The group of isometries of a Banach space and duality

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Outline

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   - Relationship with semigroups of operators

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   - Numerical index and duality
Notation and objective

Basic notation

- $X$ Banach space over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$).
- $S_X$ unit sphere, $B_X$ unit ball,
- $X^*$ dual space,
- $L(X)$ bounded linear operators,
- $\text{Iso}(X)$ surjective linear isometries,
- $T^* \in L(X^*)$ adjoint operator of $T \in L(X)$.

Main Objective

We **construct** a real Banach space $X$ such that

- $\text{Iso}(X)$ does not contain uniformly continuous one-parameter semigroups.
- But $\text{Iso}(X^*)$ contains infinitely many uniformly continuous one-parameter semigroups.
The tool: numerical range of operators

F. F. Bonsall and J. Duncan
*Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras.*

F. F. Bonsall and J. Duncan
*Numerical Ranges II.*

H. P. Rosenthal
The Lie algebra of a Banach space.
Hilbert spaces

Hilbert space Numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix
  \[ W(A) = \left\{ (Ax \mid x) : x \in \mathbb{K}^n, (x \mid x) = 1 \right\}. \]
- $H$ real or complex Hilbert space, $T \in L(H)$,
  \[ W(T) = \left\{ (Tx \mid x) : x \in H, \|x\| = 1 \right\}. \]

Some properties

$H$ Hilbert space, $T \in L(H)$:

- $W(T)$ is convex.
- In the complex case, $\overline{W(T)}$ contains the spectrum of $T$.
- If, moreover, $T$ is normal, $\overline{W(T)} = \overline{\text{co} \ Sp(T)}$. 
Banach spaces

Banach space numerical range (Bauer 1962; Lumer, 1961)

$X$ Banach space, $T \in L(X)$,

$$V(T) = \left\{ x^*(Tx) : x^* \in S_{X^*}, \; x \in S_X, \; x^*(x) = 1 \right\}$$

Some properties

$X$ Banach space, $T \in L(X)$:

- $V(T)$ is connected (not necessarily convex).
- In the complex case, $\overline{V(T)}$ contains the spectrum of $T$.
- Actually,

$$\overline{\text{co}}\; Sp(T) = \bigcap \overline{\text{co}}\; V(T),$$

the intersection taken over all numerical ranges $V(T)$ corresponding to equivalent norms on $X$.
Numerical radius

\(X\) real or complex Banach space, \(T \in L(X)\),

\[v(T) = \sup \{ | \lambda | : \lambda \in V(T) \} .\]

- \(v\) is a seminorm with \(v(T) \leq \|T\|\).
- \(v(T) = v(T^*)\) for every \(T \in L(X)\).

Numerical index (Lumer, 1968)

\(X\) real or complex Banach space,

\[n(X) = \inf \{ v(T) : T \in L(X), \|T\| = 1 \} \]

\[= \max \{ k \geq 0 : k\|T\| \leq v(T) \ \forall \ T \in L(X) \} .\]

Remarks

- \(n(X) = 1\) iff \(v(T) = \|T\|\) for every \(T \in L(X)\).
- If there is \(T \neq 0\) with \(v(T) = 0\), then \(n(X) = 0\).
- The converse is not true.
A motivating example

A real or complex $n \times n$ matrix. TFAE:

- $A$ is skew-adjoint (i.e. $A^* = -A$).
- $\text{Re}(Ax \mid x) = 0$ for every $x \in H$.
- $B = \exp(\rho A)$ is unitary for every $\rho \in \mathbb{R}$ (i.e. $B^* B = \text{Id}$).

In term of Hilbert spaces

$H$ ($n$-dimensional) Hilbert space, $T \in L(H)$. TFAE:

- $\text{Re} W(T) = \{0\}$.
- $\exp(\rho T) \in \text{Iso}(H)$ for every $\rho \in \mathbb{R}$.

For general Banach spaces

$X$ Banach space, $T \in L(X)$. TFAE:

- $\text{Re} V(T) = \{0\}$.
- $\exp(\rho T) \in \text{Iso}(X)$ for every $\rho \in \mathbb{R}$.
Characterizing uniformly continuous semigroups of operators

**Theorem**

Let $X$ be a real or complex Banach space, $T \in L(X)$. TFAE:

- $\text{Re } V(T) = \{0\}$.
- $\| \exp(\rho T) \| \leq 1$ for every $\rho \in \mathbb{R}$.
- $\{ \exp(\rho T) : \rho \in \mathbb{R}_0^+ \} \subset \text{Iso}(X)$.
- $T$ belongs to the tangent space of $\text{Iso}(X)$ at $\text{Id}$, i.e., there exists a function $f : [-1, 1] \rightarrow \text{Iso}(X)$ with $f(0) = \text{Id}$ and $f'(0) = T$.
- $\lim_{\rho \to 0} \frac{\| \text{Id} + \rho T \| - 1}{\rho} = 0$, i.e., the derivative or the norm of $L(X)$ at $\text{Id}$ in the direction of $T$ is null.

**Main consequence for us**

If $X$ is a real Banach space with $n(X) > 0$, then $\text{Iso}(X)$ is “small”:

- it does not contain any uniformly continuous one-parameter semigroups,
- the tangent space of $\text{Iso}(X)$ at $\text{Id}$ is zero.
The example

M. Martín
The group of isometries of a Banach space and duality.
preprint.
The main example

The construction

$E$ separable Banach space. We construct a Banach space $X(E)$ such that

$$n(X(E)) = 1 \quad \text{and} \quad X(E)^* \equiv E^* \oplus_1 L_1(\mu)$$

The main consequence

Take $E = \ell_2$ (real). Then

- $n(X(\ell_2)) = 1$, so $\text{Iso}(X(\ell_2))$ is “small”.
- Since $X(\ell_2)^* \equiv \ell_2 \oplus_1 L_1(\mu)$, given $S \in \text{Iso}(\ell_2)$, the operator

$$T = \begin{pmatrix} S & 0 \\ 0 & \text{Id} \end{pmatrix}$$

is a surjective isometry of $X(\ell_2)^*$.
- Therefore, $\text{Iso}(X(\ell_2)^*)$ contains infinitely many semigroups of isometries.
Sketch of the construction I

Define (viewing \( E \hookrightarrow C[0, 1] \))

\[
Y = \left\{ f \in C([0, 1] \times [0, 1]) : f(\cdot, 0) = 0 \right\} \\
X(E) = \left\{ f \in C([0, 1] \times [0, 1]) : f(\cdot, 0) \in E \right\}
\]

We need

\[
X(E)^* \equiv E^* \oplus_1 L_1(\mu) \quad \& \quad n(X(E)) = 1
\]

Proving that \( X(E)^* \equiv E^* \oplus_1 L_1(\mu) \)

- \( Y \) is an \( M \)-ideal of \( C([0, 1] \times [0, 1]) \), so \( Y \) is an \( M \)-ideal of \( X(E) \).
- This means that \( X(E)^* \equiv Y^\perp \oplus_1 Y^* \).
- \( Y^* \equiv L_1(\mu) \) for some measure \( \mu \); \( Y^\perp \equiv (X(E)/Y)^* \).
- Define \( \Phi : X(E) \longrightarrow E \) by \( \Phi(f) = f(\cdot, 0) \).
  - \( \|\Phi\| \leq 1 \) and \( \ker \Phi = Y \).
  - \( \tilde{\Phi} : X(E)/Y \longrightarrow E \) is a surjective isometry since:
    - \( \{ g \in E : \|g\| < 1 \} \subseteq \Phi(\{ f \in X(E) : \|f\| < 1 \}) \).
- Therefore, \( Y^\perp \equiv (X(E)/Y)^* \equiv E^* \).
Sketch of the construction II

Define (viewing $E \hookrightarrow C[0,1]$)

\[ Y = \{ f \in C([0,1] \times [0,1]) : f(\cdot,0) = 0 \} \]
\[ X(E) = \{ f \in C([0,1] \times [0,1]) : f(\cdot,0) \in E \} \]

We need

\[ X(E)^* \equiv E^* \oplus_1 L_1(\mu) \quad \& \quad n(X(E)) = 1 \]

Proving that $n(X(E)) = 1$

- Fix $T \in L(X(E))$. Find $f_0 \in X(E)$ and $\xi_0 \in ]0,1] \times [0,1]$ such that $|[Tf_0](\xi_0)| \sim \|T\|$.
- Consider the non-empty open set
  \[ V = \{ \xi \in ]0,1] \times [0,1] : f_0(\xi) \sim f_0(\xi_0) \} \]
  and find $\varphi : [0,1] \times [0,1] \rightarrow [0,1]$ continuous with $\text{supp}(\varphi) \subset V$ and $\varphi(\xi_0) = 1$.
- Write $f_0(\xi_0) = \lambda \omega_1 + (1-\lambda)\omega_2$ with $|\omega_i| = 1$, and consider the functions
  \[ f_i = (1-\varphi)f_0 + \varphi \omega_i \quad \text{for} \quad i = 1,2. \]
- Then, $f_i \in Y \subset X(E)$, $\|f_i\| \leq 1$, and
  \[ \|f_0 - (\lambda f_1 + (1-\lambda)f_2)\| = \|\varphi f_0 - \varphi f_0(\xi_0)\| \sim 0. \]
- Therefore, there is $i \in \{1,2\}$ such that $|[T(f_i)](\xi_0)| \sim \|T\|$, but now $|f_i(\xi_0)| = 1$.
- Equivalently,
  \[ |\delta_{\xi_0}(T(f_i))| \sim \|T\| \quad \text{and} \quad |\delta_{\xi_0}(f_i)| = 1, \]
  meaning that $v(T) \sim \|T\|$.
Some related results


Isometries in finite-dimensional spaces

Theorem

Let $X$ be a finite-dimensional real space. TFAE:

- $\text{Iso}(X)$ is infinite.
- $n(X) = 0$.
- There is $T \in L(X)$, $T \neq 0$, with $\nu(T) = 0$.

Examples of spaces of this kind

2. $X_{\mathbb{R}}$, the real space subjacent to any complex space $X$.
3. An absolute sum of any real space and one of the above.
4. Moreover, if $X = X_0 \oplus X_1$ where $X_1$ is complex and
   
   \[ \|x_0 + e^{i\theta}x_1\| = \|x_0 + x_1\| \quad (x_0 \in X_0, \ x_1 \in X_1, \ \theta \in \mathbb{R}). \]

(Note that the other 3 cases are included here)

Question

Can every Banach space $X$ with $n(X) = 0$ be decomposed as in 4?
Infinite-dimensional case

There is an infinite-dimensional real Banach space $X$ with $n(X) = 0$ but $X$ is polyhedral. In particular, $X$ does not contain $\mathbb{C}$ isometrically.

The example is

$$X = \left[ \bigoplus_{n \geq 2} X_n \right]_{c_0}$$

$X_n$ is the two-dimensional space whose unit ball is the regular polygon of $2n$ vertices.

Note

Such an example is not possible in the finite-dimensional case.
Finite-dimensional case

$X$ finite-dimensional real space. TFAE:

- $n(X) = 0$.
- $X = X_0 \oplus X_1 \oplus \cdots \oplus X_n$ such that
  - $X_0$ is a (possible null) real space,
  - $X_1, \ldots, X_n$ are non-null complex spaces,

there are $\rho_1, \ldots, \rho_n$ rational numbers, such that

$$\|x_0 + e^{i\rho_1 \theta} x_1 + \cdots + e^{i\rho_n \theta} x_n\| = \|x_0 + x_1 + \cdots + x_n\|$$

for every $x_i \in X_i$ and every $\theta \in \mathbb{R}$.

Example

$$X = (\mathbb{R}^4, \| \cdot \|), \|(a, b, c, d)\| = \frac{1}{4} \int_0^{2\pi} \left| \text{Re} \left( e^{2it} (a + ib) + e^{it} (c + id) \right) \right| \, dt.$$  
Then $n(X) = 0$ but the unique possible decomposition is $X = \mathbb{C} \oplus \mathbb{C}$ with

$$\|e^{it} x_1 + e^{2it} x_2\| = \|x_1 + x_2\|.$$
The Lie-algebra of a Banach space

**Lie-algebra**

$X$ real Banach space, $\mathcal{Z}(X) = \{ T \in L(X) : v(T) = 0 \}$.

- When $X$ is finite-dimensional, $\text{Iso}(X)$ is a Lie-group and $\mathcal{Z}(X)$ is the tangent space (i.e. its Lie-algebra).

**Remark**

If $\dim(X) = n$, then

$$0 \leq \dim(\mathcal{Z}(X)) \leq \frac{n(n-1)}{2}.$$  

**An open problem**

Given $n \geq 3$, which are the possible $\dim(\mathcal{Z}(X))$ over all $n$-dimensional $X$'s?

**Observation (Javier Merí, PhD)**

When $\dim(X) = 3$, $\dim(\mathcal{Z}(X))$ cannot be 2.
Numerical index of Banach spaces

Numerical index (Lumer, 1968)

Let $X$ be a real or complex Banach space, then

$$n(X) = \inf \{\nu(T) : T \in L(X), \|T\| = 1\} = \max\{k \geq 0 : k\|T\| \leq \nu(T) \forall T \in L(X)\}.$$ 

Some examples

1. $C(K), L_1(\mu)$ have numerical index 1.
2. $H$ Hilbert space, dim$(H) > 1$, then
   $$n(H) = 0 \quad \text{real case} \quad n(H) = \frac{1}{2} \quad \text{complex case}.$$ 
3. $n(L_p[0, 1]) = n(\ell_p)$ but both are unknown.
4. If $X_n$ is the two-dimensional space such that $B_{X_n}$ is a $2n$-polygon, then
   $$n(X_n) = \tan\left(\frac{\pi}{2n}\right) \quad \text{if } n \text{ is even} \quad n(X_n) = \sin\left(\frac{\pi}{2n}\right) \quad \text{if } n \text{ is odd}.$$ 
5. If $X$ is a $C^*$-algebra or the predual of a von Neumann algebra, then
   $n(X) = 1$ if the algebra is commutative and $n(X) = 1/2$ otherwise.
Numerical index and duality

Proposition

X Banach space.

- \( \nu(T^*) = \nu(T) \) for every \( T \in L(X) \).
- Therefore, \( n(X^*) \leq n(X) \).

Question

Is it always \( n(X) = n(X^*) \)?

Another example

- It is known: if \( X \) or \( X^* \) is a \( C^* \)-algebra, then \( n(X) = n(X^*) \).
- Consider \( Y = X(K(\ell_2)) \). Then
  \[
  n(Y) = 1 \quad \text{and} \quad Y^* \equiv K(\ell_2)^* \oplus_1 L_1(\mu).
  \]

Then, \( Y^{**} \equiv L(\ell_2) \oplus_\infty L_\infty(\mu) \) is a \( C^* \)-algebra but \( n(Y^*) \leq n(K(\ell_2)) = 1/2 \).
Numerical index and duality

Remark

In the example $n(X(\ell_2)) > n(X(\ell_2)^*)$, one finds that $X(\ell_2)^*$ has another predual (namely, $\ell_2 \oplus \infty Y$) for which the numerical index coincides with the numerical index of its dual.

Open problems

We look for sufficient conditions assuring the equality between the numerical index of a Banach space and the one of its dual.

1. Asplundness is not such a property.
2. What’s about RNP?
3. What’s about if $X^*$ has a unique predual?
4. What’s about if $X$ does not contain a copy of $c_0$?

Theorem

If $X$ is a separable Banach space containing (an isomorphic copy of) $c_0$, then there is an equivalent norm $|\cdot|$ on $X$ such that

$$n((X, |\cdot|)^*) < n((X, |\cdot|)).$$
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