and triangle equivalences between (quotients of) derived categories.

Let A be a differential graded(=dg) algebra, T an arbitrary right dg A-module and let $B = \mathrm{REnd}_A(T)$. In this work we contribute to the enlightenment of this relationship with:

1) A theorem which characterizes those A-modules T such that $\mathrm{RHom}_A(T, ?) : \mathcal{D}A \to \mathcal{D}B$ is fully faithful and preserves compact objects (§ 1). For this we use a derived version of the classical notion of faithfully balanced bimodule.
2) A theorem which studies the link between generalized tilting modules and the modules appearing in the theorem mentioned above (§ 2).

1. Special localizations

Let $\mathrm{thick}_{\mathcal{D}A}(T)$ be the smallest full subcategory of $\mathcal{D}A$ containing T and closed under shifts, extensions and direct summands.

Theorem 1.1. *The following statements are equivalent:*

1) $\mathrm{RHom}_A(T, ?) : \mathcal{D}A \to \mathcal{D}B$ *is fully faithful and preserves compact objects.*
2) *A belongs to* $\mathrm{thick}_{\mathcal{D}A}(T)$.
3) *The natural map $A \to \mathrm{RHom}_{B^{op}}(T, T)$ is a quasi-isomorphism and T is compact regarded as an object of $\mathcal{D}(B^{op})$.*

Remark 1.2. a) Notice that a good tilting module T satisfies 2), and so part 1) will always be true. Hence, the former theorem gives an alternative proof of the main result of [2].
b) The theorem is a particular case of a more general result stated in terms of dg categories and arbitrary bimodules.

2. Tilting

Let now A be an ordinary algebra, T a right A-module and $\mathrm{End}_A(T) = B$.

Proposition 2.1. *If* $\mathrm{Ext}^i_A(T,T) = 0$ *for each* $i > 0$*, the following conditions are equivalent:*

1) $A \in \mathrm{thick}_{\mathcal{D}A}(T)$,
2) *A admits a coresolution*

$$0 \to A \to T_0 \to T_1 \to \cdots \to T_n \to 0$$

in $\mathrm{Mod}\,A$*, where each* T_i *is a direct summand of a finite coproduct of copies of* T.

Corollary 2.2. *If a generalized tilting module T is such that* $\mathrm{RHom}_A(T,?) : \mathcal{D}A \to \mathcal{D}B$ *is fully faithful and preserves compact objects, then T is a good tilting module.*

Theorem 2.3. *The following assertions hold:*

1) *There are* 1*-tilting modules T such that* $\mathrm{RHom}_A(T,?) : \mathcal{D}A \to \mathcal{D}B$ *is not fully faithful.*
2) *There are right A-modules T satisfying the following conditions:*
   a) $\mathrm{pd}_A T \leq 1$,
   b) *the functor* $\mathrm{RHom}_A(T,?) : \mathcal{D}A \to \mathcal{D}B$ *is fully faithful and preserves compact objects,*
   c) *T does not satisfy condition c) of Definition 0.1, and so it is not a tilting module.*

References

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Universidad de Murcia, Departamendo de Didáctica de las CC. Matemáticas y Sociales
E-mail address: `pedronz@um.es`

Universidad de Murcia, Departamento de Matemáticas
E-mail address: `saorinc@um.es`