THE HOCHSCHILD COHOMOLOGY RING OF PREPROJECTIVE ALGEBRAS OF TYPE $\mathbb{L}_n$

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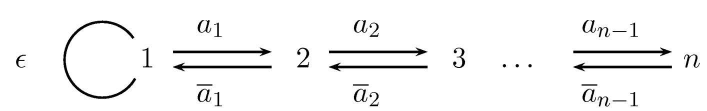
Abstract. We compute the Hochschild Cohomology of a finite-dimensional preprojective algebra of generalized Dynkin type $\mathbb{L}_n$ over a field of arbitrary characteristic. In particular, we describe the ring structure of the classical and stable Hochschild Cohomology ring under the Yoneda product by giving an explicit presentation by generators and relations.

Introduction

The aim of this work is to determine, via generators and relations, the structure of the classical and stable Hochschild cohomology rings of a preprojective algebra of type $\mathbb{L}_n$. These are exactly the finite-dimensional preprojective algebras whose graphs are not of Dynkin type. For preprojective algebras of Dynkin type, the structure of the Hochschild cohomology ring is known for type $\mathbb{A}_n$ in arbitrary characteristic ([4], [5]) and, in the case of a field of characteristic zero, for types $\mathbb{D}_n$ and $\mathbb{E}$ [7].

Given a nonoriented finite graph Δ, with Δ_0 and Δ_1 as sets of vertices and edges, respectively, the *preprojective algebra* of (type) Δ, denoted $P(\Delta)$, is the algebra given by quiver and relations as follows. The quiver $Q := Q_\Delta$ of $P(\Delta)$ has the same vertices and the same loops as Δ. Then, for each edge i——j in Δ which is not a loop, Q will have two opposite arrows $a : i \to j$ and $\bar{a} : j \to i$. Convening that $\bar{a} = a$ whenever a is a loop, the algebra is subject to as many relations as vertices in Δ, namely, one relation $\sum_{i(a)=i} a\bar{a} = 0$ per each $i \in Q_0 = \Delta_0$ (here $i(a)$ denotes the initial vertex of a).

Let Λ be the preprojective algebra of generalized Dynkin type $\mathbb{L}_n$ over an algebraically closed field K. That is, Λ is the algebra given by the quiver



and relations

$$\epsilon^2 + a_1\bar{a}_1 = 0, \quad \bar{a}_i a_i + a_{i+1}\bar{a}_{i+1} = 0 \;\; \forall i = 1, \ldots, n-1, \quad \bar{a}_{n-1}a_{n-1} = 0$$

By classical theory of derived functors, for each pair M, N of Λ-modules, one can compute the K-vector space $Ext^n_\Lambda(M, N)$ of n-extensions as the n-th cohomology space of the complex $Hom_\Lambda(P^\bullet, N)$, where

$$P^{\bullet} : ...P^{-n-1} \stackrel{d^{-n-1}}{\longrightarrow} P^{-n} \longrightarrow ... \longrightarrow P^{-1} \stackrel{d^{-1}}{\longrightarrow} P^0 \twoheadrightarrow M \rightarrow 0$$

is a projective resolution of M.

Suppose that L, M, N are Λ-modules, that $P^{\bullet}$ and $Q^{\bullet}$ are projective resolutions of L and M, respectively, and that m, n are natural numbers. If $\delta \in Ext^n_{\Lambda}(L, M)$ and $\epsilon \in Ext^m_{\Lambda}(M, N)$, then we can choose a $\tilde{\delta} \in Hom_{\Lambda}(P^{-n}, M)$, belonging to the kernel of the transpose map $(d^{-n-1})^* : Hom_{\Lambda}(P^{-n}, M) \longrightarrow Hom_{\Lambda}(P^{-n-1}, M)$ of the differential $d^{-n-1} : P^{-n-1} \longrightarrow P^{-n}$ of $P^{\bullet}$, which represents δ. Similarly we can choose an $\tilde{\epsilon} \in Hom_{\Lambda}(Q_m, N)$ which represents ϵ. Due to the projectivity of the P^i, there is a non-unique sequence of morphisms of Λ-modules $\delta^{-k} : P^{-n-k} \longrightarrow Q^{-k}$ $(k = 0, 1, ...,)$ making commute the following diagram

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & P^{-n-k} & \longrightarrow \cdots \longrightarrow & P^{-n-1} & \longrightarrow & P^{-n} & & \longrightarrow \cdots \longrightarrow & P^0 & \longrightarrow & L & \longrightarrow & 0 \\
& \downarrow{\scriptstyle \delta^{-k}} & & \downarrow{\scriptstyle \delta^{-1}} & & \downarrow{\scriptstyle \delta^{0}} & \searrow{\scriptstyle \tilde{\delta}} & & & & & & \\
\cdots \longrightarrow & Q^{-k} & \longrightarrow \cdots \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \longrightarrow & M & \longrightarrow & 0 & & &
\end{array}$$

Then the composition $\tilde{\epsilon} \circ \delta^{-m} : P^{-m-n} \longrightarrow N$ is in the kernel of $(d^{-m-n})^*$ and, thus, it represents an element of $Ext^{m+n}_{\Lambda}(L, N)$. This element is denoted by $\epsilon\delta$ and does not depend on the choices made. It is called the *Yoneda product* of ϵ and δ. It is well-known that the map

$$Ext^m_{\Lambda}(M, N) \times Ext^n_{\Lambda}(L, N) \longrightarrow Ext^{m+n}_{\Lambda}(L, N) \qquad ((\epsilon, \delta) \rightsquigarrow \epsilon\delta)$$

is K-bilinear.

When $M = N$ in the above setting, the vector space $Ext^*_{\Lambda}(M, M) = \oplus_{i \geq 0} Ext^i_{\Lambda}(M, M)$ inherites a structure of graded K-algebra, where the multiplication of homogeneous elements is the Yoneda product. We are specifically interested in a particular case of this situation. Namely, when we replace Λ by its enveloping algebra $\Lambda^e = \Lambda \otimes \Lambda^{op}$ and replace M by Λ, viewed as Λ^e-module (i.e. as a Λ-bimodule). Then $HH^i(\Lambda) := Ext^i_{\Lambda^e}(\Lambda, \Lambda)$ is called the *i-th Hochschild cohomology space*, for each $i \geq 0$. The corresponding graded algebra $Ext^*_{\Lambda^e}(\Lambda, \Lambda)$, called the *Hochschild cohomology ring (or algebra)* of Λ, is denoted by $HH^*(\Lambda)$ and turns out to be graded commutative (see [8]). That is if $\epsilon \in HH^i(\Lambda)$ and $\delta \in HH^j(\Lambda)$ are homogeneous elements then $\epsilon\delta = (-1)^{ij}\delta\epsilon$.

An important common feature of the preprojective algebras of generalized Dynkin type is that, except for $\Delta = \mathbb{A}_1$, $P(\Delta)$ is $(\Omega-)$periodic of period at most 6 (thus self-injective) where Ω is the Heller's syzygy operator (see [9] and [6] for the Dynkin cases and [3] for the case $\mathbb{L}_n$). That means that there exists $m \in \mathbb{Z}^+$, called the period of Λ, such that $\Omega^m_{\Lambda^e}(\Lambda) \cong \Lambda$ and this clearly makes it possible to compute the Hochschild cohomology groups. The multiplicative structure of the Hochschild cohomology ring $HH^*(\Lambda)$ for a self-injective finite dimensional algebra Λ is of great interest in connection with the study of varieties of modules and with questions about its relationship with the Yoneda algebra of Λ.

The three main results are the following, from which all the desired structures (classical and stable Hochschild homology and cohomology) are described.

Teorema 0.1. *Let $\Lambda = P(\mathbb{L}_n)$ and suppose that $Char(K) \neq 2$ and $Char(K)$ does not divide $2n+1$. Then $HH^*(\Lambda)$ is the graded commutative algebra given by*

a) <u>*Generators*</u>*: $x_0, x_1, \dots x_n, y, z_1, \dots, z_n, \gamma, h$ with degrees $deg(x_i) = 0$, $deg(y) = 1$, $deg(z_j) = 2$, $deg(\gamma) = 4$ and $deg(h) = 6$.*

b) <u>*Relations*</u>*:*

i) $x_i\xi = 0$, *for each $i = 1, \dots, n$ and each generator ξ.*

ii) $x_0^n = y^2 = x_0 z_j = 0$ $(j = 1, \dots, n)$

iii) $z_j z_k = (-1)^{k-j+1}(2j-1)(n-k+1)x_0^{n-1}\gamma$ *for* $1 \leq j \leq k \leq n$.

iv) $z_j\gamma = (-1)^j(n-j+1)x_0^{n-1}h$ $(j = 1, \dots, n)$

v) $\gamma^2 = z_1 h$

Teorema 0.2. *Let $\Lambda = P(\mathbb{L}_n)$ and suppose that $Char(K)$ divides $2n+1$. Then $HH^*(\Lambda)$ is the graded commutative algebra given by*

a) <u>*Generators*</u>*: $x_0, x_1, \dots, x_n, y, z_1, \dots, z_n, t_1, t_2, \dots t_{n-1}, \gamma, h$ with degrees $deg(x_i) = 0$, $deg(y) = 1$, $deg(z_j) = 2$, $deg(t_k) = 3$, $deg(\gamma) = 4$ and $deg(h) = 6$.*

b) <u>*Relations*</u>*:*

i) $x_i\xi = 0$ *for each $i = 1, \dots, n$ and each generator ξ.*

ii) $x_0^n = y^2 = x_0 z_j = x_0 t_i = y t_i = t_i t_k = 0$ $(j = 1, \dots, n \quad i, k = 1, \dots n-1)$

iii) $z_j z_k = (-1)^{k-j+1}(2j-1)(n-k+1)x_0^{n-1}\gamma$ *for* $1 \leq j \leq k \leq n$.

iv) $z_j\gamma = (-1)^j(n-j+1)x_0^{n-1}h$

v) $\gamma^2 = z_1 h$

vi) $y z_j = (-1)^{j-1}(2j-1)y z_1$

vii) $z_k t_j = \delta_{jk} x_0^{n-1} y\gamma$ $(k = 1, \dots, n \quad j = 1 \dots, n-1)$

viii) $t_j\gamma = \delta_{1j} x_0^{n-1} y h$ $(j = 1, \dots, n)$.

Teorema 0.3. *Let us assume that $Char(K) = 2$ and $\Lambda = P(\mathbb{L}_n)$. Then $HH^*(\Lambda)$ is the commutative algebra, graded by the natural numbers, given by:*

a) <u>*Generators*</u>*: $x_0, x_1, \dots, x_n, y_0, y_1, z_2, \dots, z_n, h$ with degrees $deg(x_i) = 0$, $deg(y_j) = 1$, $deg(z_k) = 2$ and $deg(h) = 3$.*

b) <u>*Relations*</u> *(given in ascending degree):*

i) $x_0^n = x_i x_j = 0$ *$(i, j = 0, \dots, n$ and $(i,j) \neq (0,0))$*

ii) $x_i y_0 = \delta_{i1} x_0^{n-1} y_1$
$x_i y_1 = 0$

iii) $x_i z_k = \delta_{ik} x_0^{n-1} y_0 y_1$
$x_0 y_0^2 = 0 = y_1^2$

iv) $y_0^3 = n x_0^{n-1} h$
$y_0^2 y_1 = \sum_{j=1}^n (n-j+1) x_j h$
$y_0 z_k = (n-k+1) x_0^{n-1} h$
$y_1 z_k = \sum_{j=1}^n (n - max(j,k) + 1) x_j h$

v) $z_k z_l = (n - max(l,k) + 1) x_0^{n-1} y_0 h$,

where $i = 1, ..., n$ and $k, l = 2, ..., n$ in the relations ii)-v).

The proof for these theorems basically consists of computing the structure of each $HH^i(\Lambda)$ as a module over $Z(\Lambda) = HH^0(\Lambda)$ which makes it possible to give a canonical basis of each $HH^i(\Lambda)$.

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