DUALITY FOR GROUPOIDS AND WEAK HOPF ALGEBRAS

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ABSTRACT. We prove that the category of finite grupoids is anti-equivalent to the category of commutative semisimple finite-dimensional weak Hopf algebras over an algebraically closed field of characteristic 0. This result is obtained as a consequence of our main result, which shows that there is an adjoint pair of functors between the category of grupoids with finitely many objects and the category of weak Hopf algebras.

INTRODUCTION

The category of finite groups is anti-equivalent to the category of commutative semisimple finite-dimensional Hopf algebras over an algebraically closed field of characteristic 0 (see [1, Theorem 3.4.2]). This basic result could be understood as the simplest discrete version of Tannaka's duality theorem. Weak Hopf algebras were introduced in [2, 4] in the framework of the theory of quantum groupoids, so they have been treated basically as genuine noncommutative since their inception. In particular, the extension of the aforementioned duality for finite grupoids seems have not been explored. The aim of this communication is to study this topic and, more generally, to explain a precise relationship between the category of (set theoretical) grupoids and that of weak Hopf algebras.

1. THE GROUPOID WEAK HOPF ALGEBRA

We work over a field K of characteristic 0, and algebras and coalgebras are meant over K. The tensor product of vector spaces over K is denoted by $\otimes$. For an excelent introduction to the theory of Hopf algebras, we refer to [1]. A systematic account of the fundamentals of the theory of weak Hopf algebras is [3].

Definition 1.1. A *weak bialgebra* is a quintuple $(H, \mu, \eta, \Delta, \varepsilon)$, where (H, μ, η) is an associative unital algebra with multiplication $\mu : H \otimes H \to H$ and unit $\eta : K \to H$, and (H, Δ, ε) is a coassociative counital coalgebra with comultiplication $\Delta : H \to H \otimes H$ and counit $\varepsilon : H \to K$ such that the following compatibility conditions hold.

(a) The comultiplication Δ is multiplicative (equivalently, the multiplication μ is comultiplicative.

(b) For all $x, y, z \in H$

$$\varepsilon(xyz) = \varepsilon(xy_{(1)})\varepsilon(y_{(2)}z)$$
$$\varepsilon(xyz) = \varepsilon(xy_{(2)})\varepsilon(y_{(1)}z)$$

SUPPORT FROM THE MINISTERIO DE CIENCIA E INNOVACIÓN AND FEDER, GRANT MTM2010-20940-C02-01 IS GRATEFULLY ACKNOWLEDGED.

(c)

$$(\Delta \otimes H) \circ \Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1))$$
$$(\Delta \otimes H) \circ \Delta(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$$

Example 1.2. Consider a small category $\mathcal{C}$, with a finite set of objects $\mathcal{C}_0$. Let $\mathcal{C}_1$ denote the set of arrows, and $s, t : \mathcal{C}_1 \to \mathcal{C}_0$ the source and target maps. By $K\mathcal{C}$ we denote the weak Hopf algebra built on the k–vector space with basis $\mathcal{C}_1$. The coalgebra structure on $K\mathcal{C}$ is determined by $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ for every $g \in \mathcal{C}_1$. The multiplication is given on $\mathcal{C}_1$ by $gh = g \circ h$ if $s(g) = t(h)$, and $gh = 0$ if $s(g) \neq t(h)$, and it is extended to $K\mathcal{C}$ by linearity. The unit for this product is given by $\sum_{x \in \mathcal{C}_0} 1_x$, where 1_x denotes the identity morphism at the object x. Some straightfoward computations show that $K\mathcal{C}$ is a weak bialgebra.

Definition 1.3. A weak bialgebra $(H, \mu, \eta, \Delta, \varepsilon, S)$ is a *weak Hopf algebra* if there exists a linear map $S : H \to H$, called *the antipode*, satisfying that for all $x, y \in H$

$$\begin{aligned} x_1 S(x_{(2)}) &= \varepsilon(1_{(1)}x)1_{(2)}, \\ S(x_{(1)})x_{(2)} &= 1_{(1)}\varepsilon(x1_{(2)}), \\ S(x_{(1)})x_{(2)}S(x_{(3)}) &= S(x) \end{aligned}$$

Example 1.4. Let $\mathcal{G}$ be a grupoid over a finite set, that is, a category $\mathcal{G}$ with a finite set of objects $\mathcal{G}_0$ such that every arrow is an isomorphism. The weak bialgebra $K\mathcal{G}$ defined in Example 1.2 is a weak Hopf algebra in this case because the linear map $S : K\mathcal{G} \to K\mathcal{G}$ defined by $S(g) = g^{-1}$ for every $g \in \mathcal{G}_1$ is an antipode. We refer to $K\mathcal{G}$ as the groupoid weak Hopf algebra of the groupoid $\mathcal{G}$.

Attached to any weak Hopf algebra H we have two relevant maps $\sqcap^L, \sqcap^R : H \to H$ defined by

$$\sqcap^L(x) = \varepsilon(1_{(1)}x)1_{(2)}, \ \sqcap^R(x) = 1_{(1)}\varepsilon(x1_{(2)}),$$

Their images $H^L = \sqcap^L(H), H^R = \sqcap^R(H)$ are separable K–subalgebras of H (see [3, Proposition 2.11]).

2. An adjoint pair

Let Gpd denote the category whose objects are the grupoids with finitely many objects, and with functors between them as morphisms. The construction of Example 1.4 give the objects part of a functor $K(-) : \mathsf{Gpd} \to \mathsf{WHA}_K$, where WHA_K is the category of weak Hopf algebras over K. Morphisms in WHA_K are multiplicative homomorphisms of coalgebras $f : H \to H'$ between weak Hopf algebras H, H' such that $f\sqcap^L = \sqcap^L f$ and $f\sqcap^R = \sqcap^R f$. Our main result is the following.

Theorem 2.1. *The functor $K(-) : \mathsf{Gpd} \to \mathsf{WHA}_K$ has a right adjoint $G(-) : \mathsf{WHA}_K \to \mathsf{Gpd}$.*

The proof of Theorem 2.1 requires the construction of the groupoid $G(H)$ for any weak Hopf algebra H. The arrows of $G(H)$ are defined as

$$G(H)_1 = \{g \in H : \Delta(g) = g \otimes g, \varepsilon(g) = 1, S^2(g) = g\}$$

The objects are then given by

$$G(H)_0 = \sqcap^L(G(H)_1) = \sqcap^R(G(H)_1),$$

and the source and target maps, used to specify which morphisms of $G(H)$ will be composable, are the restrictions to $G(H)_1$ of $\sqcap^R$ and $\sqcap^L$, respectively, to $G(H)_0$. The resulting category $G(H)$ is a groupoid, as the inverse of each $g \in G(H)_1$ is given by $S(g)$.

Theorem 2.1 allows to identify Gpd as a subcategory of WHA_K, according to the following corollary.

Corollary 2.2. *The category* Gpd *is equivalent to the full subcategory of* WHA_K *of all cosemisimple pointed weak Hopf algebras.*

When K is algebraically closed, the subcategory of WHA_K equivalent to Gpd has an alternative description:

Corollary 2.3. *If* K *is algebraically closed, then the category* Gpd *is equivalent to the full subcategory of* WHA_K *of all cosemisimple cocommutative weak Hopf algebras.*

3. Duality

The dual $H^* = Hom_K(H, K)$ of each finite-dimensional Hopf algebra H is a weak Hopf algebra (see [3]). A consequence of Corollary 2.3 is the following duality theorem for finite grupoids.

Theorem 3.1. *Let* Gpd^f *denote the category of finite grupoids, and consider the category* CWH^f_K *of finite-dimensional commutative semisimple weak Hopf algebras. If* K *is algebraically closed, then these categories are anti-equivalent by the contravariant functors:*

$$\Phi : \mathsf{Gpd}^f \to \mathsf{CWH}^f_K, G \mapsto (KG)^*$$

$$\Psi : \mathsf{CWH}^f_K \to \mathsf{Gpd}^f, H \mapsto G(H^*)$$

We consider Theorem 3.1 as a starting point to study duality between grupoids and weak Hopf algebras from an algebraic point of view. To deal with non finite grupoids, the first step is to consider the finite dual H^o of a weak Hopf algebra H of any dimension as the pertinent generalization of H^*. This should be helpful to study duality for algebraic groupoids, for instance, and as a guideline to extend it to genuine quantum grupoids without strict finiteness conditions. This is left for future work.

Aknowledgement: *We thank to Gabriella Böhm for helpful discussions.*

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