

ZASSENHAUS CONJECTURE FOR TORSION UNITS

MAURICIO CAICEDO, LEO MARGOLIS, AND ÁNGEL DEL RÍO

ABSTRACT. Zassenhaus Conjecture for torsion units states that every augmentation one torsion unit of the integral group ring of a finite group G is conjugate to an element of G in the units of rational group algebra $\mathbb{Q}G$. This conjecture has been proved for nilpotent groups, metacyclic groups and some other families of groups. We prove the conjecture for cyclic-by-abelian groups.

INTRODUCTION

Let G be a finite group and $\mathbb{Z}G$ denotes the integral group ring of G with coefficients over the ring of integers \mathbb{Z} . In the 1960s Hans Zassenhaus established a series of conjectures about the finite subgroups of units of augmentation one of $\mathbb{Z}G$, these units are usually called normalized units. Namely he conjectured that every finite group of normalized units of $\mathbb{Z}G$ is conjugate to a subgroup of G in the units of $\mathbb{Q}G$. These conjecture is usually denoted (ZC3), while the version of (ZC3) for the particular case of subgroups of normalized units with the same cardinality as G is usually denoted (ZC2). These conjectures have important consequences. For example, a positive solution of (ZC2) implies a positive solution for the Isomorphism and Automorphism Problems (see [Seh93] for details). The most celebrated positive result for Zassenhaus Conjectures is due to Weiss [Wei91] who proved (ZC3) for nilpotent groups. However Roggenkamp and Scott founded a counterexample to the Automorphism Problem, and henceforth to (ZC2) (see [Rog91] and [Kli91]). Later Hertweck [Her01] provided a counterexample to the Isomorphism Problem.

The only conjecture of Zassenhaus that is still up is the version for cyclic subgroups namely:

Zassenhaus Conjecture for Torsion Units (ZC1). If G is a finite group then every normalized torsion unit of $\mathbb{Z}G$ is conjugate in $\mathbb{Q}G$ to an element of G .

Besides the family of nilpotent groups, (ZC1) has been proved for some concrete groups [BH08, BHK04, HK06, LP89, LT91, Her08b], for groups having a Sylow subgroup with an abelian complement [Her06], for some families of cyclic-by-abelian groups [LB83, LT90, LS98, MRSW87, PMS84, PMRS86, dRS06, RS83] and some classes of metabelian groups not necessarily cyclic-by-abelian [MRSW87, SW86]. Other results on Zassenhaus Conjectures can be found in [Seh93, Seh01] and [Seh03, Section 8]

The latest and most general result for (ZC1) on the class of cyclic-by-abelian groups is due to Hertweck [Her08a] who proved (ZC1) for finite groups of the form $G = AX$ with A a cyclic normal subgroup of G and X an abelian subgroup of G . This includes the class of metacyclic groups that was not covered in previous results.

We prove (ZC1) for arbitrary cyclic-by-abelian groups.

Theorem. Let G be a finite cyclic-by-abelian group. Then every normalized torsion unit of $\mathbb{Z}G$ is conjugate in $\mathbb{Q}G$ to an element of G .

REFERENCES

- [BH08] V. Bovdi and M. Hertweck, *Zassenhaus conjecture for central extensions of S_5* , J. Group Theory **11** (2008), no. 1, 63–74. MR 2381018 (2009a:20010)
- [BHK04] V. Bovdi, C. Höfert, and W. Kimmerle, *On the first Zassenhaus conjecture for integral group rings*, Publ. Math. Debrecen **65** (2004), no. 3-4, 291–303. MR 2107948 (2006f:20009)
- [dRS06] Á. del Río and S.K. Sehgal, *Zassenhaus conjecture (ZC1) on torsion units of integral group rings for some metabelian groups*, Arch. Math. (Basel) **86** (2006), no. 5, 392–397. MR 2229354 (2007c:16064)
- [Her01] M. Hertweck, *A counterexample to the isomorphism problem for integral group rings*, Ann. of Math. **154** (2001), 115–138.
- [Her06] ———, *On the torsion units of some integral group rings*, Algebra Colloq. **13** (2006), no. 2, 329–348. MR 2208368 (2006k:16049)
- [Her08a] ———, *Torsion units in integral group rings of certain metabelian groups*, Proc. Edinb. Math. Soc. (2) **51** (2008), no. 2, 363–385. MR 2465913 (2009j:16027)
- [Her08b] ———, *Zassenhaus conjecture for A_6* , Proc. Indian Acad. Sci. Math. Sci. **118** (2008), no. 2, 189–195. MR 2423231 (2009c:20010)
- [HK06] C. Höfert and W. Kimmerle, *On torsion units of integral group rings of groups of small order*, Groups, rings and group rings, Lect. Notes Pure Appl. Math., vol. 248, Chapman & Hall/CRC, Boca Raton, FL, 2006, pp. 243–252. MR 2226199 (2007d:16077)
- [Kli91] L. Klingler, *Construction of a counterexample to a conjecture of Zassenhaus*, Comm. Algebra **19** (1991), no. 8, 2303–2330. MR 1123126 (92i:20004)
- [LB83] I.S. Luthar and A.K. Bhandari, *Torsion units of integral group rings of metacyclic groups*, J. Number Theory **17** (1983), no. 2, 270–283. MR 716946 (85c:20004)
- [LP89] I.S. Luthar and I.B.S. Passi, *Zassenhaus conjecture for A_5* , Proc. Indian Acad. Sci. Math. Sci. **99** (1989), no. 1, 1–5. MR 1004634 (90g:20007)
- [LS98] I.S. Luthar and P. Sehgal, *Torsion units in integral group rings of some metacyclic groups*, Res. Bull. Panjab Univ. Sci. **48** (1998), no. 1-4, 137–153 (1999). MR 1773990 (2001f:16065)
- [LT90] I.S. Luthar and P. Trama, *Zassenhaus conjecture for certain integral group rings*, J. Indian Math. Soc. (N.S.) **55** (1990), no. 1-4, 199–212. MR 1088139 (92b:20008)
- [LT91] ———, *Zassenhaus conjecture for S_5* , Comm. Algebra **19** (1991), no. 8, 2353–2362. MR 1123128 (92g:20003)
- [MRSW87] Z. Marciniak, J. Ritter, S. K. Sehgal, and A. Weiss, *Torsion units in integral group rings of some metabelian groups. II*, J. Number Theory **25** (1987), no. 3, 340–352. MR 880467 (88k:20019)
- [PMRS86] C. Polcino Milies, J. Ritter, and S.K. Sehgal, *On a conjecture of Zassenhaus on torsion units in integral group rings. II*, Proc. Amer. Math. Soc. **97** (1986), no. 2, 201–206. MR 835865 (87i:16013)
- [PMS84] C. Polcino Milies and S.K. Sehgal, *Torsion units in integral group rings of metacyclic groups*, J. Number Theory **19** (1984), no. 1, 103–114. MR 751167 (86i:16009)
- [Rog91] K.W. Roggenkamp, *Observations on a conjecture of Hans Zassenhaus*, Groups—St. Andrews 1989, Vol. 2, London Math. Soc. Lecture Note Ser., vol. 160, Cambridge Univ. Press, Cambridge, 1991, pp. 427–444. MR 1123997 (92g:20004)
- [RS83] J. Ritter and S.K. Sehgal, *On a conjecture of Zassenhaus on torsion units in integral group rings*, Math. Ann. **264** (1983), no. 2, 257–270. MR 711882 (85e:16014)
- [Seh93] S.K. Sehgal, *Units in integral group rings*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 69, Longman Scientific & Technical, Harlow, 1993, With an appendix by Al Weiss. MR 1242557 (94m:16039)
- [Seh01] ———, *Zassenhaus conjecture*, Encyclopaedia of mathematics. Supplement. Vol. III (M. Hazewinkel, ed.), Kluwer Academic Publishers, Dordrecht, 2001, pp. 453–454. MR 1935796 (2003j:00009)

- [Seh03] ———, *Group rings*, Handbook of algebra, Vol. 3, North-Holland, Amsterdam, 2003, pp. 455–541. MR 2035104 (2005d:16044)
- [SW86] S.K. Sehgal and A. Weiss, *Torsion units in integral group rings of some metabelian groups*, J. Algebra **103** (1986), no. 2, 490–499. MR 864426 (88f:20015)
- [Wei91] A. Weiss, *Torsion units in integral group rings*, J. Reine Angew. Math. **415** (1991), 175–187. MR 1096905 (92c:20009)

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE MURCIA, 30100 MURCIA, SPAIN
E-mail address: mauriciojc02@hotmail.com

FACHBEREICH MATHEMATIK, UNIVERSITÄT STUTTGART, PFAFFENWALDRING 57, 70569 STUTTGART,
GERMANY
E-mail address: leo.imsueden@yahoo.com

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE MURCIA, 30100 MURCIA, SPAIN
E-mail address: adelrio@um.es