The second numerical index of Banach spaces

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Where is contained?

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Outline of the lecture

Introduction

Section 1

Some notation

X real or complex Banach space

B_X closed unit ball

S_X unit sphere

X^* topological dual

$\mathcal{L}(X)$ Banach algebra of all bounded linear operators from X to X

$\mathcal{L}(X,Y)$ Banach space of all bounded linear operators from X to Y

$\Pi(X)$ $= \{(x, x^*) \in S_X \times S_{X^*} \ : \ x^*(x) = 1\}$

Definitions

Numerical range and numerical radius (Bauer, Lumer, early 60's)

X Banach space, $T \in \mathcal{L}(X)$

$$\begin{aligned} V(T) &= \{x^*(Tx) \,:\, (x, x^*) \in \Pi(X)\} \\ v(T) &= \sup\{|\lambda| \,:\, \lambda \in V(T)\} \\ &= \sup\{|x^*(Tx)| \,:\, x \in S_X,\, x^* \in S_{X^*},\, x^*(x) = 1\} \end{aligned}$$

Obviously one has $v(T) \leqslant \|T\|$

Numerical index (Lumer, 1968)

X Banach space

$$n(X) = \inf\{v(T) \,:\, T \in S_{\mathcal{L}(X)}\} = \max\{k \geqslant 0 \,:\, k\|T\| \leqslant v(T)\}$$

- $0 \leqslant n(X) \leqslant 1$;
- v and $\|\cdot\|$ are equivalent norms iff $n(X) > 0$;

Some known results

- $n(C(K)) = n(L_1(\mu)) = 1$ (Duncan-McGregor-Pryce-White, 1970)
- $n(X) = 1$ iff $\max\limits_{|w|=1} \|\mathrm{Id} + w\,T\| = 1 + \|T\| \ \forall T \in \mathcal{L}(X)$ (Duncan et al., 1970)
- $\{n(X) \ : \ X$ is a complex Banach space$\} = [1/\mathrm{e}, 1]$
  $\{n(X) \ : \ X$ is a real Banach space$\} = [0, 1]$ (Duncan et al., 1970)
- $n(L_p(\mu)) = \inf\{n(\ell_p^m) \ : \ m \in \mathbb{N}\}$ for μ so that $\dim(L_p(\mu)) = \infty$
  (Aksoy-Eddari-Khamsi, 2007)
- $n(L_p(\mu)) > 0$ for $p \neq 2$ (Martín-Merí-Popov, 2011)
- Let $\{X_\lambda \ : \ \lambda \in \Lambda\}$ be an arbitrary family of Banach spaces. Then

$$n\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{c_0}\Big) = n\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_1}\Big) = n\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_\infty}\Big) = \inf\{n(X_\lambda) \ : \ \lambda \in \Lambda\}$$

$$n\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_p}\Big) \leqslant \inf\{n(X_\lambda) \ : \ \lambda \in \Lambda\}$$

(Martín-Payá, 2000)

Some known results

- Let X be a Banach space, L locally compact Hausdorff, K compact Hausdorff, Ω completely regular Hausdorff, and μ positive measure. Then

$$n\big(C_0(L,X)\big) = n\big(C_b(\Omega,X)\big) = n\big(L_1(\mu,X)\big) = n(X) \quad \text{(Martín-Payá, 2000)}$$

$$n\big(L_\infty(\mu,X)\big) = n(X) \quad \text{(Martín-Villena, 2003)}$$

$$n\big(C_w(K,X)\big) = n(X) \quad \text{(López-Martín-Merí, 2007)}$$

- $n(\cdot)$ is continuous with respect to the Banach-Mazur distance (Finet-Martín-Payá, 2003)
- $n(X^*) \leqslant n(X)$ holds for every Banach space X and the inequality can be strict (Boyko-Kadets-Martín-Werner,2007)
- X real with $\dim(X) = \infty$ and $n(X) = 1 \implies \ell_1 \subset X^*$ (Avilés-Kadets-Martín-Merí-Shepelska, 2010)

Possible extension I: Spear operators

Spear operators (Ardalani, 2014)

$G \in \mathcal{L}(X,Y)$ is a spear operator iff $\max\limits_{|w|=1} \|G + w\,T\| = 1 + \|T\| \ \forall T \in \mathcal{L}(X,Y)$.

★ Some examples:

- Id_X when $n(X) = 1$,
- the Fourier transform,
- the inclusion $A(\mathbb{D}) \hookrightarrow C(\mathbb{T})$.

★ Some consequences:

- $G : X \longrightarrow Y$ real spear with infinite rank $\implies X^* \supset \ell_1$,
- $G : X \longrightarrow Y$ real spear, infinite rank, X RNP $\implies Y \supset c_0$ or $Y \supset \ell_1$,
- $G : X \longrightarrow Y$ spear, $\dim(X) > 1$
  $\implies X^*$ not strictly convex nor smooth, B_X does not contain WLUR points.

V. Kadets, M. Martín, J. Merí, A. Pérez
Spear operators between Banach spaces
Lecture Notes of Mathematics 2205 (2018)

Possible extension II: numerical index with respect to an operator

Numerical range and radius with respect to G (Ardalani, 2014)

$G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$, $T \in \mathcal{L}(X,Y)$

$$V_G(T) = \bigcap_{\delta>0} \overline{\{y^*(Tx) \ : \ x \in S_X,\, y^* \in S_{Y^*},\, \operatorname{Re} y^*(Gx) > 1-\delta\}}$$

$$v_G(T) = \sup\{|\lambda| \ : \ \lambda \in V_G(T)\}$$

★ $v_{\mathrm{Id}}(T) = v(T)\ \forall T \in \mathcal{L}(X)$

Numerical index with respect to G

$$n_G(X,Y) = \inf\{v_G(T) \ : \ T \in S_{\mathcal{L}(X,Y)}\} = \max\{k \geqslant 0 \ : \ k\|T\| \leqslant v_G(T)\}$$

★ $n_{\mathrm{Id}}(X,X) = n(X)$.

Work in progress...

The second numerical index

Section 2

The base field does matter for the numerical index

(Bohnenblust-Karlin, Glickfeld-1970)

$n(X) \geqslant 1/\mathrm{e}$ for every complex Banach space X

Examples in the real case

- $n(H) = 0$ for H real Hilbert space with $\dim(H) \geqslant 2$
- $n(X_{\mathbb{R}}) = 0$ for X complex Banach space
- But there is X such that $n(X) = 0$ and v is a norm (Martín-Payá, 2000)

In the first two cases there is $T \in \mathcal{L}(X) \setminus \{0\}$ with $v(T) = 0$:

- $(x_1, x_2, x_3, \ldots) \longmapsto (-x_2, x_1, 0, \ldots)$,
- $x \longmapsto ix$

Observation

$$v(T) = 0 \iff \exp(\rho T) \text{ is an onto isometry for every } \rho \in \mathbb{R}$$

The second numerical index

Lie Algebra

X real Banach space

$$\mathcal{Z}(X) := \{S \in \mathcal{L}(X) \ : \ v(S) = 0\} \qquad \text{(it is a closed subspace of } \mathcal{L}(X)\text{)}$$

Then, for all $T + \mathcal{Z}(X) \in \mathcal{L}(X)/\mathcal{Z}(X)$ we may consider two norms:

$$\|T + \mathcal{Z}(X)\| := \inf\big\{\|T - S\| \ : \ S \in \mathcal{Z}(X)\big\}$$
$$v(T + \mathcal{Z}(X)) := \inf\big\{v(T - S) \ : \ S \in \mathcal{Z}(X)\big\} = v(T)$$

It is immediate that $v(T) \leqslant \|T + \mathcal{Z}(X)\|$ for every $T \in \mathcal{L}(X)$

Second numerical index

$$\begin{aligned} n'(X) &:= \inf\big\{v(T) \ : \ T \in \mathcal{L}(X)\,, \|T + \mathcal{Z}(X)\| = 1\big\} \\ &= \max\big\{k \geqslant 0 \ : \ k\|T + \mathcal{Z}(X)\| \leqslant v(T) \ \forall T \in \mathcal{L}(X)\big\} \end{aligned}$$

Obviously $0 \leqslant n'(X) \leqslant 1$

The second numerical index

Observations

- If $\mathcal{Z}(X) = \{0\}$ (in particular if $n(X) > 0$), then $n'(X) = n(X)$
- $n(X) \leqslant n'(X)$ (observe that $v(T) \leqslant \|T + \mathcal{Z}(X)\| \leqslant \|T\|$)
- On $\mathcal{L}(X)/\mathcal{Z}(X)$, both $\| \cdot + \mathcal{Z}(X)\|$ and $v(\cdot)$ are norms

Some examples

- $n'(X) > 0$ when X is finite-dimensional
- But there is a Banach space X with $n(X) = 0$ and $n'(X) = 0$

Further observation

There is no third numerical index

Main example

Theorem

Let H be a Hilbert space. Then, $n'(H) = 1$.

Proof

Fixed $T \in \mathcal{L}(H)$ we have to show that

$$v(T) = \|T + \mathcal{Z}(H)\| \quad \left(= \left\| \frac{T + T^*}{2} \right\| \right)$$

Facts

- $S \in \mathcal{Z}(H) \iff S = -S^*$
- $T = T^* \implies v(T) = \|T\|$

Absolute norm on $\mathbb{R}^m$ and absolute sum of Banach spaces

Absolute norm

A norm $\|\cdot\|$ on $\mathbb{R}^m$ is absolute if

- $\|(a_1,\dots,a_m)\| = \big\|(|a_1|,\dots,|a_m|)\big\|$ for every $(a_1,\dots,a_m)\in\mathbb{R}^m$.
- $\|e_k\| = 1$ for every $k=1,\dots,m$ where $e_k = (0,\dots,0,\underbrace{1}_{k},0,\dots,0)$.

Absolute sum

Let E be $\mathbb{R}^m$ endowed with an absolute norm. We write $\big[X_1\oplus\cdots\oplus X_m\big]_E$ for the E-sum of the Banach spaces $X_1,\dots,X_m$. That is, the space $X_1\times\cdots\times X_m$ endowed with the complete norm $\|(x_1,\dots,x_n)\| = \big\|(\|x_1\|,\dots,\|x_m\|)\big\|_E$.

When E is $\mathbb{R}^2$ endowed with an absolute norm $\|\cdot\|_a$ we just write $X_1\oplus_a X_2 = [X_1\oplus X_2]_E$.

Relationship of n' with absolute sums

Proposition

Let $X = X_1 \oplus_a X_2$, where $\oplus_a \neq \oplus_2$ is an absolute sum. Then,

$$n'(X) \leqslant \min\{n'(X_1), n'(X_2)\}.$$

Corollary

Let $\{X_\lambda \,:\, \lambda \in \Lambda\}$ be a family of Banach spaces, $1 \leqslant p \leqslant \infty$ with $p \neq 2$. Then

$$n'\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_p}\Big) \leqslant \inf\{n'(X_\lambda) \,:\, \lambda \in \Lambda\}.$$

Examples (equality does not hold)

$$n'(\ell_2^2 \oplus_\infty \mathbb{R}) \leqslant \tfrac{\sqrt{3}}{2} < 1 \qquad \text{and} \qquad n'(\ell_2^2 \oplus_1 \mathbb{R}) \leqslant \tfrac{\sqrt{3}}{2} < 1$$

Relationship of n' with absolute sums

Proposition

Let X_1, X_2 be Banach spaces and write $X = X_1 \oplus_\infty X_2$ or $X = X_1 \oplus_1 X_2$.

- If $n(X_1) > 0$ and $n(X_2) > 0$, then $n'(X) = n(X) = \min\{n(X_1), n(X_2)\}$.
- If $n(X_1) > 0$ and $n(X_2) = 0$, then $n'(X) \geqslant \min\left\{n(X_1), \frac{n'(X_2)}{n'(X_2)+1}\right\}$.
- If $n(X_1) = 0$ and $n(X_2) = 0$, then

$$n'(X) \geqslant \min\left\{\frac{n'(X_1)}{n'(X_1)+1}, \frac{n'(X_2)}{n'(X_2)+1}\right\}.$$

Example

$$\tfrac{1}{2} \leqslant n'(\ell_2^2 \oplus_\infty \mathbb{R}) \leqslant \tfrac{\sqrt{3}}{2} \qquad \text{and} \qquad \tfrac{1}{2} \leqslant n'(\ell_2^2 \oplus_1 \mathbb{R}) \leqslant \tfrac{\sqrt{3}}{2}$$

A family of examples

Example

For every $\theta \in (0, 1/2]$, there is a four-dimensional Banach space X_θ such that $n(X_\theta) = 0$ and $n'(X_\theta) = \theta$.

Let Y_θ be a two-dimensional space with $n(Y_\theta) = \theta$ and take $X_\theta = Y_\theta \oplus_\infty \ell_2^2$. Then:

- $n(X_\theta) \leqslant n(\ell_2^2) = 0$
- $n'(X_\theta) \leqslant n'(Y_\theta) = n(Y_\theta) = \theta$
- $n'(X_\theta) \geqslant \min\left\{ n(Y_\theta), \dfrac{n'(\ell_2^2)}{n'(\ell_2^2)+1} \right\} = \min\left\{\theta, \frac{1}{2}\right\} = \theta$

More examples (low dimensions)

- $\dim(X) = 2,\ n(X) = 0 \implies n'(X) = 1,$
- $\{n'(X) \ : \ n(X) = 0,\ \dim(X) = 3\} \supset [1/\mathrm{e}, 1/2]$ and it is NOT an interval,
- $\{n'(X) \ : \ n(X) = 0,\ \dim(X) = 4\} \supset (0, 1/2].$

n' is not continuous with respect Banach-Mazur distance

Example

For $1 < p < \infty$, let $X_p = \ell_p^2 \oplus_p \ell_2^2$ (observe that $n(X_p) = 0$ for every p).

- Then $n'(X_p) \leqslant n'(\ell_p^2) = n(\ell_p^2) \quad$ for $p \neq 2$
- Therefore $\lim\limits_{p\to 2} n'(X_p) \leqslant \lim\limits_{p\to 2} n(\ell_p) = 0$
- On the other hand $n'(X_2) = n'(\ell_2^4) = 1$

Another example

For $1 < p < \infty$, let $X_p = \ell_p^2 \oplus_1 \ell_2^2$ (observe that $n(X_p) = 0$ for every p).

- Then $n'(X_p) \leqslant n'(\ell_p^2) = n(\ell_p^2) \quad$ for $p \neq 2$
- Therefore $\lim\limits_{p\to 2} n'(X_p) \leqslant \lim\limits_{p\to 2} n(\ell_p) = 0$
- On the other hand $\frac{1}{2} \leqslant n'(X_2) < 1$

Observation

Continuity of $n'(\cdot)$ holds if $\mathcal{Z}(X)$ does not change

Relationship of the indices with absolute sums (revisited)

Let E be $\mathbb{R}^m$ endowed with an absolute norm, let $X_1, \ldots, X_m$ be Banach spaces and $X = \left[X_1 \oplus \cdots \oplus X_m\right]_E$.

Diagonal operator

$S \in \mathcal{L}(X)$ is diagonal if $P_k S I_j = 0$ for $j, k \in \{1, \ldots, m\}$ with $j \neq k$.
If moreover $S \in \mathcal{Z}(X)$, it follows that

$$P_k S I_k \in \mathcal{Z}(X_k) \qquad \forall k \in \{1, \ldots, m\}.$$

Positive elements and positive operators

- $a \in E$ is positive if $a_k \geqslant 0$ for every $k = 1, \ldots, m$.
- $U \in \mathcal{L}(E)$ is positive if $U(a)$ is positive for every positive $a \in E$.

Relationship of the indices with absolute sums (revisited)

Let E be $\mathbb{R}^m$ endowed with an absolute norm, let $X_1, \ldots, X_m$ be Banach spaces and $X = \left[X_1 \oplus \cdots \oplus X_m\right]_E$.

Proposition (Martín-Merí-Popov-Randrianantoanina, 2011)

$$n\left(X\right) \leqslant \min\left\{n(X_1), \ldots, n(X_m)\right\}$$

Proposition

Suppose that every $S \in \mathcal{Z}(X)$ is diagonal. Then,

$$n'\left(X\right) \leqslant \min\left\{n'(X_1), \ldots, n'(X_m)\right\}$$

Relationship of the indices with positive operators on E

Let E be $\mathbb{R}^m$ endowed with an absolute norm, let $X_1, \ldots, X_m$ be Banach spaces and $X = \left[X_1 \oplus \cdots \oplus X_m\right]_E$.

Proposition

For every positive operator $U \in \mathcal{L}(E)$ one has

$$n(X) \leqslant \frac{v(U)}{\|U\|}$$

Proposition

Suppose that every $S \in \mathcal{Z}(X)$ is diagonal. Let $U \in \mathcal{L}(E)$ be a positive operator so that there is $a \in E$ satisfying $\|Ua\| = \|U\|$ and $\operatorname{supp}(Ua) \cap \operatorname{supp}(a) = \emptyset$. Then,

$$n'(X) \leqslant \frac{v(U)}{\|U\|}$$

Spaces with absolute norm and $n'(X) = 1$

Theorem

Let X be $\mathbb{R}^m$ endowed with an absolute norm. Suppose that $n(X) = 0$ and $n'(X) = 1$. Then, X is a Hilbert space.

Observation

The result is more general and it can be extended to Banach spaces with (long) one-unconditional basis.

Sketch of the proof

- As $n(X) = 0$ and X has an absolute norm, we may use a result of Rosenthal (80's) to obtain:
  ★ There are $\ell \in \mathbb{N}$, Hilbert spaces $H_1, \ldots, H_\ell$, and $E = (\mathbb{R}^\ell, |\cdot|_a)$ such that $X = \left[H_1 \oplus \cdots \oplus H_\ell\right]_E$ and
  - $\dim(H_1) \geqslant 2$
  - Every $S \in \mathcal{Z}(X)$ is diagonal (with respect to this decomposition)
- Suppose for contradiction that $\ell \neq 1$.

Spaces with absolute structure and $n'(X) = 1$

Sketch of the proof

- For $k \in \{2, \ldots, \ell\}$ let $E_{1,k}$ be the linear span of $\{e_1, e_k\}$ in E. Using that $n'(X) = 1$, it is possible to prove that $E_{1,k} = \ell_2^2$.
- Fix $j \in \{2, \ldots, \ell\}$ and show that the positive operator $U \in \mathcal{L}(E)$ given by $U = e_1^* \otimes e_j$ satisfies

$$\|U\| = 1 \qquad \text{and} \qquad v(U) < 1.$$

- Therefore $n'(X) \leqslant \dfrac{v(U)}{\|U\|} < 1$ which gives the desired contradiction.

Vector valued spaces

Proposition

Let X be a Banach space, L locally compact Hausdorff, K compact Hausdorff, Ω completely regular Hausdorff, and μ positive measure. Then

- $n'\big(C_0(L,X)\big) \leqslant n'(X)$
- $n'\big(C_w(K,X)\big) \leqslant n'(X)$
- $n'\big(C_b(\Omega,X)\big) \leqslant n'(X)$
- $n'\big(L_1(\mu,X)\big) \leqslant n'(X)$
- $n'\big(L_\infty(\mu,X)\big) \leqslant n'(X)$

Example

Let K be a compact Hausdorff topological space with at least two points. Then

$$n'(C(K,\ell_2^2)) \leqslant \frac{\sqrt{3}}{2} < 1.$$

Duality

Observation

Let X be a Banach space. If every element in $\mathcal{Z}(X^*)$ is the transpose of an element in $\mathcal{Z}(X)$ then $n'(X^*) \leqslant n'(X)$

Proposition

Suppose that one of the following holds

- The norm of X^* is Fréchet-smooth on a dense set (e.g. $X = \ell_\infty$);
- B_X is the closed convex hull of the $w - \|\cdot\|$ continuity points of Id (in particular, X RNP, X CPCP, X LUR, X has a Kadec norm, $X = X_1 \widetilde{\otimes_\pi} X_2$ where X_1, X_2 RNP, or $X = L(R)$ where R is reflexive);
- $X^* \not\supseteq \ell_1$;
- X is isomorphic to a subspace of a separable L-embedded space;
- X is the (unique) predual of a von Neumann algebra.

Then $n'(X^*) \leqslant n'(X)$

Duality II

On the other hand,

Example

Given $0 \leqslant \alpha \leqslant \beta \leqslant 1/2$, there is a Banach space $X_{\alpha,\beta}$ with $n(X_{\alpha,\beta}) = 0$ such that

$$n'(X_{\alpha,\beta}) = \beta \quad \text{and} \quad n'(X^*_{\alpha,\beta}) = \alpha.$$

An application to BPB-property for numerical radius

Section 3

An application

Definition (Guirao-Kozhushkina, 2013; Kim-Lee-Martín,2014)

X Banach space.

- X has the Bishop-Phelps-Bollobás property for numerical radius if for every $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ such that whenever $T \in L(X)$ and $(x, x^*) \in \Pi(X)$ satisfy $v(T) = 1$ and $|x^*Tx| > 1 - \eta(\varepsilon)$, there exist $S \in L(X)$ and $(y, y^*) \in \Pi(X)$ such that

$$v(S) = |y^*Sy| = 1, \quad \|T - S\| < \varepsilon, \quad \|x - y\| < \varepsilon, \quad \text{and} \quad \|x^* - y^*\| < \varepsilon.$$

- X has the weak-Bishop-Phelps-Bollobás property for numerical radius if for every $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ such that whenever $T \in L(X)$ and $(x, x^*) \in \Pi(X)$ satisfy $v(T) = 1$ and $|x^*Tx| > 1 - \eta(\varepsilon)$, there exist $S \in L(X)$ and $(y, y^*) \in \Pi(X)$ such that

$$v(S) = |y^*Sy|, \quad \|T - S\| < \varepsilon, \quad \|x - y\| < \varepsilon, \quad \text{and} \quad \|x^* - y^*\| < \varepsilon.$$

An application

Proposition (Kim-Lee-Martín,2014)

X Banach space with $n(X) > 0$. Then, the weak-Bishop-Phelps-Bollobás property for numerical radius implies the Bishop-Phelps-Bollobás property for numerical radius

Actually...

X Banach space with $n'(X) > 0$. Then, the weak-Bishop-Phelps-Bollobás property for numerical radius implies the Bishop-Phelps-Bollobás property for numerical radius

Proposition (Kim-Lee-Martín,2014)

X uniformly convex and uniformly smooth $\implies$ X has the weak-Bishop-Phelps-Bollobás property for numerical radius

Corollary

Hilbert spaces have the Bishop-Phelps-Bollobás property for numerical radius

Open problems

Section 4

Some open problems

- Which is the set of values of $n'(X)$ for Banach spaces X with $n(X) = 0$? Does it cover the interval $[0, 1]$?
  - We know that it covers the interval $[0, 1/2]$ and contains 1. It can be done (except for the value cero) with four-dimensional spaces.
  - If $\dim(X) = 2$ and $n(X) = 0$ then $X = \ell_2^2$.
  - If $\dim(X) = 3$ and $n(X) = 0$ then $X = \ell_2^2 \oplus_a \mathbb{R}$. In this case we know that it is NOT an interval.
- Is $n'(X \oplus_2 Y) \leqslant \min\{n'(Y), n'(W)\}$?
- Let μ be a positive measure, X a Banach space and $1 < p < \infty$. Is it true that $n'(L_p(\mu, X)) \leqslant n'(X)$?
- Is $n'(X^*) \leqslant n'(X)$ for every Banach space X?
- Are Hilbert spaces the unique Banach spaces X with $n(X) = 0$ and $n'(X) = 1$?
- X complex, what is the meaning of $n'(X_{\mathbb{R}})$?
- $X = \mathbb{C} \oplus_a \mathbb{C}$. What is the value of $n'(X_{\mathbb{R}})$?
  - $\oplus_a = \oplus_2 \implies n'(X_{\mathbb{R}}) = 1$,
  - $\oplus_a = \oplus_1 \implies \frac{1}{2} \leqslant n'(X_{\mathbb{R}}) \leqslant \frac{\sqrt{3}}{2}$.