PRIME FUZZY IDEALS OVER NONCOMMUTATIVE RINGS

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Abstract. In this paper we introduce prime fuzzy ideals over a noncommutative ring. This notion of primeness is equivalent to level cuts being crisp prime ideals. It also generalizes the one provided by Kumbhojkar and Bapat in 1993, which lacks this equivalence in a noncommutative setting. Semiprime fuzzy ideals over a noncommutative ring are also defined and characterized as intersection of primes. This allows us to introduce the fuzzy prime radical and contribute to establish the basis of a Fuzzy Noncommutative Ring Theory. This work is going to appear in [17].

Introduction

Since the well-known paper of Rosenfeld [19] dealing with fuzzy sets of a group, many researchers have focused on giving an algebraic structure to the universe space, defining algebraic topics on a fuzzy environment and studying their properties. See, for instance, [7, 8, 10, 5, 13, 20, 26, 11, 18, 12, 1, 21].

Focusing on the structure of ring, the early paper of Liu [10] defining fuzzy ideals initiated the investigation of rings by expanding the class of ideals with these fuzzy objects. Some years later, during the late eighties and nineties, many works of different authors were published in order to develop a Fuzzy Ring Theory. Most of these authors limit their attention to commutative rings or, even, fail to mention that this requirement is necessary. Nevertheless, the commutativity condition becomes too restrictive when we realize that noncommutative rings can be found in a wide range of knowledge areas as Particle Physics, Quantum Field Theory, Gauge Theory, Cryptography or Coding Theory.

In this work we study the notion of primeness on fuzzy ideals clarifying relationships between various definitions appearing in the literature, and we propose a new definition of primeness over arbitrary rings. It is generally accepted that the concept of fuzzy primeness considered in [5, 6] is the most appropriate since, in commutative rings, it is equivalent to the level cuts being crisp prime ideals. Nevertheless, when working over arbitrary rings, this is no longer valid in previous papers. Our definition fills in this gap. It generalizes the one given in [5, 6] and verifies the aforementioned property.

Throughout R will be an arbitrary ring with unity. We recall from [10] that a (two-sided) fuzzy ideal over R is a fuzzy set $I : R \to [0, 1]$ satisfying the following properties:

(1) $I(x - y) \geq I(x) \wedge I(y)$ for any $x, y \in R$, i.e., it is an additive fuzzy subgroup.
(2) $I(xy) \geq I(x) \vee I(y)$ for any $x, y \in R$.

For any $\alpha \leq I(0)$, we may consider the α-cut, I_α, as the subset $I_\alpha = \{x \in R \text{ such that } I(x) \geq \alpha\}$. It is easy to prove that I is a fuzzy ideal if and only if I_α is an ideal of R for any $I(1) < \alpha \leq I(0)$. For any fuzzy set F, the fuzzy ideal generated by F will be the least ideal containing F, i.e., the intersection of all fuzzy ideals I satisfying that $F \leq I$. We shall denote it by $\langle F \rangle$.

1. A little survey about fuzzy primeness

Accordingly with crisp Ring Theory, the notion of prime ideal was one of the first concepts under consideration in its fuzzy version. In [20] the product of fuzzy ideals is defined as:

$$IJ(x) = \bigvee_{x=\sum_i a_i b_i} \bigwedge_i (I(a_i) \wedge J(b_i)),$$

and Zahedi proposes the following definition in [24].

Definition 1.1 (D1)**.** A non-constant fuzzy ideal $P : R \to [0,1]$ is said to be *prime* if, whenever $IJ \leq P$ for some fuzzy ideals I and J, it satisfies that $I \leq P$ or $J \leq P$.

Malik and Mordeson [13], completing the work of Mukherjee and Sen [16], and Swamy and Swamy [20] for L-fuzzy ideals, give a nice characterization of all D1-prime fuzzy ideals.

Lemma 1.2. [13, 20] *A fuzzy ideal $P : R \to [0,1]$ is D1-prime if and only if P has the form*

$$P(x) = \begin{cases} 1 & \text{if } x \in Q, \\ t & \text{otherwise,} \end{cases}$$

where Q is a crisp prime ideal of R and $0 \leq t < 1$.

Albeit the former result makes fuzzy D1-primeness clear and transparent, the reader may figure out a subtle problem about its usefulness: there is no plenty more D1-prime fuzzy ideals than crisp ones. Hence, the additional information about the ring provided by D1-prime fuzzy ideals is quite reduced. Additionally, as pointed out in [4], the notion seems to be too strong since there are fuzzy ideals whose level cuts are prime, despite of they are not D1-prime.

Although the first inconvenient becomes much more difficult to resolve, an evident solution to the second one is to define primeness as the property that it should verify.

Definition 1.3 (D2)**.** A non-constant fuzzy ideal $P : R \to [0,1]$ is said to be *prime* if P_α is prime for any $P(0) \geq \alpha > P(1)$.

As the reader may think in reading Definition 1.3, it only translates our problem since now we need a characterization which makes operational D2-primeness.

In [5], Kumbhojkar and Bapat deal with prime fuzzy ideals from an element-like perspective and they state primeness looking forward similarities with standard commutative algebra.

Definition 1.4 (D3)**.** A non-constant fuzzy ideal $P : R \to [0,1]$ is said to be *prime* if, for any $x, y \in R$, whenever $P(xy) = P(0)$, then $P(x) = P(0)$ or $P(y) = P(0)$.

D2-primeness implies D3-primeness, although the converse does not hold. For this reason, the authors give a stronger notion which they call strongly primeness.

Definition 1.5 (D4)**.** A non-constant fuzzy ideal $P : R \to [0,1]$ is said to be *prime* if, for any $x, y \in R$, $P(xy) = P(x)$ or $P(xy) = P(y)$.

Obviously, D3-primeness is weaker than D4-primeness, but we are interested in the relation between D2-primeness and D4-primeness.

Lemma 1.6. *A fuzzy ideal P is D4-prime if and only if each level cut P_α is completely prime for all $P(0) \geq \alpha > P(1)$.*

Hence, D4-primeness implies D2-primeness. Although the converse result does not hold for arbitrary rings.

In order to obtain an element-like definition, in [24], Zahedi develops two notions involving the so-called singletons. We recall the reader that, given an element $x \in R$ and $t \in (0, 1]$, the singleton x_t is the fuzzy set defined by $x_t(x) = t$ and zero otherwise.

Definition 1.7 (D0)**.** A non-constant fuzzy ideal $P : R \to [0, 1]$ is said to be *prime* if, whenever $x_t y_s \leq P$ for some singletons x_t and y_s, then $x_t \leq P$ or $y_s \leq P$. This is called completely prime by Zahedi in [24, Definition 2.7].

Definition 1.8 (D0′)**.** A non-constant fuzzy ideal $P : R \to [0, 1]$ is said to be *prime* if, whenever $\langle x_t \rangle \langle y_s \rangle \leq P$ for some singletons x_t and y_s, then $x_t \leq P$ or $y_s \leq P$.

In [24, Theorem 4.9], D0′-primeness is proven to be equivalent to D1-primeness. In [6, Theorem 3.5], D0-primeness is proven to be equivalent to D1-primeness, see also [15, Theorem 2.6]. Nevertheless, the proof is no longer valid when R is an arbitrary non-commutative ring. As the reader may see, in a general setting, there is no characterization of primeness consistent with the level cuts.

2. Fuzzy primeness over noncommutative rings

Definition 2.1. Let R be an arbitrary ring with unity. A non-constant fuzzy ideal $P : R \to [0, 1]$ is said to be *prime* if, for any $x, y \in R$, $\bigwedge P(xRy) = P(x) \vee P(y)$.

Proposition 2.2. *Let R be an arbitrary ring with unity and $P : R \to [0, 1]$ be a non-constant fuzzy ideal of R. The following conditions are equivalent:*

1. *P is prime.*
2. *P_α is prime for all $P(0) \geq \alpha > P(1)$.*

Moreover, if R is commutative, any of these statements is equivalent to P being D4-prime.

Definition 2.3. Let R be an arbitrary ring with unity. A non-constant fuzzy ideal $P : R \to [0, 1]$ is said to be *semiprime* if $\bigwedge P(xRx) = P(x)$ for all $x \in R$.

Proposition 2.4. *Let R be an arbitrary ring with unity and $P : R \to [0, 1]$ be a non-constant fuzzy ideal of R. The following conditions are equivalent:*

1. *P is semiprime.*
2. *P_α is semiprime for all $P(0) \geq \alpha > P(1)$.*

Theorem 2.5. *A fuzzy ideal is semiprime if and only if it is the intersection of prime fuzzy ideals.*

Corollary 2.6. *Let R be a ring with unity and I be a non-constant fuzzy ideal over R. The following fuzzy ideals coincide:*

1. *The intersection F_1 of all semiprime fuzzy ideals containing I.*

(2) *The intersection F_2 of all prime fuzzy ideals containing I.*
(3) *The fuzzy ideal F_3 given by $F_3(x) = \bigvee\{t \in [0,1]$ such that $x \in \mathrm{Rad}(I_t)\}$.*

For any non-constant fuzzy ideal I, we define the fuzzy prime radical of I, $\mathrm{FRad}(I)$, as any of the fuzzy ideals described in Corollary 2.6. Hence, the following result is immediate.

Corollary 2.7. *P is a semiprime fuzzy ideal if and only if $\mathrm{FRad}(P) = P$.*

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