Numerical index of vector-valued function spaces*

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Abstract

We show that the numerical index of a c_0-, l_1-, or l_∞-sum of Banach spaces is the infimum numerical index of the summands. Moreover, we prove that the spaces $C(K,X)$ and $L_1(\mu,X)$ (K any compact Hausdorff space, μ any positive measure) have the same numerical index as the Banach space X. We also observe that these spaces have the so-called Daugavet property whenever X has the Daugavet property.

1 Introduction

The numerical index of a Banach space is the greatest constant of equivalence between the numerical radius and the usual norm in the Banach algebra of all bounded linear operators on the space. Let us recall the relevant definitions. Given a real or complex Banach space X, we write B_X for the closed unit ball and S_X for the unit sphere of X. The dual space will be denoted by X^* and $L(X)$ will be the Banach algebra of all bounded linear operators on X. The *numerical range* of such an operator T is the subset $V(T)$ of the scalar field defined by

$$V(T) = \{x^*(Tx) \,:\, x \in S_X,\, x^* \in S_{X^*},\, x^*(x) = 1\}.$$

The *numerical radius* of T is then given by

$$v(T) = \sup\{|\lambda| \,:\, \lambda \in V(T)\}.$$

It is clear that v is a seminorm on $L(X)$, and $v(T) \leq \|T\|$ for every $T \in L(X)$. Very often, v is actually a norm and it is equivalent to the operator norm $\|\cdot\|$. Thus,

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it is natural to consider the so called *numerical index* of the space X, namely the constant $n(X)$ defined by

$$n(X) = \inf\{v(T) \,:\, T \in S_{L(X)}\}.$$

Equivalently, $n(X)$ is the greatest constant $k \geq 0$ such that $k\|T\| \leq v(T)$ for every $T \in L(X)$. Note that $0 \leq n(X) \leq 1$, and $n(X) > 0$ if and only if v and $\|\cdot\|$ are equivalent norms on $L(X)$.

The concept of numerical index was first suggested by G. Lumer in 1968. At that time, it was known that a Hilbert space of dimension greater than 1 has numerical index $1/2$ in the complex case, and 0 in the real case. Two years later, J. Duncan, C. McGregor, J. Pryce, and A. White [9] proved that L-spaces and M-spaces have numerical index 1. They also determined the range of values of the numerical index. More precisely, for a real Banach space X, $n(X)$ can be any number in the interval $[0,1]$, while $\{n(X) \,:\, X \text{ complex Banach space}\} = [1/e, 1]$. The remarkable result that $n(X) \geq 1/e$ for every complex Banach space X goes back to H. Bohnenblust and S. Karlin [4] (see also [10]). The disk algebra is another example of a Banach space with numerical index 1 [6, Theorem 32.9]. Necessary conditions for a real Banach space to have numerical index 1 were investigated in [15]. For general information and background on numerical ranges we refer to the books by F. Bonsall and J. Duncan [5, 6]. Further developments in the Hilbert space case can be found in [11].

In this paper we compute the numerical index for some classes of Banach spaces. First, we prove that the numerical index of a c_0-, l_1-, or l_∞-sum of Banach spaces is the infimum numerical index of the summands. As an application of this result, we exhibit an example of a real Banach space X such that the numerical radius is a norm on $L(X)$ but it is not equivalent to the usual operator norm, i.e. $n(X) = 0$. We did not find explicit examples of this kind in the previous literature. Our main results deal with spaces of vector-valued functions. We prove that

$$n(C(K,X)) = n(L_1(\mu,X)) = n(X)$$

for every real or complex Banach space X, with no restrictions on the compact Hausdorff space K or the positive measure μ. It is worth mentioning here some results by Å. Lima [14] showing that one cannot expect a general result on the numerical index of injective or projective tensor products. The situation for spaces of the form $L_p(\mu,X)$ is much more complicated. Even the computation of $n(l_p)$ for $1 < p < \infty$, $p \neq 2$, is an open problem.

The numerical radius is related to another quantitative characteristic of an operator T, the *Daugavet equation*:

$$\text{(DE)} \qquad \|Id + T\| = 1 + \|T\|.$$

Quite a lot of attention has been paid to this equation, starting from the result by I. Daugavet [7] that it is satisfied by all compact operators on $C[0,1]$. Actually $C[0,1]$ is the first example of a Banach space with the so-called *Daugavet property*, an interesting property deeply studied in [12, 13]. Following [12], we say that a Banach space X has the Daugavet property if (DE) holds for all rank-one operators $T \in L(X)$, and it then follows that (DE) is actually satisfied by all weakly compact operators on X [12, Théorème 4]. For further information on this subject we refer to [2, 12, 13, 19] and the references therein. As for the relation to the numerical radius, it is easy to deduce from an old result by G. Lumer (see [16, Lemma 12] or [5, Lemma 9.2]) that the equality $v(T) = \|T\|$ is equivalent to the following weak form of (DE):

$$\text{(wDE)} \qquad \max\{\|Id + \lambda T\| \,:\, |\lambda| = 1\} = 1 + \|T\|.$$

Therefore, a Banach space X satisfies $n(X) = 1$ if and only if (wDE) holds for every $T \in L(X)$. With this relation in mind, one may browse through papers on the Daugavet property to find some relevant examples of Banach spaces with numerical index 1. For instance, it follows from results by D. Werner [18] that all function algebras have numerical index 1.

We remark that the Daugavet property and having numerical index 1 are independent properties. Indeed, $n(c_0) = 1$ although c_0 fails the Daugavet property, while it will follow from the results in this paper that $n(C([0,1], l_2)) = n(l_2) < 1$ in spite of the fact that $C([0,1], l_2)$ has the Daugavet property [13]. Nevertheless, some ideas coming from papers on the Daugavet property have been very helpful in our proofs, and conversely, we managed to adapt one of our arguments to get a result on the Daugavet property: it passes from X to $C(K,X)$ with no restriction on K.

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2 Numerical index of sums

Our first goal will be to show that the numerical index of c_0-, l_1-, and l_∞-sums can be computed in terms of the summands in the expected way. Given an arbitrary family $\{X_\lambda \,:\, \lambda \in \Lambda\}$ of Banach spaces, we denote by $[\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}$ (resp. $[\oplus_{\lambda\in\Lambda} X_\lambda]_{l_1}$, $[\oplus_{\lambda\in\Lambda} X_\lambda]_{l_\infty}$) the c_0-sum (resp. l_1-sum, l_∞-sum) of the family. In case

Λ has just two elements, we use the simpler notation $X \oplus_\infty Y$ or $X \oplus_1 Y$. For countable sums of copies of a space X we write $c_0(X)$, $l_1(X)$ or $l_\infty(X)$.

Proposition 1. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of Banach spaces. Then*

$$n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{l_1}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{l_\infty}\Big) = \inf_\lambda n(X_\lambda).$$

Proof. Let Z denote either $X \oplus_\infty Y$ or $X \oplus_1 Y$ for any Banach spaces X and Y. We first check that $n(Z) \leq n(X)$ by showing that $n(Z) \leq v(S)$ for any $S \in L(X)$ with $\|S\| = 1$. For such an operator S, let $T \in L(Z)$ be given by $T(x,y) = (Sx, 0)$. Then $\|T\| = 1$ and since

$$n(Z) \leq v(T),$$

for every $\varepsilon > 0$ we may find $z = (x,y) \in S_Z$ and $z^* = (x^*, y^*) \in S_{Z^*}$ such that

$$x^*(x) + y^*(y) = \|x^*\|\|x\| + \|y^*\|\|y\| = 1 \tag{1}$$

and

$$n(Z) - \varepsilon \leq |z^*(Tz)| = |x^*(Sx)| \leq \left| \frac{x^*}{\|x^*\|}\left(S\left(\frac{x}{\|x\|}\right)\right)\right|$$

(we may assume that $n(Z) > 0$) implying that $n(Z) - \varepsilon \leq v(S)$, and therefore $n(Z) \leq v(S)$, because $x^*(x) = \|x^*\|\|x\|$ by (1).

For fixed $\lambda_0 \in \Lambda$ one has clearly

$$[\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0} = \left[\oplus_{\lambda\neq\lambda_0} X_\lambda\right]_{c_0} \oplus_\infty X_{\lambda_0},$$

so

$$n\left([\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}\right) \leq n(X_{\lambda_0})$$

and it follows that

$$n\left([\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}\right) \leq \inf_\lambda n(X_\lambda).$$

The same argument works for l_1- and l_∞- sums.

The proof of the reverse inequalities will follow ideas from the proof of [19, Theorem 1] by P. Wojtaszczyk. We first work with the c_0- or the l_∞-sum. If Z denotes any of these sums, an operator $T \in L(Z)$ can be seen as a family $(T_\lambda)_{\lambda\in\Lambda}$ where $T_\lambda \in L(Z, X_\lambda)$ for every λ, and $\|T\| = \sup_\lambda \|T_\lambda\|$. Given $\varepsilon > 0$, we find $\lambda_0 \in \Lambda$ such that $\|T_{\lambda_0}\| > \|T\| - \varepsilon$, and write $X = X_{\lambda_0} \oplus_\infty Y$ where $Y = \left[\oplus_{\lambda\neq\lambda_0} X_\lambda\right]_{c_0}$ or $Y = \left[\oplus_{\lambda\neq\lambda_0} X_\lambda\right]_{l_\infty}$. Since B_Z is the convex hull of $S_{X_{\lambda_0}} \times S_Y$, we may find $x_0 \in S_{X_{\lambda_0}}$ and $y_0 \in S_Y$ such that,

$$\|T_{\lambda_0}(x_0, y_0)\| > \|T\| - \varepsilon.$$

Now fix $x_0^* \in X_{\lambda_0}^*$ with $\|x_0^*\| = x_0^*(x_0) = 1$ and consider the operator $S \in L(X_{\lambda_0})$ defined by

$$Sx = T_{\lambda_0}(x, x_0^*(x)y_0) \qquad (x \in X_{\lambda_0}).$$

We clearly have

$$\|S\| \geq \|Sx_0\| = \|T_{\lambda_0}(x_0, y_0)\| > \|T\| - \varepsilon,$$

so we may find $x \in X_{\lambda_0}$, $x^* \in X_{\lambda_0}^*$ such that

$$\|x\| = \|x^*\| = x^*(x) = 1 \qquad \text{and} \qquad |x^*(Sx)| \geq n(X_{\lambda_0})[\|T\| - \varepsilon].$$

Now $z = (x, x_0^*(x)y_0) \in S_Z$, $z^* = (x^*, 0) \in S_{Z^*}$ satisfy $z^*(z) = 1$ and

$$\text{(2)} \qquad |z^*(Tz)| = |x^*(T_{\lambda_0}(x, x_0^*(x)y_0))| = |x^*(Sx)| \geq n(X_{\lambda_0})[\|T\| - \varepsilon].$$

It follows that

$$v(T) \geq \inf_\lambda n(X_\lambda)\|T\|$$

and so $n(Z) \geq \inf_\lambda n(X_\lambda)$ as required.

The proof for the l_1-sum is somehow the dual of the above argument. If $Z = [\oplus_{\lambda\in\Lambda} X_\lambda]_{l_1}$, an operator $T \in L(Z)$ may also be seen as a family $(T_\lambda)_{\lambda\in\Lambda}$ where now $T_\lambda \in L(X_\lambda, Z)$ for all λ, and again $\|T\| = \sup_\lambda \|T_\lambda\|$. Given $\varepsilon > 0$, find $\lambda_0 \in \Lambda$ such that $\|T_{\lambda_0}\| > \|T\| - \varepsilon$, and write $Z = X_{\lambda_0} \oplus_1 Y$, $T_{\lambda_0} = (A, B)$ where $A \in L(X_{\lambda_0})$ and $B \in L(X_{\lambda_0}, Y)$. Now we choose $x_0 \in S_{X_{\lambda_0}}$ such that

$$\|T_{\lambda_0} x_0\| = \|Ax_0\| + \|Bx_0\| > \|T\| - \varepsilon,$$

find $a_0 \in S_{X_{\lambda_0}}$, $y^* \in S_{Y^*}$ satisfying

$$\|Ax_0\| a_0 = Ax_0 \qquad \text{and} \qquad y^*(Bx_0) = \|Bx_0\|,$$

and define an operator $S \in L(X_{\lambda_0})$ by

$$Sx = Ax + y^*(Bx)a_0 \qquad (x \in X_{\lambda_0}).$$

Then

$$\|S\| \geq \|Sx_0\| = \Big\| Ax_0 + \|Bx_0\| a_0 \Big\| = \|Ax_0\| + \|Bx_0\| > \|T\| - \varepsilon,$$

so we may find $x \in X_{\lambda_0}$, $x^* \in X_{\lambda_0}^*$ such that

$$\|x\| = \|x^*\| = x^*(x) = 1 \qquad \text{and} \qquad |x^*(Sx)| \geq n(X_{\lambda_0})[\|T\| - \varepsilon].$$

For $z = (x, 0) \in S_Z$ and $z^* = (x^*, x^*(a_0)y^*) \in S_{Z^*}$ we clearly have $z^*(z) = 1$ and

$$\text{(3)} \qquad |z^*(Tz)| = |x^*(Ax) + x^*(a_0)y^*(Bx)| = |x^*(Sx)| \geq n(X_{\lambda_0})[\|T\| - \varepsilon].$$

The desired inequality $n(Z) \geq \inf_\lambda n(X_\lambda)$ follows. □

Remarks 2.

(a) By the proof of Proposition 1, we also have

$$n\left([\oplus_{\lambda\in\Lambda}X_\lambda]_{l_p}\right) \leq \inf_{\lambda} n(X_\lambda)$$

for $1 < p < \infty$ and any family $\{X_\lambda : \lambda \in \Lambda\}$ of Banach spaces. The exact computation of the numerical index for l_p-sums is an open problem. Even, the exact value of $n(l_p)$ for $1 < p < \infty$, $p \neq 2$, seems to be unknown.

(b) In [3] the reader may find a result on l_∞-sums which sounds like one of the assertions in Proposition 1, but dealing with the so-called *numerical index of a numerical range space* (see also [3] for the definitions). We shall not go into the details but simply mention that, for any Banach space X, the pair $(L(X), Id)$ is the motivating example of a numerical range space, whose numerical index is precisely $n(X)$. By [3, Lemma 5.11], the suitably defined l_∞-sum of a family of numerical range spaces has numerical index (as a numerical range space) equal to the infimum numerical index of the summands. However, the space of operators on an l_∞-sum of Banach spaces is much larger than the l_∞-sum of the spaces of operators on the summands, and therefore, the statement on l_∞-sums in Proposition 1 is independent on the result in [3].

Examples 3.

(a) *For each $t \in [0,1]$ in the real case (resp. $t \in [1/e, 1]$ in the complex case) there is a Banach space X isomorphic to c_0 with $n(X) = t$.* Indeed, by [9, Theorems 3.5, 3.6] there is a two-dimensional real (resp. complex) space Y with $n(Y) = t$, and we just take $X = c_0(Y)$. The same argument works for l_1 or l_∞ in place of c_0.

(b) *There exists a real Banach space X such that the numerical radius is a norm on $L(X)$ but not equivalent to the operator norm. Moreover, X can be chosen to be isomorphic to c_0, l_1, or l_∞.* Certainly, by [9, Theorem 3.6] we can find for every natural number n, a two-dimensional real space X_n with $n(X_n) = 1/n$. If we consider $X = [\oplus_{n\in\mathbb{N}}X_n]_{c_0}$ (or $X = [\oplus_{n\in\mathbb{N}}X_n]_{l_1}$, or $X = [\oplus_{n\in\mathbb{N}}X_n]_{l_\infty}$), then $n(X) = 0$ by Proposition 1. Nevertheless, since $n(X_n) > 0$ for every n, inequality (2) (or (3) for the l_1 case) shows that $v(T) > 0$ for every nonzero $T \in L(X)$.

In his 1991 paper [1], Y. Abramovich asked if a space made by combining L-spaces and M-spaces with l_1- and l_∞-sums has the property that equality (wDE) in the introduction holds for all bounded linear operators in the space. As we already explained, this is equivalent to asking if such a space has numerical index 1. The following obvious consequence of Proposition 1 includes an affirmative answer to Abramovich's question.

Corollary 4. *The class of Banach spaces with numerical index* 1 *is stable under* c_0*-*, l_1*-, and* l_∞*-sums.*

The above corollary is somehow analogous to the result by P. Wojtaszczyk [19, Theorem 1] that the Daugavet property is stable under c_0-, l_1-, and l_∞-sums. As we already mentioned, our proof of Proposition 1 borrows some of his ideas.

3 Spaces of vector-valued functions

The fact that $n(c_0(X)) = n(X)$ will now be generalized to spaces of vector-valued continuous functions. Given a compact Hausdorff space K and a Banach space X, we consider the Banach space $C(K,X)$ of all continuous functions from K into X, endowed with its natural supremum norm.

Theorem 5. *Let K be a compact Hausdorff space and X a Banach space. Then,*

$$n\,(C(K,X)) = n(X)$$

Proof. To show that $n(C(K,X)) \geq n(X)$, we fix $T \in L(C(K,X))$ with $\|T\| = 1$ and prove that $v(T) \geq n(X)$. Given $\varepsilon > 0$, we may find $f_0 \in C(K,X)$ with $\|f_0\| = 1$ and $t_0 \in K$ such that

$$\|[T f_0](t_0)\| > 1 - \varepsilon. \tag{4}$$

Denote $y_0 = f_0(t_0)$ and find a continuous function $\varphi : K \longrightarrow [0,1]$ such that $\varphi(t_0) = 1$ and $\varphi(t) = 0$ if $\|f_0(t) - y_0\| \geq \varepsilon$. Now write $y_0 = \lambda x_1 + (1-\lambda)x_2$ with $0 \leq \lambda \leq 1$, $x_1, x_2 \in S_X$, and consider the functions

$$f_j = (1-\varphi)\,f_0 + \varphi x_j \in C(K,X) \qquad (j = 1,2).$$

Then $\|\varphi f_0 - \varphi y_0\| < \varepsilon$ meaning that

$$\|f_0 - (\lambda f_1 + (1-\lambda)\,f_2)\| < \varepsilon,$$

and, using (4), we must have

$$\|[T f_1](t_0)\| > 1 - 2\varepsilon \qquad \text{or} \qquad \|[T f_2](t_0)\| > 1 - 2\varepsilon. \tag{5}$$

By making the right choice of $x_0 = x_1$ or $x_0 = x_2$ we get $x_0 \in S_X$ such that

$$\|\,[T\,((1-\varphi)\,f_0 + \varphi x_0)]\,(t_0)\| > 1 - 2\varepsilon. \tag{6}$$

Next we fix $x_0^* \in S_{X^*}$ with $x_0^*(x_0) = 1$, denote

$$\Phi(x) = x_0^*(x)(1-\varphi) f_0 + \varphi x \in C(K,X) \qquad (x \in X),$$

and consider the operator $S \in L(X)$ given by

$$Sx = [T(\Phi(x))](t_0) \qquad (x \in X).$$

Since, by (6),

$$\|S\| \geq \|Sx_0\| > 1 - 2\varepsilon,$$

we may find $x \in S_X$, $x^* \in S_{X^*}$ such that

$$x^*(x) = 1 \qquad \text{and} \qquad |x^*(Sx)| \geq n(X)[1-2\varepsilon].$$

Now, define $g \in S_{C(K,X)}$ by $g = \Phi(x)$, for this x, and consider the functional $g^* \in S_{C(K,X)^*}$ given by

$$g^*(h) = x^*(h(t_0)) \qquad (h \in C(K,X)).$$

Since $g(t_0) = x$, we have $g^*(g) = 1$ and

$$|g^*(Tg)| = |x^*(Sx)| \geq n(X)[1-2\varepsilon].$$

Hence $v(T) \geq n(X)$, as required.

For the reverse inequality, take an operator $S \in L(X)$ with $\|S\| = 1$, and define $T \in L(C(K,X))$ by

$$[T(f)](t) = S(f(t)) \qquad (t \in K,\ f \in C(K,X)).$$

Then $\|T\| = 1$, so $v(T) \geq n(C(K,X))$. By [5, Theorem 9.3] the numerical radius of T is given by

$$v(T) = \sup\left\{ \left|x^*\big([Tf](t)\big)\right| \ :\ f \in S_{C(K,X)},\ t \in K,\ x^* \in S_{X^*},\ x^*(f(t)) = 1 \right\}.$$

Therefore, given $\varepsilon > 0$, we may find $f \in S_{C(K,X)}$, $x^* \in S_{X^*}$, and $t \in K$, such that $x^*(f(t)) = 1$ and

$$n(C(K,X)) - \varepsilon < \left|x^*\big([Tf](t)\big)\right| = |x^*(S(f(t)))|.$$

It clearly follows that $v(S) \geq n(C(K,X))$, so $n(X) \geq n(C(K,X))$. $\square$

Remark 6. The first part of the above proof can be easily adapted to get a stability result for the Daugavet property, that is, $C(K,X)$ has this property whenever X has it. Just assume that $T \in L(C(K,X))$ with $\|T\| = 1$ is a rank-one operator, build $S \in L(X)$ exactly as in the above proof, and note that S is a rank-one operator as well. The Daugavet property of X gives an $x \in S_X$ which satisfies

$$\|x + Sx\| > 1 + \|S\| \ - \varepsilon > 2 - 3\varepsilon.$$

Now define again the function $g \in S_{C(K,X)}$ as in the above proof, and note that

$$\|Id + T\| \geq \left\| \left[(Id + T)(g)\right](t_0) \right\| = \|x + Sx\| > 2 - 3\varepsilon.$$

Recall that $C(K)$ has the Daugavet property if and only if K is perfect, and, in such a case, $C(K,X)$ has the Daugavet property for any Banach space X (see [13]). Conversely, if $C(K,X)$ has the Daugavet property and K has an isolated point, then we can write $C(K,X) = X \oplus_\infty Z$ for some Banach space Z, and it clearly follows that X has the Daugavet property (see [19]). Therefore, we may summarize the situation as follows: *$C(K,X)$ has the Daugavet property if and only if K is perfect or X has the Daugavet property.*

Remark 7. Very often, when working with the numerical index of a Banach space, we are only able to use operators of a simple kind, say finite-rank or compact operators. So one may wonder if the numerical index can be computed by using only this kind of operators. For instance, it was observed in [15] that an Asplund space X satisfies $n(X) = 1$ as soon as $v(T) = \|T\|$ for every rank-one operator on X, and the same is true if X satisfies the Radon-Nikodým property. Nevertheless, the above theorem shows that, in general, one cannot estimate the numerical index by using only weakly compact operators. Just take $X = C([0,1],H)$ where H is a Hilbert space with dimension greater than one. Then X satisfies the Daugavet property (see [13]), so $v(T) = \|T\|$ for every weakly compact operator T on X, but the above theorem tells us that $n(X) = n(H) < 1$.

Given a measure space (Ω, Σ, μ) we now consider the Banach space $L_1(\mu, X)$ of Bochner-integrable functions f from Ω into a Banach space X with its natural norm

$$\|f\| = \int_\Omega \|f(t)\| \, d\mu(t).$$

We generalize the fact that $n(l_1(X)) = n(X)$ as follows.

Theorem 8. *Let (Ω, Σ, μ) be a measure space. Then, for every Banach space X,*

$$n\left(L_1(\mu, X)\right) = n(X).$$

Proof. Let us first note that it is enough to deal with finite measures, since $L_1(\mu,X)$ is isometrically isomorphic to an l_1-sum of spaces $L_1(\mu_i,X)$ for suitable finite measures μ_i, and Proposition 1 applies. So, we assume $\mu(\Omega)<\infty$, and fix $T\in L(L_1(\mu,X))$ with $\|T\|=1$ to prove that $v(T)\geq n(X)$, which will give us the inequality $n(L_1(\mu,X))\geq n(X)$. Given $\varepsilon>0$ we find $f\in S_{L_1(\mu,X)}$ such that $\|Tf\|>1-\varepsilon$ and use [8, Lemma III.2.1] to get a partition π of Ω into a finite family of disjoint measurable sets with positive measure satisfying

$$\|f-E_\pi f\|<\varepsilon \qquad \text{and} \qquad \|Tf-E_\pi Tf\|<\varepsilon,$$

where E_π is the contractive projection given by

$$E_\pi g=\sum_{A\in\pi}\left(\frac{1}{\mu(A)}\int_A g\,d\mu\right)\chi_A \qquad (g\in L_1(\mu,X)).$$

We clearly have

$$\begin{aligned}\|E_\pi TE_\pi f\| &\geq \|E_\pi Tf\|-\|E_\pi Tf-E_\pi TE_\pi f\|\\ &\geq \|E_\pi Tf\|-\|f-E_\pi f\|\\ &\geq \|Tf\|-\|Tf-E_\pi Tf\|-\|f-E_\pi f\|\\ &>1-3\varepsilon.\end{aligned}$$

Now, let Y be the range of E_π and let $S\in L(Y)$ be the restriction to Y of the operator $E_\pi T$. The above inequality shows that $\|S\|>1-3\varepsilon$. Moreover, Y is isometric to a finite l_1-sum of copies of X, so $n(Y)=n(X)$ by Proposition 1. It follows that $v(S)>n(X)[1-3\varepsilon]$ (if $n(X)=0$ the required inequality holds trivially), so we may find $h\in S_Y$, $h^*\in S_{Y^*}$ such that

$$h^*(h)=1 \qquad \text{and} \qquad |h^*(Sh)|>n(X)[1-3\varepsilon].$$

By taking $g^*=E_\pi^*(h^*)\in B_{L_1(\mu,X)^*}$ we immediately have

$$g^*(h)=1 \qquad \text{and} \qquad |g^*(Th)|=|h^*(Sh)|>n(X)[1-3\varepsilon],$$

which shows that $v(T)>n(X)[1-3\varepsilon]$, and the required inequality follows.

To prove the reverse inequality, take an operator $S\in L(X)$ with $\|S\|=1$, and define $T\in L(L_1(\mu,X))$ by

$$[T(f)](t)=T(f(t)) \qquad (t\in\Omega,\ f\in L_1(\mu,X))$$

which clearly satisfies $\|T\|=1$. So, given $\varepsilon>0$ we may find $f\in S_{L_1(\mu,X)}$ and $f^*\in S_{L_1(\mu,X)^*}$ such that

$$f^*(f)=1 \qquad \text{and} \qquad |f^*(Tf)|\geq n(L_1(\mu,X))[1-\varepsilon].$$

We may arrange that f has the form

$$f = \sum_{i=1}^{n} \frac{x_i}{\mu(A_i)} \chi_{A_i},$$

where $x_1, x_2, \ldots, x_n$ are nonzero vectors in X with $\sum_{i=1}^{n} \|x_i\| = 1$, and $A_1, A_2, \ldots, A_n$ are disjoint subsets of Ω with positive measure. Moreover, f^* can be taken of the form

$$f^*(h) = \sum_{i=1}^{n} x_i^* \int_{A_i} h \, d\mu \qquad (h \in L_1(\mu, X)),$$

where $x_i^* \in S_{X^*}$ and $x_i^*(x_i) = \|x_i\|$ for $i = 1, 2, \ldots, n$, All this follows from [5, Theorem 9.3] and the denseness of simple functions in $L_1(\mu, X)$. We clearly have

$$\sum_{i=1}^{n} \|x_i\| \left| x_i^* \left(S \frac{x_i}{\|x_i\|} \right) \right| \geq \left| \sum_{i=1}^{n} x_i^*(S x_i) \right| = |f^*(Tf)| \geq n(L_1(\mu, X)) [1 - \varepsilon],$$

and since $\sum_{i=1}^{n} \|x_i\| = 1$, we must have

$$\left| x_k^* \left(S \frac{x_k}{\|x_k\|} \right) \right| \geq n(L_1(\mu, X))[1 - \varepsilon]$$

for some $k \in \{1, 2, \ldots, n\}$. It follows that $n(X) \geq n(L_1(\mu, X))$. $\square$

Remark 9. One can discuss the Daugavet property in spaces $L_1(\mu, X)$ in the same way as we did for spaces $C(K, X)$ in Remark 6, but this time the similar result follows easily from [13] and [19]. If X has the Daugavet property and μ is a positive measure we may write $L_1(\mu, X)$ in the form $L_1(\nu, X) \oplus_1 [\oplus_{i \in I} X]_{l_1}$ for suitable set I and atomless measure ν. Then $L_1(\nu, X)$ has the Daugavet property by [13], and the l_1-sum of spaces with the Daugavet property has it by [19]. Conversely, if $L_1(\mu, X)$ has the Daugavet property and μ has an atom, then $L_1(\mu, X)$ can be written as $X \oplus_1 Z$ for convenient space Z, and it clearly follows that X has the Daugavet property [19]. Therefore, $L_1(\mu, X)$ *has the Daugavet property if and only if* μ *is atomless or* X *has the Daugavet property.*

Since $C(K, X) = C(K) \otimes_\varepsilon X$ and $L_1(\mu, X) = L_1(\mu) \otimes_\pi X$ where $\otimes_\varepsilon$ and $\otimes_\pi$ denote, respectively, the injective and projective tensor products, one may wonder if Theorems 5 and 8 might be special cases of a general result giving $n(X \otimes_\varepsilon Y)$ and $n(X \otimes_\pi X)$ as a function of $n(X)$ and $n(Y)$. To conclude this paper, we use an example due to Å. Lima [14] to show that such a general result cannot be expected, even in the finite-dimensional case.

Example 10. *There exist Banach spaces X and Y with $n(X) = n(Y) = 1$, such that $n(X \otimes_\varepsilon X) < 1$, $n(Y \otimes_\pi Y) < 1$, and $n(X \otimes_\pi X) = n(Y \otimes_\varepsilon Y) = 1$.* Indeed, let $X = l_1^4$ and $Y = X^* = l_\infty^4$, that is, the real four-dimensional l_1 and l_∞ spaces respectively. It is clear that

$$n(X \otimes_\pi X) = n(Y \otimes_\varepsilon Y) = n(X) = n(Y) = 1.$$

To see that $n(X \otimes_\varepsilon X) < 1$ and $n(Y \otimes_\pi Y) < 1$, let us first recall the result by C. McGregor [17, Theorem 3.1] that a finite-dimensional space Z satisfies $n(Z) = 1$ if and only if $|z^*(z)| = 1$ for all extreme points $z \in B_Z$ and $z^* \in B_{Z^*}$. Now, take $Z = Y \otimes_\pi Y$ and note that

$$Z^* \equiv X \otimes_\varepsilon X \equiv L(Y,X).$$

It follows from results by Å. Lima [14, Lemma 3.2, Proposition 2.4, and Theorem 2.3] that Z (hence also Z^*) does not satisfy McGregor's condition on the extreme points, so $n(Z) < 1$ and $n(Z^*) < 1$, as claimed.

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