Norm equalities for operators

Miguel Martín

(joint work with Vladimir Kadets and Javier Merí)

Departamento de Análisis Matemático

http://www.ugr.es/local/mmartins

March 4th, 2006 – La Manga del Mar Menor, Murcia
Introduction

In a Banach space $X$ with the Radon-Nikodým property the unit ball has many denting points.

$x \in S_X$ is a denting point of $B_X$ if for every $\varepsilon > 0$ one has

$$x \notin \overline{co}(B_X \setminus (x + \varepsilon B_X)).$$

$C[0, 1]$ and $L_1[0, 1]$ have an extremely opposite property: for every $x \in S_X$ and every $\varepsilon > 0$

$$\overline{co} \left( B_X \setminus (x + (2 - \varepsilon)B_X) \right) = B_X.$$

This geometric property is equivalent to a property of operators on the space.
The Daugavet property

The Daugavet equation

\[ X \text{ Banach space, } T \in L(X) \]
\[ \| \text{Id} + T \| = 1 + \| T \| \]  
(DE)

The Daugavet property

A Banach space \( X \) is said to have the **Daugavet property** iff every rank-one operator on \( X \) satisfies (DE).

- Then, every weakly compact operator on \( X \) satisfies (DE).
- **Geometric characterization**: \( X \) has the Daugavet property iff for each \( x \in S_X \)
  \[ \overline{\text{co}} \left( B_X \setminus (x + (2 - \varepsilon)B_X) \right) = B_X. \]

Introduction and motivation

The equations

Some questions

The Daugavet property

Daugavet type inequalities

Some questions

The Daugavet property

Some propaganda

Suppose $X$ has the Daugavet property. Then:

- $X$ does not have the Radon-Nikodým property.
  
  \textit{(Wojtaszczyk, 1992)}

- Every weakly-open subset of $B_X$ has diameter 2.
  
  \textit{(Shvidkoy, 2000)}

- $X$ contains a copy of $\ell_1$. $X^*$ contains a copy of $L_1[0,1]$.
  
  \textit{(Kadets–Shvidkoy–Sirotkin–Werner, 2000)}

- $X$ does not have unconditional basis.
  
  \textit{(Kadets, 1996)}

- $X$ does not embed into a unconditional sum of Banach spaces without a copy of $\ell_1$.
  
  \textit{(Shvidkoy, 2000)}
### Daugavet type inequalities

#### Commutative $L_p$ spaces

- **Benyamini–Lin, 1985:**
  For every $1 < p < \infty$, $p \neq 2$, there exists $\psi_p : (0, \infty) \to (0, \infty)$ such that
  \[ \| \text{Id} + T \| \geq \psi_p(\| T \|) \]
  for every compact operator $T$ on $L_p[0, 1]$.

- If $p = 2$, then there is a non-null compact $T$ on $L_2[0, 1]$ such that
  \[ \| \text{Id} + T \| = 1. \]

- **Boyko–Kadets, 2004:**
  If $\psi_p$ is the best possible function above, then
  \[ \lim_{p \to 1^+} \psi_p(t) = t \quad (t > 0). \]
## Daugavet type inequalities

### Non-commutative $\mathcal{L}_p$ spaces

- **Oikhberg, 2002:**
  For every $1 < p < \infty$, $p \neq 2$, there exists $k_p > 0$ such that
  \[
  \| \text{Id} + T \| \geq 1 + k_p \min\{\| T \|, \| T \|^2\}
  \]
  for every compact $T$ on $\mathcal{L}_p(\tau)$.

### Spaces of operators

- **Oikhberg, 2005:**
  If $K(\ell_2) \subseteq X \subseteq L(\ell_2)$, then
  \[
  \| \text{Id} + T \| \geq 1 + \frac{1}{8\sqrt{2}} \| T \|
  \]
  for every compact $T$ on $X$. 
The questions

Is any of the previous inequalities an equality?

Even more, is there any norm equality valid for all compact operators on any of the above spaces?

Main question

Study the possibility of finding norm equalities for operators in the spirit of Daugavet equation, valid for all rank-one operators on a Banach space.

We will study three cases:

1. \(|\text{Id} + T| = f(|T|)| for arbitrary \(f\).
2. \(|g(T)| = f(|T|)| for analytic \(g\) and arbitrary \(f\).
3. \(|\text{Id} + g(T)| = f(|g(T)|)| for analytic \(g\) and continuous \(f\).
Equalities of the form $\|\text{Id} + T\| = f(\|T\|)$

**Proposition**

Let $X$ be a real or complex space, $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ an arbitrary function, $a, b \in K$. If the norm equality

$$\|a \text{Id} + b T\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, then

$$f(t) = |a| + |b| t \quad (t \in \mathbb{R}_0^+)$$

If $a \neq 0$, $b \neq 0$, then $X$ has the Daugavet property.

Then, we have to look for Daugavet-type equalities in which $\text{Id} + T$ is replaced by something different.
### Theorem

Let $X$ be a real or complex Banach space with $\dim(X) \geq 2$. Suppose that the norm equality

$$\|g(T)\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, where

- $g : K \to K$ is analytic,
- $f : \mathbb{R}_0^+ \to \mathbb{R}$ is arbitrary.

Then, there are $a, b \in K$ such that

$$g(\zeta) = a + b \zeta \quad (\zeta \in K).$$

### Corollary

Only three norm equalities of the form

$$\|g(T)\| = f(\|T\|)$$

are possible:

- $b = 0$: $\|a \text{Id}\| = |a|$,  
- $a = 0$: $\|b T\| = |b| \|T\|$,  
  (trivial cases)
- $a \neq 0, b \neq 0$: $\|a \text{Id} + b T\| = |a| + |b| \|T\|$,  
  (Daugavet property)
Equalities of the form $\|\Id + g(T)\| = f(\|g(T)\|)$

**Remark**

If $X$ has the Daugavet property and $g$ is analytic, then

$$\|\Id + g(T)\| = |1 + g(0)| - |g(0)| + \|g(T)\|$$

for every rank-one $T \in L(X)$.

- Our aim here is not to show that $g$ has a suitable form,
- but it is to see that for every $g$ another simpler equation can be found.
- From now on, we have to separate the complex and the real case.
Equalities of the form $\|\text{Id} + g(T)\| = f(\|g(T)\|)$

- **Complex case:**

**Proposition**

$X$ complex, $\dim(X) \geq 2$. Suppose that

$$\|\text{Id} + g(T)\| = f(\|g(T)\|)$$

for every rank-one $T \in L(X)$, where

- $g : \mathbb{C} \rightarrow \mathbb{C}$ analytic non-constant,
- $f : \mathbb{R}^+_0 \rightarrow \mathbb{R}$ continuous.

Then

$$\|(1 + g(0))\text{Id} + T\| = |1 + g(0)| - |g(0)| + \|g(0)\text{Id} + T\|$$

for every rank-one $T \in L(X)$.

We obtain two different cases:

- $|1 + g(0)| - |g(0)| \neq 0$ or
- $|1 + g(0)| - |g(0)| = 0$. 

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Equalities of the form $\|\text{Id} + g(T)\| = f(\|g(T)\|)$. Complex case

**Theorem**

If $\text{Re } g(0) \neq -1/2$ and

$$\|\text{Id} + g(T)\| = f(\|g(T)\|)$$

for every rank-one $T$, then $X$ has the Daugavet property.

**Theorem**

If $\text{Re } g(0) = -1/2$ and

$$\|\text{Id} + g(T)\| = f(\|g(T)\|)$$

for every rank-one $T$, then exists $\theta_0 \in \mathbb{R}$ s.t.

$$\|\text{Id} + e^{i\theta_0} T\| = \|\text{Id} + T\|$$

for every rank-one $T \in L(X)$.

**Example**

If $X = C[0, 1] \oplus_2 C[0, 1]$, then

- $\|\text{Id} + e^{i\theta} T\| = \|\text{Id} + T\|$
  for every $\theta \in \mathbb{R}$, rank-one $T \in L(X)$.
- $X$ does not have the Daugavet property.
Equalities of the form \( \|\text{Id} + g(T)\| = f(\|g(T)\|) \). Real case

- **Real case:**

**Remarks**
- The proofs are not valid (we use Picard’s Theorem).
- They work when \( g \) is onto.
- But we do not know what is the situation when \( g \) is not onto, even in the easiest examples:
  - \( \|\text{Id} + T^2\| = 1 + \|T^2\| \),
  - \( \|\text{Id} - T^2\| = 1 + \|T^2\| \).

**Example**
- \( g(0) = -1/2 \):

If \( X = C[0, 1] \oplus_2 C[0, 1] \), then
  - \( \|\text{Id} - T\| = \|\text{Id} + T\| \)
    for every rank-one \( T \in L(X) \).
  - \( X \) does not have the Daugavet property.
1. Study the real or complex spaces for which the equality

\[ \| \text{Id} + T \| = \| \text{Id} - T \| \]

holds for every rank-one operator.

2. Study the real spaces \( X \) for which the equality

\[ \| \text{Id} + T^2 \| = 1 + \| T^2 \| \]

holds for every rank-one operator \( T \) on \( X \).

3. Is there any real space \( X \) with \( \dim(X) > 1 \) such that

\[ \| \text{Id} + T^2 \| = 1 + \| T^2 \| \]

for every operator?