Quantum Fluctuations of Vector Fields and the Primordial Curvature Perturbation in the Universe

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“What he [a scientist] is really seeking is to learn something new that has a certain fundamental kind of significance: a hitherto unknown lawfulness in the order of nature, which exhibits *unity* in a *broad range* of phenomena. Thus, he wishes to find in the reality in which he lives a certain oneness and totality, or wholeness, constituting a kind of *harmony* that is felt to be beautiful. In this respect, the scientist is perhaps not basically different from the artist, the architect, the music composer, etc., who all want to *create* this sort of thing in their work.”

David Bohm

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Abstract

The successes and fine-tuning problems of the Hot Big Bang theory of the Universe are briefly reviewed. Cosmological inflation alleviates those problems substantially and give rise to the primordial curvature perturbation with the properties observed in the Cosmic Microwave Background. It is shown how application of the quantum field theory in the exponentially expanding Universe leads to the conversion of quantum fluctuations into the classical field perturbation. The δN formalism is reviewed and applied to calculate the primordial curvature perturbation ζ for three examples: single field inflation, the end-of-inflation and the curvaton scenarios.

The δN formalism is extended to include the perturbation of the vector field. The latter is quantized in de Sitter space-time and it is found that in general the particle production process of the vector field is anisotropic. This anisotropy is parametrized by introducing two parameters p and q, which are determined by the conformal invariance breaking mechanism. If any of them are non-zero, generated ζ is statistically anisotropic. Then the power spectrum of ζ and the non-linearity parameter f_{NL} have an angular modulation.

This formalism is applied for two vector curvaton models and the end-of-inflation scenario. It is found that for $p \neq 0$, the magnitude of f_{NL} and the direction of its angular modulation is correlated with the anisotropy in the spectrum. If $|p| \gtrsim 1$, the anisotropic part of f_{NL} is dominant over the isotropic one. These are distinct observational signatures; their detection would be a smoking gun for a vector field contribution to ζ.

In the first curvaton model the vector field is non-minimally coupled to gravity and in the second one it has a time varying kinetic function and mass. In the former, only statistically anisotropic ζ can be generated, while in the latter, isotropic ζ may be realized too. Parameter spaces for these vector curvaton scenarios are large enough for them to be realized in the particle physics models. In the end-of-inflation scenario f_{NL} have similar properties to the vector curvaton scenario with additional anisotropic term.

Contents

1. The Hot Big Bang and Inflationary Cosmology

In the first Chapter of this thesis we start by reviewing briefly the Hot Big Bang (HBB) model of the Universe. It is one of the greatest achievements of the last century in understanding the structure and evolution of the Universe from the first second until today, 13.7×10^9 years later. Predictions of the HBB model are in very impressive agreement with the observed distribution of the large scale structure and with the abundance measurements of the light elements. However, in addition to the dark matter and dark energy problems, the HBB model suffers from the need to fine tune initial conditions. The latter motivates us to look at the earlier stage of the evolution of the Universe. Currently the most popular and most predictive paradigm for this epoch is the inflationary scenario, which is introduced in section 1.5.

1.1. Kinematics of HBB

The HBB model relies on a hypothesis called the Cosmological Principle which states that the Universe is spatially homogeneous and isotropic on sufficiently large scales.

The physical model of the Universe, is divided into two parts. One part describes the large scale behavior of the system and possesses high degree of symmetries so that mathematical models become simple and equations relatively easy to calculate. This is a background model. The second part deals with the deviations from the simplistic description of the background. These deviations are considered to be small compared to the background values. They don't influence the large scale behavior of the system: only the region much smaller than the scale on which background is defined.

The Cosmological Principle is a hypothesis about the properties of the background distribution of matter in the Universe. The background is defined as the smeared-out distribution of matter, with smearing performed on large enough scales so that the distribution appears smooth. However, a priory it is not clear that such scales do exist. It might be that probing larger and larger cosmological scales, we constantly discover new structures. This would happen if galaxies are distributed hierarchically at all distances

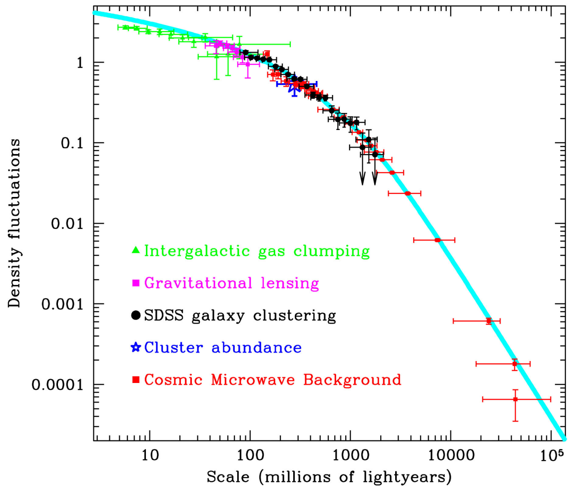

Figure 1.1.: Density fluctuations as the function of the size of the smoothing scale. The thick line represents a model with a scale invariant power spectrum and cosmological parameters $\Omega_m = 0.28$, $H = 72\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ and $\Omega_b/\Omega_m = 0.16$, $\tau = 0.17$, were Ω_m and Ω_b is the dark matter and baryon density parameters respectively, H is the Hubble parameter in physical units and τ is the optical depth [1].

as in the fractal Universe. In such a Universe probing larger and larger distances we find galaxies, clusters of galaxies, clusters of clusters of galaxies and so on. However, in the real Universe there is a scale at which the hierarchical structure stops and the Universe may be considered smooth. From the Figure 1.1 one can see that the Universe looks smoother and smoother if we probe it on larger scales. At around few hundreds Mpcs, which correspond to the size of largest superclusters, perturbations becomes smaller than the background value. At these scales separation of the matter distribution into the smooth background value and small perturbations is well justified.

The isotropy hypothesis of the Cosmological Principle is supported by observations too. The strongest evidence comes from the measurements of the temperature irregularities of the Cosmic Microwave Background (CMB). The COBE satellite was the first to measure these irregularities [2]. They showed that the anisotropy in the temperature distribution is only of order $\Delta T/T \sim 10^{-5}$. In addition the evidence for the isotropy of the Universe is further supported by the galaxy redshift surveys, measurements of peculiar velocities of galaxies, distribution of radio galaxies, X-ray background and the Lyman-α forest [3]. Another assumption of the cosmological principle, the homogeneity of the Universe, is an inevitable conclusion if we assume the validity of the Copernican Principle. This principle states that our location in the Universe is not central or somehow special. Combined with the evidence of the isotropy the outcome of the Copernican Principle is that the Universe is isotropic around every point. It can be shown that the last statement leads to the conclusion of spatial homogeneity.

Accepting the validity of the Cosmological Principle we can find the metric for the homogeneous and isotropic Universe. This can be done using only geometric considerations [4] giving the proper time interval as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}x^2}{1-Kx^2} + x^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right]. \tag{1.1}$$

This metric is expressed in spherical coordinates and is called the Friedmann-Robertson-Walker (FRW) metric. t in Eq. (1.1) is the coordinate time and spatial coordinates $l(t) \equiv a(t)x$ are decomposed into the comoving coordinates x which are constant in time and the time dependent scale factor $a(t)$ which parametrizes the evolution of the Universe, i.e. its expansion or contraction. In this metric, K parametrizes the curvature of space-time: if $K<0$, the Universe is spatially open, if $K>0$ it is closed and if $K=0$ it is flat. As will be seen later, the inflationary paradigm predicts $K \approx 0$, which is in a very good agreement with observations. Therefore, in Chapters 2 and 3 we consider only the flat Universe in order to dispense with the unnecessary complications related with the curvature term. Furthermore, instead of using the spherical coordinate system in Eq. (1.1) in many situations it will be more convenient to use the Cartesian coordinate system. Then the flat ($K=0$) FRW metric in Eq. (1.1) takes a simple form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\,\mathrm{d}x^i\mathrm{d}x^j. \tag{1.2}$$

1.2. Dynamics of the HBB

From Eq. (1.1) we have seen that the evolution of the isotropic and homogeneous Universe may be described by only one parameter, the scale factor $a(t)$. To determine the dynamics of $a(t)$ we have to specify the energy content of the Universe. In most situations it may be well approximated by an ideal fluid whose energy-momentum tensor is

$$T_{\nu\mu} = (\rho + p)\,u_\mu u_\nu + p g_{\mu\nu}, \tag{1.3}$$

where u_μ is the velocity four-vector of the fluid, $g_{\mu\nu}$ is the metric and ρ, p are the energy density and pressure of the fluid respectively.

Using the FRW metric in Eq. (1.1) and the energy momentum conservation law $\nabla_\nu T^{\mu\nu} = 0$, where ∇_ν is the covariant derivative, we find

$$\dot{\rho} = -3H\left(\rho + p\right), \tag{1.4}$$

where H is the Hubble parameter defined as

$$H \equiv \frac{\dot{a}}{a} \tag{1.5}$$

and the dot denotes the derivative with respect to the coordinate time t. Eq. (1.4) can also be rewritten as

$$a\frac{\mathrm{d}\rho}{\mathrm{d}a} = 3\left(\rho + p\right). \tag{1.6}$$

For the perfect fluid the pressure is uniquely related to the energy density which is conveniently parametrized by the equation of state. Assuming a barotropic fluid we have

$$p = w\rho, \tag{1.7}$$

where w is called the barotropic parameter. For different kinds of perfect fluid w will have different values, for example, for non-relativistic pressureless matter (sometimes called 'dust') $w = 0$, for radiation (relativistic particles) $w = 1/3$ or $w = -1$ for the vacuum energy. Using the equation of state, from the continuity equation (1.6) it is easy to find the evolution of the energy density of the perfect fluid by integration:

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}, \tag{1.8}$$

where '0' denotes initial values. Hence, it is clear that the energy density scales as $\rho \propto a^{-3}$ for the pressureless matter, $\rho \propto a^{-4}$ for the relativistic matter and $\rho =$ constant for the vacuum energy.

To find how the content of the Universe determines the time evolution of the scale factor $a\left(t\right)$ a theory of gravity must be assumed. For the purpose of this thesis it will be enough to consider only Einstein's theory of General Relativity (GR) with the field equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = m_{\mathrm{Pl}}^2 T_{\mu\nu}, \tag{1.9}$$

where $R_{\mu\nu}$ and R are the Ricci tensor and scalar respectively. Using temporary component of Einstein's field equation we can find the Friedmann equation. With the spatially homogeneous and isotropic metric in Eq. (1.1) it becomes

$$H^2 = \frac{\rho}{3m_{\mathrm{Pl}}^2} - \frac{K}{a^2}. \tag{1.10}$$

Furthermore, the acceleration of the Universe is obtained using the spatial components of Eq. (1.9). Together with Eq. (1.10) they give

$$\frac{\ddot{a}}{a} = -\frac{\rho + 3p}{6m_{\rm Pl}^2}, \tag{1.11}$$

which can be expressed in terms of the Hubble parameter as

$$\dot{H} + H^2 = -\frac{\rho + 3p}{6m_{\rm Pl}^2}. \tag{1.12}$$

It is often useful to introduce the parameter Ω, which is related to the curvature of the space time [5]:

$$\Omega - 1 \equiv \frac{K}{a^2 H^2}. \tag{1.13}$$

In the Einstein gravity this quantity measures the energy density of the Universe ρ relative to the energy density of the flat Universe $\rho_{\rm c}$ for given H, called the critical energy density. This can be seen from the Friedmann equation (1.10): for zero curvature $K = 0$, the critical energy density is

$$\rho_{\rm c} = 3m_{\rm Pl}^2 H^2. \tag{1.14}$$

Plugging this back into the Friedmann equation and using Eq. (1.13), we find that in the Einstein's gravity

$$\Omega = \frac{\rho}{\rho_{\rm c}}. \tag{1.15}$$

If the energy density of the Universe is critical, $\Omega = 1$, the Friedmann equation becomes

$$H^2 = \frac{\rho}{3m_{\rm Pl}^2}. \tag{1.16}$$

In accord with the comment above the equation (1.2) for the most of this thesis we consider only $\Omega = 1$ and the Friedmann equation of the form in Eq. (1.16).

The early Universe is dominated by radiation, as can be seen from the scaling laws below Eq. (1.8). In this era it is useful to express Eq. (1.16) in terms of the temperature of relativistic particle species. To do this let us remember from thermodynamics that the energy density ρ of the weakly interacting gas of particles is given in terms of the internal degrees of freedom $g_{\rm dof}$ and its phase space distribution function $f(p)$ as [6]

$$\rho = \frac{g_{\rm dof}}{(2\pi)^3} \int E(p)\, f(p)\, {\rm d}^3 p, \tag{1.17}$$

where p is the magnitude of the momentum of the particle and E is its total energy

$E^2 = p^2 + m^2$. For particles in thermal equilibrium the distribution function $f(p)$ is

$$f(p) = \frac{1}{\exp\left(\frac{E-\mu}{T}\right) \pm 1}, \tag{1.18}$$

where μ is the chemical potential. The '+' sign here corresponds to the Fermi-Dirac species and '−' sign to the Bose-Einstein species.

In the early Universe when it is dominated by the relativistic particles, i.e. $T \gg m$, we may take the limit $T \gg \mu$. Inserting Eq. (1.18) into Eq. (1.17) and integrating it we obtain

$$\rho_\gamma = \frac{\pi^2}{30} g_*(T)\, T^4, \tag{1.19}$$

where ρ_γ denotes the energy density of the relativistic particles and g_* are the number of effectively massless degrees of freedom:

$$g_*(T) = \sum_{i=\text{bosons}} g_i + \frac{7}{8} \sum_{i=\text{fermions}} g_i. \tag{1.20}$$

Note that g_* is a function of the temperature because in this sum we included only relativistic species, i.e. particles with the mass $m \ll T$. For example, at temperatures $T \ll \text{MeV}$ only photons and three neutrino species are relativistic giving $g_* = 3.36$. At temperatures $T > 300\,\text{GeV}$ all particles of the Standard Model are relativistic resulting in $g_* = 106.75$ [6].

Finally inserting Eq. (1.19) into Eq. (1.16) we find that in the flat, radiation dominated Universe the Hubble parameter is related to the temperature as

$$H = \pi \sqrt{\frac{g_*}{90}} \frac{T^2}{m_{\text{Pl}}}. \tag{1.21}$$

1.3. Big Bang Nucleosynthesis

Arguably the biggest success of the HBB theory is the explanation for the origin of chemical elements in the Universe. According to this theory the lightest of them were created during the first three minutes after the Big Bang, when the Universe content was in a state of a very hot plasma. This process is called the Big Bang Nucleosynthesis (BBN). Due to its immense importance for the modern cosmology in predicting the abundances of light chemical elements and being a very sensitive method to constraint new theories of particle physics, in this section we give a summary of BBN.

To describe the creation of light elements in the early Universe the crucial parameter is the reaction rate Γ of some process under consideration. For illustrative purposes let us consider, for example, interactions of particles. Then Γ would represent the interaction rate per particle. The crucial quantity here is the ratio Γ/H, where H is the Hubble parameter. At the epoch of BBN the Universe is dominated by the matter which satisfies the strong energy condition, so that H^{-1} represents the age of the Universe. In this case $\Gamma/H < 1$ means that on average less than one particle interacted throughout the history of the Universe. In other words, we can say that particles are decoupled. If, on the other hand $\Gamma/H > 1$, particles have interacted many times and it is safe to assume that they are in thermal equilibrium. During the radiation dominated stage the expansion rate of the Universe is proportional to the temperature squared, $H \propto T^2$ (see Eq. (1.21)), while the reaction rates are typically proportional to $\Gamma \propto T^s$. In the adiabatically expanding Universe the temperature decreases as $T \propto a^{-1}$. Hence, we can write $\Gamma/H \propto a^{2-s}$, from which we see that if $s > 2$ the process which was in equilibrium at some initial time, i.e. $\Gamma/H > 1$, it will fall out of equilibrium at later times. If the process we are interested is the interaction of particles, then we can say that after being in equilibrium, particles "freeze-out" when Γ/H becomes smaller than one, i.e. the number density is not affected by interactions. Then BBN can be roughly divided into three stages depending on which processes are in thermal equilibrium.

When the temperature of the Universe was around 10 MeV, which corresponds to the age of 10^{-2} s, the ratio of neutrons and protons is controlled by the weak interactions:

$$n \longleftrightarrow p + e^- + \bar{\nu}, \quad n + \nu_e \longleftrightarrow p + e^-, \quad n + e^+ \longleftrightarrow p + \bar{\nu}_e. \tag{1.22}$$

where ν_e and $\bar{\nu}_e$ are the electron neutrino and antineutrino, and e^-, e^+ are the electron and the positron. If the rate of these interactions are much more rapid than the expansion of the Universe, i.e. $\Gamma_{n\leftrightarrow p}/H > 1$, the species involved in these interactions are in a thermal equilibrium, which means that the neutron to proton ratio evolves according to

$$\frac{n}{p} = \mathrm{e}^{-Q/T}, \tag{1.23}$$

where $Q \equiv m_n - m_p = 1.29\,\mathrm{MeV}$ is the mass difference of neutrons and protons. As we can see for the energies high above MeV, the number of neutrons and protons are almost the same, $(n/p) \approx 1$.

The reaction rates for the processes in Eq. (1.22) can be calculated using Fermi theory

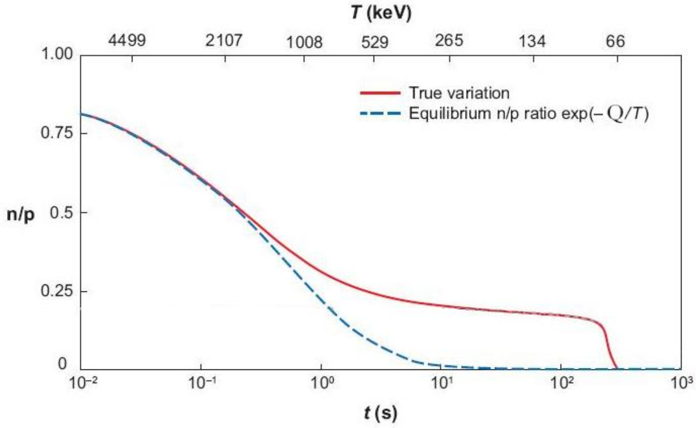

Figure 1.2.: Evolution of the n/p ratio. The solid red line represents the true variation while the dashed blue one represents an equilibrium evolution. BBN starts at $T \sim 0.1\,\mathrm{MeV}$ which results in the steep decline of the red line at these energies. (Figure adapted from Ref. [7])

for the weak interactions, which gives [6]

$$\Gamma_{n\leftrightarrow p} = \begin{cases} \tau_n^{-1}\,(T/m_e)^3\,\mathrm{e}^{-Q/T} & T \ll Q,\ m_e \\ \simeq 2G_F^2 T^5 & T \gg Q,\ m_e, \end{cases} \tag{1.24}$$

where τ_n is the neutron halflife and $G_\mathrm{F} = 1.1664 \times 10^{-5}\,\mathrm{GeV}^{-2}$ is the Fermi constant. When the reaction rate $\Gamma_{n\leftrightarrow p}$ falls bellow the Hubble expansion rate, i.e. when $\Gamma_{n\leftrightarrow p}/\mathrm{H} \lesssim 1$, processes in Eq. (1.22) depart from the equilibrium and the number of neutrons and protons “freezes-out”. The approximate temperature of the freeze-out can be calculated using Eq. (1.21) and considering that $T \gtrsim m_e$:

$$\frac{\Gamma_{n\leftrightarrow p}}{H} \sim \left(\frac{1}{g_*^{1/6}}\frac{T}{0.8\,\mathrm{MeV}}\right)^3. \tag{1.25}$$

In the Standard Model of particle physics with the three (almost) massless neutrino species the number of relativistic degrees of freedom is $g_* = 10.75$. Thus, the freeze-out temperature is found to be

$$T_\mathrm{fr} \sim 1\,\mathrm{MeV}, \tag{1.26}$$

which corresponds to about 1 s. The ratio n/p at this moment can be calculated from

Eq. (1.23)

$$\frac{n}{p} \approx \frac{1}{6}. \tag{1.27}$$

However, this is not a true "freeze-out" because the n/p ratio is not constant but decreases slowly. This happens because of occasional weak interactions among neutrons, protons, $e^{\pm}$ and ν_e, $\bar{\nu}_e$ eventually dominated by the free neutron β decay. However, this decrease is much slower than the equilibrium value given in Eq. (1.23) (see Figure 1.2).

When the nucleosynthesis starts at about 0.1 MeV (corresponding to about 180 s), the neutron to proton ration had been decreased to

$$\frac{n}{p} \approx \frac{1}{7}. \tag{1.28}$$

The main product of BBN is the helium-4, ^{4}He. The production of heavier elements is very subdominant because there are no stable nuclei with the mass number 5 or 8 and hence no elements form through reactions such as $n +^4$ He, $p +^4$ He or ^{4}He $+^4$ He. In addition reactions such as $\mathrm{T}+^4\mathrm{He} \longleftrightarrow \gamma+^7\mathrm{Li}$ and $^3\mathrm{He}+^4\mathrm{He} \longleftrightarrow \gamma+^7\mathrm{Be}$ are suppressed because of the large Coulomb barriers. The formation of ^{4}He in principle could proceed directly through the four body collision. But the very low number densities of neutrons and protons renders this type of reactions negligible. Hence, the element formation must start with the production of deuterium through the two-body collision:

$$p + n \longleftrightarrow \mathrm{D} + \gamma. \tag{1.29}$$

Although the binding energy of the deuterium is $\Delta_{\mathrm{D}} = 2.23\,\mathrm{MeV}$, the formation of this element becomes effective only at much smaller temperatures. This is because of a large number of energetic photons which destroy deuterium. So D nuclei can start forming without being immediately photo-dissociated only when the number of such photons per baryon falls below unity, which occurs at the temperature $T < 0.1\,\mathrm{MeV}$ [8]. Therefore, this period is called the *deuterium bottleneck*. But once deuterium starts forming, the whole set of reactions sets in producing other heavier elements.

The final number density of ^{4}He depends on the whole nuclear network only very weakly. And it is a very good approximation to assume that all neutrons which didn't β decay will end up being bound into the ^{4}He atoms. Hence, the helium mass fraction Y_p can be calculated very easily just by power counting:

$$\mathrm{Y}_p \simeq \frac{2n}{n+p} = \frac{2\,(n/p)}{1+(n/p)} \simeq 0.25. \tag{1.30}$$

Number	Reaction	Number	Reaction
1	τ_n	9	$^3\mathrm{He}\,(\mathrm{T},\gamma)\,^7\mathrm{Be}$
2	$p\,(n,\gamma)\,d$	10	$\mathrm{T}\,(\mathrm{T},\gamma)\,^7\mathrm{Li}$
3	$\mathrm{D}\,(p,\gamma)\,^3\mathrm{He}$	11	$^7\mathrm{Be}\,(n,p)\,^7\mathrm{Li}$
4	$\mathrm{D}\,(\mathrm{D},n)\,^3\mathrm{He}$	12	$^7\mathrm{Li}\,(p,\mathrm{T})\,^4\mathrm{He}$
5	$\mathrm{D}\,(\mathrm{D},p)\,\mathrm{T}$	13	$^4\mathrm{He}\,(\mathrm{D},\gamma)\,^6\mathrm{Li}$
6	$^3\mathrm{He}\,(n,p)\,\mathrm{T}$	14	$^6\mathrm{Li}\,(p,\mathrm{T})\,^3\mathrm{He}$
7	$\mathrm{T}\,(\mathrm{D},n)\,^4\mathrm{He}$	15	$^7\mathrm{Be}\,(n,\mathrm{T})\,^4\mathrm{He}$
8	$^3\mathrm{He}\,(\mathrm{D},p)\,^4\mathrm{He}$	16	$^7\mathrm{Be}\,(\mathrm{D},p)\,2^4\mathrm{He}$

Table 1.1.: The most relevant reactions of BBN. Here, numbers of reactions correspond to the ones in Figure 1.3 (adapted from [8]).

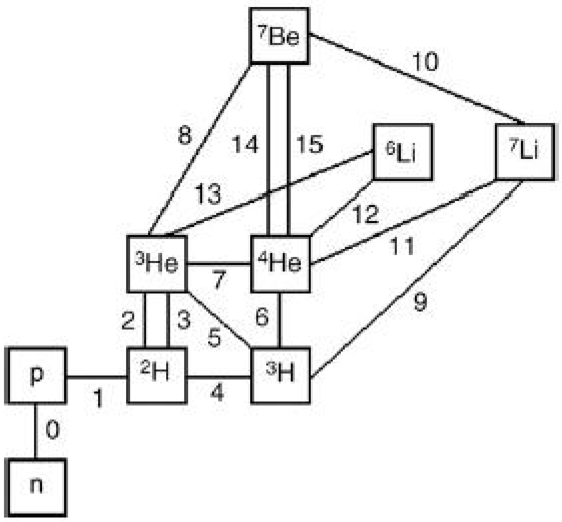

Figure 1.3.: The network of most relevant reactions of BBN. The numbers represent reactions in Table 1.1 [8].

The essential parameter for the processes of BBN is the number density of baryons. To quantify it, one usually uses the ratio of baryons over photons defined as

$$\eta_{10} \equiv 10^{10} \frac{n_{\rm B}}{n_\gamma}. \tag{1.31}$$

At temperatures somewhat below $T \lesssim 0.3\,\text{MeV}$, all the positrons have annihilated with the electrons and hence the number of baryons and photons in a comoving volume does not change. Therefore, η_{10} must stay constant from BBN through recombination until today. And one can relate this value to the energy density parameter for the baryons Ω_B today [9]:

$$\eta_{10} = \frac{273.45\Omega_B h^2}{1 - 0.007\mathrm{Y}_p} \left(\frac{2.725\,\mathrm{K}}{T_{\rm CMB}}\right)^3 \left(\frac{6.708 \times 10^{-45}\,\mathrm{MeV}^{-2}}{G}\right), \tag{1.32}$$

where $T_{\rm CMB}$ is the photon temperature today and G is Newton's gravitational constant.

The most relevant reactions for the BBN are summarized in Table 1.1 and Figure 1.3. The precise final abundances of each element, including ^{4}He, are calculated numerically, solving a system of coupled kinetic equations for each element as well as Einstein equations, including the covariant conservation of total energy momentum tensor and conservation of baryon number and electric charge. Results are given in Figure 1.4 together with observationally inferred values for some elements.

1.4. The Problem of Initial Conditions of the Hot Big Bang

The HBB cosmology is very successful in explaining the structure and evolution of the Universe after 1 s. However, in order to agree with observations the initial conditions of the HBB model have to be fine tuned. In this section we review briefly the problem of this fine tuning.

1.4.1. The Flatness Problem

Current observations agree very well with the density parameter Ω of the Universe being very close to one, i.e. the Universe is spatially flat. However, in the phase diagram the value $\Omega = 1$ is the unstable fixed point. In other words, any initially tiny departure from flatness will become larger and larger as the Universe evolves. This can be easily seen by using the Friedmann equation (1.6) and the definition of the Ω in Eq. (1.13):

$$\Omega = \frac{1}{1 - \frac{\rho}{3m_{\rm Pl}^2}\frac{K}{a^2}}. \tag{1.33}$$

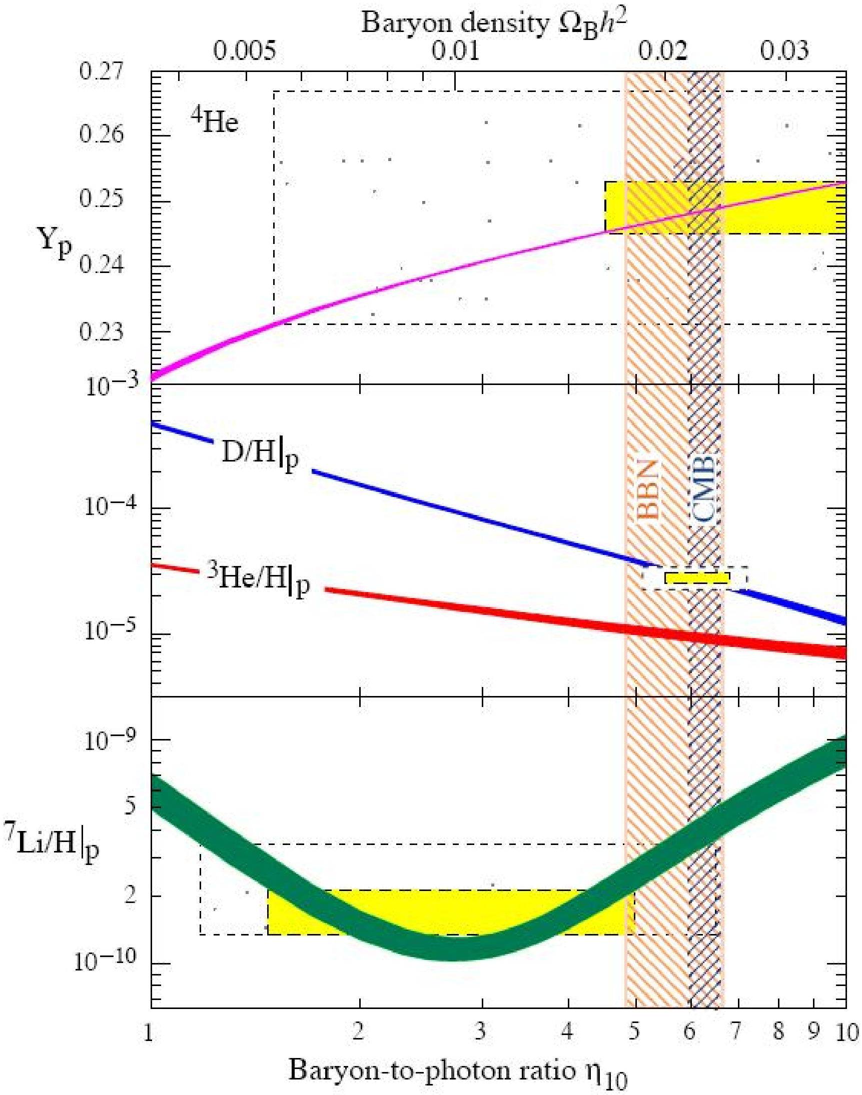

Figure 1.4.: Abundances of light elements from the standard model of BBN. The bands represent 95% CL, the boxes represent observed values (smaller - $\pm 2\sigma$ only statistical errors; larger - $\pm 2\sigma$ statistical and systematic errors), the narrow vertical column represents η_{10} value inferred from CMB observations and the wider column indicates the BBN concordance range (both at 95% CL) [10].

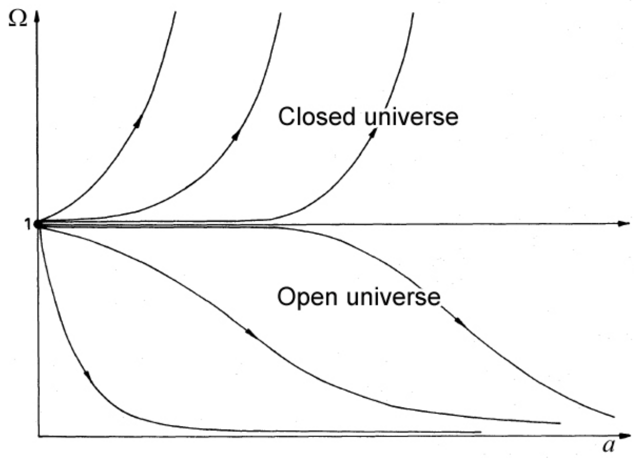

Figure 1.5.: The evolution of the density parameter of the Universe filled with an ideal fluid whose barotropic parameter is $w > -1/3$. From the figure it is clear that any initial departure from critical density will grow in time, hence the spatially flat Universe is an unstable fixed point in the phase diagram (adapted from Ref. [11]).

Assuming ρ corresponds to the energy density of a perfect fluid and using Eq. (1.8) we arrive at

$$\Omega = \frac{1}{1 - \left[\left(\Omega_{\text{ini}} - 1\right) / \Omega_{\text{ini}}\right] y^{3w+1}}, \tag{1.34}$$

where Ω_{ini} denotes the initial value of the density parameter and $y \equiv a/a_0$. From this equation it is already clear that if $w > -1/3$ any initial departure from the flat Universe with $\Omega_{\text{ini}} \neq 1$ will grow in time. For example if initially the Universe is open, $\Omega_{\text{ini}} < 1$, the energy density at any later time $y > 1$ will decrease monotonically towards zero $\Omega \to 0$, i.e. towards the empty Milne Universe. On the other hand, if initially the Universe is closed, $\Omega_{\text{ini}} > 1$, its energy density will increase rapidly and reach a singularity in a finite time. This behavior of the density parameter is illustrated in the phase diagram in Figure 1.5. Therefore, for the present Universe to be flat, $\Omega_0 \approx 1$, its initial energy density had to be extremely close to the critical value. For example, in order to reproduce the present Universe, the energy density at the time of BBN had to be

$$|\Omega_{\text{BBN}} - 1| \lesssim 10^{-16}. \tag{1.35}$$

It is extremely unlikely for Ω_{BBN} to be so close to unity by accident.

The flatness problem of the HBB is sometimes rephrased as an age problem. This can be seen from Eq. (1.34) and Figure 1.5. If initially the Universe is closed $\Omega_{\text{ini}} > 1$, very soon after its birth it recollapses without having time to form any galaxies and stars. If, on the other hand, initially the Universe is open $\Omega_{\rm ini} < 1$, it soon becomes the empty Milne Universe before the formation of any structure. In both cases the Universe does not have time to become the one we observe today.

1.4.2. The Horizon Problem

In section 1.1 we have demonstrated that the isotropy and (with mild assumptions) homogeneity of the Universe at present as well as at the Last Scattering Surface (LSS) is an observationally established fact, which justifies the use of FRW metric. However, in the HBB model this fact is highly non-trivial due to the finite age of the Universe. We expect the space-time regions to be homogeneous and isotropic on scales which could have been in a causal contact, i.e. which could have "communicated" with each other. But because the maximum velocity of the signal is finite (equal to the speed of light in vacuum) and because the age of the Universe is finite too, there is only a limited distance at which two regions could have had a causal contact.

To illustrate the horizon problem in HBB let us consider the epoch of the last scattering. From observations of the CMB we know that the fractional temperature variations at that time was of order 10^{-5}. The distance of maximal causal contact at LSS is

$$l_{\rm causal} \sim c\, t_{\rm LSS}, \tag{1.36}$$

where c is the speed of light and $t_{\rm LSS}$ is the age of the Universe at LSS. At this epoch the size of the present horizon, given by $l_{\rm today} \sim c\, t_{\rm today}$, was $a_{\rm LSS}/a_{\rm today}$ times smaller

$$l_{\rm LSS} \sim c\, t_{\rm today} \frac{a_{\rm LSS}}{a_{\rm today}}. \tag{1.37}$$

Taking into account that the Universe was matter dominated at the last scattering and remained so until very recently, i.e. $a \propto t^{2/3}$, we may compare $l_{\rm LSS}$ with the size of the causality length

$$\frac{l_{\rm LSS}}{l_{\rm causal}} \sim \left(\frac{a_{\rm today}}{a_{\rm LSS}}\right)^{1/3}. \tag{1.38}$$

Hydrogen recombined at the temperature $T_{\rm LSS} \approx 1.6 \times 10^5\,{\rm K}$. Assuming adiabatic expansion for the Universe $T \propto a^{-1}$, and using $T_{\rm CMB} \approx 2.7\,{\rm K}$ at $a = a_{\rm today}$ we find

$$\left(\frac{l_{\rm LSS}}{l_{\rm causal}}\right)^3 \sim 10^5. \tag{1.39}$$

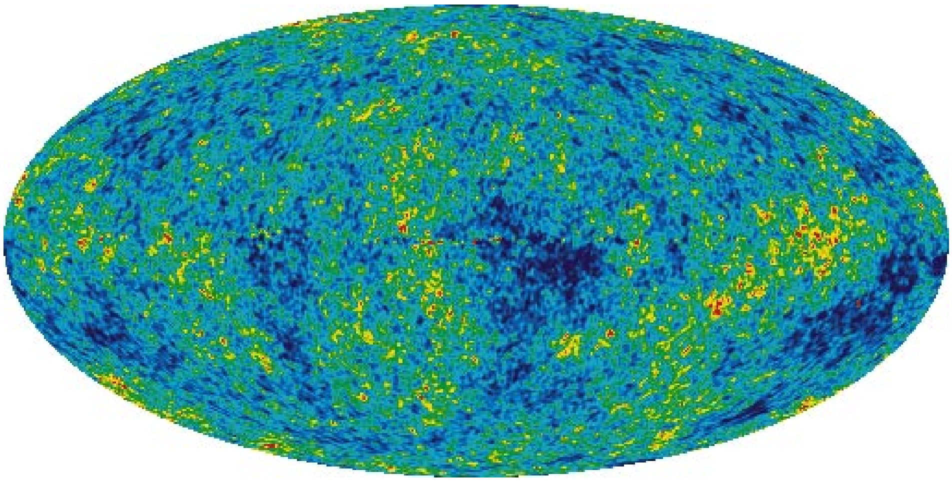

Figure 1.6.: Cosmic Microwave Background temperature variation map observed by the WMAP satellite [12].

Therefore, at the time of recombination the observable Universe consisted of at least 10^5 causally disconnected regions with the fractional temperature variation of only 10^{-5}. No physical process could have caused such extreme smoothness in so many causally disconnected regions. This constitutes the horizon problem of HBB cosmology.

1.4.3. The Origin of Primordial Perturbations

As we have seen in Figure 1.1 the Universe can be considered isotropic and homogeneous only on smoothing scales larger that a few hundreds of Mpcs. On smaller scales it is highly inhomogeneous due to the presence of structures such as stars, galaxies and galaxy clusters. It is already established that this structure in the Universe formed due to gravitational instability, when slightly denser regions collapsed onto themselves forming a complicated web distribution of galaxies. However, for this process to be initiated the existence of some primordial seed density perturbations must be postulated. Indeed, the first observational proof of such perturbations was provided by the COBE satellite [2] (see Figure 1.6 for the high resolution CMB map from WMAP measurements). Unfortunately, the properties of these seed perturbations cannot be explained within the framework of HBB cosmology.

Causality constraints require that seed perturbations could have formed only due to processes inside the causal horizon, which in the HBB model is monotonically decreasing as we go back in time. However, it was already realized in 60's and 70's that random displacements and movements of particles inside the horizon cannot produce the necessary

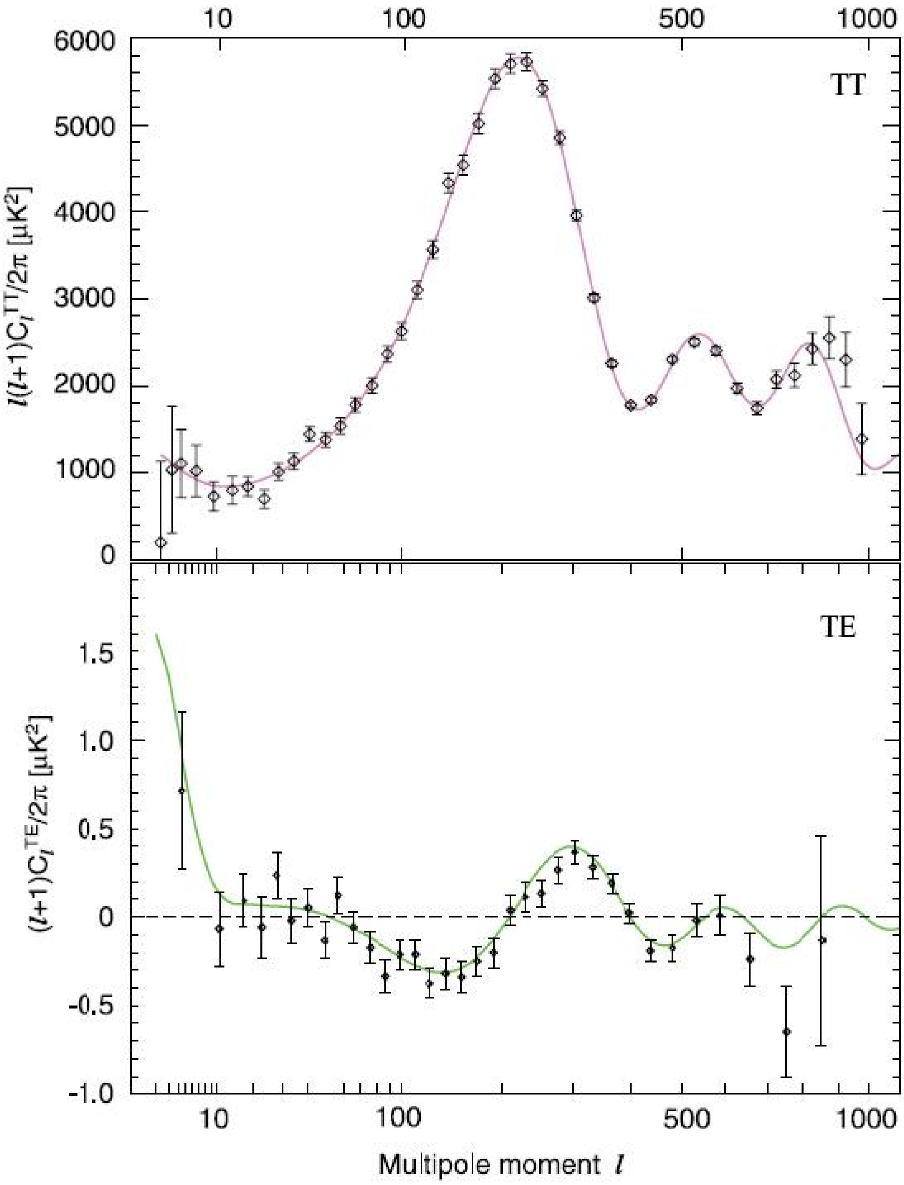

Figure 1.7.: The angular power spectrum of the CMB temperature-temperature (TT) and temperature-polarization (TE) anisotropies. Solid lines represent the best fit ΛCDM model [13].

perturbation spectrum. Zel'dovich [14, 15] and Peebles [16] were the first to realize that such processes would produce matter density perturbations with the power spectrum $P(k) \propto k^4$. Such perturbations would result in an excessive overproduction of black holes on small length scales. Indeed, assuming a smooth power law spectrum of primordial perturbations, $P(k) \propto k^n$, from the CMB observations it was found that $n \approx 0.96$ [17], very different from the $n = 4$ case.

In addition, with more precise measurements of primordial perturbations other properties became clear which cannot be explained by the standard HBB cosmology [18]. As seen in the upper graph of Figure 1.7 the CMB temperature power spectrum features so called "acoustic peaks". But more importantly this pattern is caused by adiabatic density perturbations. Such perturbations in the baryon-photon fluid upon horizon entry start oscillating only with excited cosine modes and with the same phase. Therefore, this is a strong indication that seed perturbations are present on scales larger than the horizon, i.e. they could not be created by causal processes during HBB.

Although no viable model exists, in principle one could construct a model in which causal processes mimic the pattern of adiabatic acoustic peaks [19, 20, 21]. However, even stronger proof for the superhorizon origin of primordial perturbations is provided by the temperature-polarization cross correlation function [22]. The polarization signal is not affected on its path from LSS towards us. Therefore, by measuring polarization perturbations we can be certain to be probing the era of recombination. But as clearly seen in the lower graph of Figure 1.7 on angular scales $50 < l < 200$ the temperature and polarization anticorrelates. Since these low multipoles represent superhorizon scales at LSS it is certain that primordial density perturbations were already present before entering the horizon.

From this discussion, one can see that the problem of seed perturbations in the HBB cosmology in essence is a restatement of the horizon problem. The superhorizon origin of primordial perturbations is the strongest support for the inflationary scenario.

1.5. Inflation

1.5.1. The Accelerated Expansion

The initial condition problems of standard HBB cosmology named in section 1.4 may be substantially alleviated if we postulate an accelerated expansion of the Universe at it's earliest stages. This epoch is called inflation. When we say "accelerated expansion" we mean that the distance between any two comoving points in the Universe is increasing with a positive acceleration. In other words the scale factor in Eqs. (1.1) or (1.2) obeys

$$\ddot{a} > 0. \tag{1.40}$$

This might be considered as the definition of the era when gravity is repulsive.

Instead of Eq. (1.40) we may rewrite it in the form which gives more physical interpretation. As was discussed in section 1.2, H^{-1} defines the Hubble length. Then $(aH)^{-1}$ is the comoving Hubble length. From Eq. (1.40) we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} < 0. \tag{1.41}$$

It shows that the comoving Hubble length during inflation decreases. Therefore, two points which initially were inside the Hubble radius (their comoving distance smaller than $(aH)^{-1}$) at some moment goes outside this radius. This moment is called "the horizon exit".

The condition in Eq. (1.40) can be written even in another form, which will be very useful in later sections. Substituting the derivative of the Hubble parameter $\dot{H}$ into Eq. (1.40) after some calculations we find $-\dot{H} < H^2$. When

$$\left|\dot{H}\right| \ll H^2, \tag{1.42}$$

the expansion is almost exponential. Even more so, if $\dot{H} \to 0$, it is exactly exponential, i.e. $a \propto \exp(Ht)$, and we call this de Sitter expansion.

By postulating the early phase of accelerated expansion (Eq. (1.40)) of the early Universe, the fine tuning problem of initial conditions of the HBB model discussed in section 1.4 are substantially alleviated. The crucial condition for this is Eq. (1.41).

How the period of accelerated expansion solves the flatness problem can be seen by inserting Eq. (1.41) into Eq. (1.13). Because during inflation the comoving horizon is decreasing, a^2H^2 grows with time and $|\Omega - 1|$ is driven towards zero. Therefore, $\Omega = 1$ instead of being an unstable fixed point in the HBB model, becomes an attractor during the inflationary stage. To illustrate this let us use Eq. (1.34). As we will see shortly in the next section, the accelerated expansion of the Universe may be achieved if it is dominated by the vacuum energy for which $\rho = -p$, i.e. $w = -1$. Substituting the value $w = -1$ into Eq. (1.34) we find that the density parameter approaches $\Omega \to 1$ and the phase diagram in the Figure 1.5 changes into the Figure 1.8.

The decreasing comoving horizon size during inflation has an effect that initially any two causally connected points within the Hubble radius at some later moment leaves the

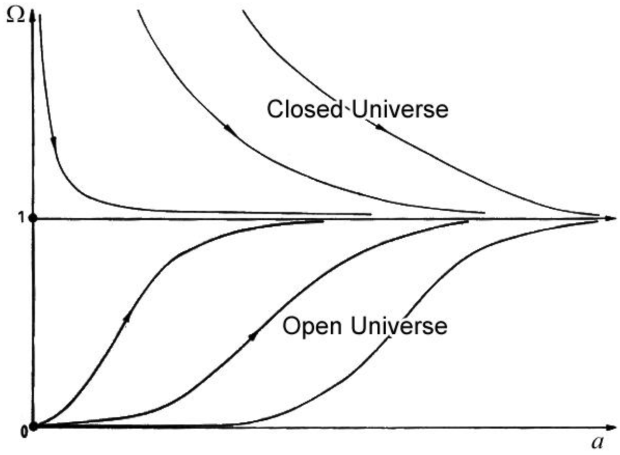

Figure 1.8.: The evolution of the density parameter for the Universe dominated by the vacuum energy with $w = -1$. Instead of being an unstable point in HBB model, $\Omega = 1$ becomes an attractor (adapted from Ref. [11]).

horizon. This makes the causally connected regions to be larger than the horizon size and the postulate of the inflationary period solves a second problem of the HBB model.

However, the most important achievement of inflation is that it explains the origin of superhorizon seed perturbations. According to the inflationary paradigm during accelerated expansion of the Universe vacuum quantum fluctuations are converted into classical perturbations. Details of this process with scalar field quantum fluctuations will be discussed in Chapter 2 and extended to the quantum fluctuations of vector fields in Chapter 3. But before that let us discuss what may cause an accelerated expansion of the Universe.

1.5.2. The Scalar Field Driven Inflation

There are several reasons why the Universe could have expanded exponentially. It might be that, at the relevant energy scales, the Einstein gravity is not a viable theory of Nature and it must be modified. Modification is such that it gives almost exponential expansion of space-time. The very first proposed model of inflation was due to this kind of modified gravity theory [23].

Another possibility is that inflation happens at the energy scales where Einstein gravity is still a viable theory of nature. Then Eq. (1.40) puts constraints on properties of matter which may be responsible for the inflationary expansion. This can be found using

Eq. (1.11):

$$\rho + 3p < 0. \tag{1.43}$$

Because the energy density ρ must always be positive, it follows that the Universe undergoes accelerated expansion if the pressure is negative enough, $p < -\rho/3$. The lower bound for pressure is determined by the dominant energy condition, which requires that $p \geq -\rho$. Should this bound be violated the propagation of energy outside the lightcone becomes possible and one cannot guarantee the stability of the vacuum [24]. Taking the extreme case $p = -\rho$ and from Eq. (1.12) we find that $\dot{H} \to 0$ and the Universe is expanding exponentially (de Sitter Universe).

The equation of state $p \approx -\rho$ may be realized in the framework of GR if the Universe is assumed to be dominated by classical scalar fields. For simplicity we will assume that only one such field is relevant, which is then called the inflaton. The most general Lagrangian which is consistent with the GR for a single field inflation is given by [25]

$$\mathcal{L} = P\left(X, \phi\right), \tag{1.44}$$

where ϕ is the scalar field, $X \equiv \frac{1}{2}\partial_\mu \phi \partial^\mu \phi$ and P is some function. Inflationary models which study the evolution of the Universe under the influence of such fields are called k-inflation. But to make essential properties of inflationary models more transparent let us concentrate on a particular case where the field is canonically normalized. Then Eq. (1.44) becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V\left(\phi\right), \tag{1.45}$$

where $V\left(\phi\right)$ is the potential. The equation of motion of the field is obtained by requiring that the variation of the action with respect to the field vanishes. For the homogeneous component this gives

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0, \tag{1.46}$$

where $V_\phi \equiv \partial V\left(\phi\right)/\partial\phi$ is the derivative of the potential with respect to the field and because we are interested in the homogeneous part of the field we have neglected gradient terms.

The energy-momentum tensor of a scalar field may be obtained using the variation of the action with respect to $g_{\mu\nu}$ [26]

$$T_{\mu\nu} = -2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}. \tag{1.47}$$

For the scalar field with the Lagrangian of the form in Eq. (1.45) it becomes

$$T^{\mu}_{\nu} = \partial^{\mu}\phi\partial_{\nu}\phi - \delta^{\mu}_{\nu}\left[\frac{1}{2}\partial^{\sigma}\phi\partial_{\sigma}\phi - V(\phi)\right]. \tag{1.48}$$

From this equation one can notice that the energy-momentum tensor for the homogeneous scalar field (homogenized by inflation) becomes as that of the perfect fluid so that the energy density ρ and pressure p can be defined as

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{1.49}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{1.50}$$

Using the first of these relations the Friedmann equation in Eq. (1.10) becomes

$$3m_{\rm Pl}^2 H^2 = \frac{1}{2}\dot{\phi}^2 + V. \tag{1.51}$$

Differentiating it, we derive another useful expression

$$3m_{\rm Pl}^2 \dot{H} = -\dot{\phi}^2, \tag{1.52}$$

were we have used the equation of motion in Eq. (1.46) as well.

From expressions (1.49) and (1.50) it is clear that the condition $p \approx -\rho$ is satisfied if the kinetic energy of the field is negligible compared to the potential one, i.e.

$$\frac{\left|\dot{H}\right|}{H^2} \ll 1 \;\Leftrightarrow\; \dot{\phi}^2 \ll V(\phi). \tag{1.53}$$

This requirement is called "the slow-roll condition" and is fulfilled if the potential of the field is sufficiently shallow. These conditions may be conveniently rewritten in terms of slow-roll parameters ϵ and η in the following way. Because the field is slowly rolling we might also expect that the second derivative is also small, $\ddot{\phi}/H \ll \dot{\phi}$. Then, in the equation of motion (1.46), the first term is negligible

$$3H\dot{\phi} \simeq -V_{\phi}. \tag{1.54}$$

The first slow-roll parameter becomes

$$\epsilon \equiv \frac{m_{\mathrm{Pl}}^2}{2}\left(\frac{V_\phi}{V}\right)^2 \ll 1. \tag{1.55}$$

On the other hand, differentiating Eq. (1.54) and using conditions in Eq. (1.53) we define the second slow-roll parameter

$$\eta \equiv m_{\mathrm{Pl}}^2 \frac{V_{\phi\phi}}{V} \ll 1. \tag{1.56}$$

Conditions in Eq. (1.55) and (1.56) are called flatness conditions for the shape of the potential of the scalar field. The quasi exponential expansion during inflation lasts as long as those conditions are satisfied. When any of the parameters ϵ of η becomes of order one, inflation ends.

It is convenient to define a number of e-folds until the end of inflation

$$N \equiv \ln \frac{a\left(t_{\mathrm{end}}\right)}{a\left(t\right)} = \int_t^{t_{\mathrm{end}}} H \mathrm{d}t = \int_\phi^{\phi_{\mathrm{end}}} \frac{H}{\dot{\phi}} \mathrm{d}\phi = m_{\mathrm{Pl}}^{-2} \int_{\phi_{\mathrm{end}}}^{\phi} \frac{V}{V_\phi} \mathrm{d}\phi. \tag{1.57}$$

If the initial time in this equation is chosen to be when cosmological scales leaves the horizon, we may calculate how many numbers of e-folds of inflation is needed to solve the horizon and flatness problems of the HBB model. If the energy scale of inflation and the reheating temperature in Eq. (1.62) is at the supersymmetry energy scales, then $N \approx 60$ [5]. In some models, with very low reheating temperature it can go down to $N \approx 40$. If we assume the validity of Einstein's gravity after inflation and fields with canonical kinetic terms the maximum number of e-folds is $N \approx 70$.

1.5.3. The End of Inflation and Reheating

Inflation ends when the slow-roll parameters defined in Eqs. (1.55) and (1.56) become of order one. Soon after this happens, the inflaton field fast-rolls towards its VEV and starts oscillating around the minimum of its effective potential. Expanding this potential around the minimum, the leading term in the series is $V\left(\phi\right) = \frac{1}{2}m^2\left(\phi - \langle\phi\rangle\right)^2$, where $\langle\phi\rangle$ is the VEV and m is the mass of the inflaton field. Inserting this into Eq. (1.46) we see that for $m \gg H$ the field acts as the underdamped harmonic oscillator with the frequency $\omega = m$, much larger than the Hubble time. Therefore, in accord with the equation of motion of the harmonic oscillator we may write $\overline{\dot{\phi}^2} = 2\overline{V}\left(\phi\right)$, were the average values are defined over one Hubble time. Inserting this into Eq. (1.50) we find that the average pressure of the oscillating scalar field is $\overline{p} = 0$ and the energy density from Eq. (1.49) is

$\rho = \overline{\dot{\phi}^2}$. Using this, Eq. (1.46) may be rewritten as

$$\dot{\rho} + 3H\rho = 0. \tag{1.58}$$

Taking into account that the Universe is dominated by the oscillating inflaton field with the zero pressure from this equation we obtain

$$\overline{p} = 0 \quad \text{and} \quad \rho \propto a^{-3}. \tag{1.59}$$

Therefore, the oscillating inflaton field acts as the pressureless matter and the Universe evolves as dominated by the non-relativistic dust particles (inflatons) [27].

Because the field is oscillating it might be interpreted as the collection of massive inflaton particles with zero momentum. Before inflaton oscillations the temperature of the Universe is effectively zero. However, for the successful BBN, discussed in section 1.3, the Universe must be radiation dominated with the temperature above 10 MeV. Therefore, to recover the successes of the HBB cosmology, the energy stored in the inflaton field must be released to effectively massless particles. This process is known as 'reheating'. The first proposals for the mechanism to reheat the Universe were based on the single-body decays [28, 29]. During inflation such decays may be neglected because the field is not oscillating and cannot be interpreted as a collection of particles. But during the phase of coherent oscillations inflaton particles may decay into other scalar particles χ or fermions ψ through the terms in the Lagrangian such as $g\phi\chi^2$ and $h\phi\bar{\psi}\psi$, where g is the coupling constant with the dimension of mass and h is a dimensionless coupling constant. Due to these couplings the equation (1.58) must include an additional friction term Γ which parametrizes the inflaton decay into these particles

$$\dot{\rho} + (3H + \Gamma)\,\rho = 0, \tag{1.60}$$

where $\Gamma \equiv \Gamma_{\phi\rightarrow\chi\chi} + \Gamma_{\phi\rightarrow\psi\bar{\psi}}$. When the mass of the inflaton is much larger than those of χ and ψ, i.e. $m \gg m_\chi, m_\psi$, the decay rates are known to be [29, 30]

$$\Gamma_{\phi\rightarrow\chi\chi} = \frac{g^2}{8\pi m} \quad \text{and} \quad \Gamma_{\phi\rightarrow\psi\overline{\psi}} = \frac{h^2 m}{8\pi}. \tag{1.61}$$

When $H > \Gamma$ the number of produced particles is very small (see section 1.3) and they do not influence the dynamics of the Universe. However, these particles may still thermalise and their temperature becomes much larger than the temperature at reheating (given in Eq. (1.62)) [31]. At time t_{reh}, when the Hubble parameter becomes $H \sim \Gamma$, the decay processes become significant and practically all inflaton energy is transferred to the newly created particles. The temperature of the Universe at this moment may be calculated using the flat Friedman equation in Eq. (1.16) and assuming that new particles are relativistic, then from Eq. (1.19) we get

$$T_{\mathrm{reh}} \simeq g_*^{-1/4}\sqrt{\Gamma m_{\mathrm{Pl}}}, \tag{1.62}$$

where $g_* = 10^2 - 10^3$ [32] is the number of effective relativistic degrees of freedom defined in Eq. (1.20).

The mechanism of reheating described above is based on perturbative particle decay. However, in some inflationary scenarios the energy transfer from the inflaton field may be preceded by another, much more efficient process. To distinguish it from the conventional reheating, it is called 'preheating'. In the first such proposal, the parametric preheating, the inflaton field decays into relativistic particles of other fields very rapidly in short, explosive bursts due to the parametric resonance effects [33, 34]. At the second stage, these particles decay into relativistic species which finally thermalise. It should be noted, however, that it is not possible to transfer the total energy stored in the inflaton field by this process. When the amplitude of inflaton oscillations decreases below some critical value, the parametric resonance becomes inefficient. The residual oscillating inflaton field must decay through the perturbative reheating processes described above. If these processes are not efficient enough, due to the scaling law in Eq. (1.59), the residual oscillating inflaton field comes to dominate the relativistic decay products of preheating. In this situation the transfer of the inflaton energy into radiation is still dominated by the perturbative reheating processes.

2. The Origin of the Primordial Curvature Perturbation

2.1. Statistical Properties of the Curvature Perturbation

2.1.1. Random Fields

As it will become clear in section 2.2 the origin of cosmological perturbations is quantum mechanical. But quantum mechanical processes are non-deterministic: one can only predict the probability of experimental outcome. Therefore, to make quantitative descriptions of these processes one needs to use statistical methods. The same is true for cosmological perturbations. One cannot calculate exact values of perturbations at each space point, only the statistical properties may be predicted by theories and compared with observations. To quantify the properties of cosmological perturbations a very useful method is to describe them as random fields.

Let us introduce some random field β. It is assumed that our Universe is just one realization of many (hypothetical) possible universes. Then, to each of these universes one can assign a particular realization β_n from the whole ensemble β.[1] Depending on the problem to be solved, functions β_n may parametrize, for example, the spatial distribution of the density, velocity or other fields. Each of the functions β_n are realized with the probability $p(\beta_n)\,\mathrm{d}n$, where n is a continuous index and p is the probability distribution function (PDF).

Properties of the random field β are specified by the form of PDF. It is said that the random field β is statistically homogeneous if the probability $p(\beta_n)$ of the realization β_n is the same as that of realization β_m, where $\beta_n(\mathbf{x}) = \beta_m(\mathbf{x}+\mathbf{X})\ \forall\ \mathbf{X}$. In other words, probabilities are equal for realizations which differ only by the spatial translation. And β is said to be statistically isotropic at a point $\mathbf{x}$ if probabilities are equal for realizations which differs only by rotation, i.e. $p(\beta_n) = p(\beta_m)$ for $\beta_n(\mathbf{x}) = \beta_m(\mathcal{R}\mathbf{x})$, where $\mathcal{R}$ is the

[1] As is usual in the literature the notation $\beta_n(\mathbf{x})$ is used to denote two things: a function itself and the value of that function at the point $\mathbf{x}$. We will adopt the same notation here hoping that the meaning will be clear from the context and no confusion will arise. In addition, to denote a function itself (not it's value) we will use β_n too, keeping in mind that it is a function of the spatial argument $\mathbf{x}$.

rotation matrix, $\mathcal{R} \in SO(3)$. Analogously, β is parity conserving if $p(\beta_n) = p(\beta_m)$ for $\beta_n(\mathbf{x}) = \beta_m(-\mathbf{x})\ \forall\ \mathbf{x}$. For the following discussion we will consider only statistically homogeneous and parity conserving fields. Usually in cosmology it is assumed that the field of the primordial density perturbation is statistically isotropic as well. But, as we will show in Chapter 3 this might not necessarily be so.

Instead of working with PDF of the random field directly more convenient and observationally more relevant quantities are N-point correlation functions. For example the two-point correlation function is related to the PDF as

$$\langle \beta(\mathbf{x}_1)\, \beta(\mathbf{x}_2) \rangle \equiv \int p(\beta_n)\, \beta_n(\mathbf{x}_1)\, \beta_n(\mathbf{x}_2)\, \mathrm{d}n. \tag{2.1}$$

Integration over n shows that it is the average over all the ensemble. In general the two-point correlation function does not specify the PDF uniquely, one needs to calculate higher order correlators which are defined analogously.

A very powerful way to analyze correlation functions is by decomposing them into the eigenvectors of the translation operator. In flat space this corresponds to the decomposition into Fourier series. But to perform this decomposition it is necessary to chose the box of a certain size with periodic boundary conditions. In the cosmological context the choice of the box size is a very important issue. One requires that the box is large enough so that wave-vectors $\mathbf{k}$ could be treated as continuous and the Fourier series could be replaced by an integral. On the other hand, it is undesirable that the box is infinitely large. It might be that at very large distances the Universe becomes very anisotropic and inhomogeneous. This for example happens in chaotic inflationary models. Therefore, choosing too large a box one would have to take into account unknown physics. Usually it is enough for the box size to be only several orders of magnitude larger than the horizon of the observable Universe, so that $\ln(H_0 L) \sim \mathcal{O}(1)$, where L is the comoving box size and H_0 is the Hubble parameter today. Such a box is called a minimal box [5, 35]. This choice is sufficient to approximate Fourier series as integrals. And we normalize Fourier modes such that

$$\beta_n(\mathbf{x}) = \int \beta_n(\mathbf{k})\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \frac{\mathrm{d}^3 k}{(2\pi)^3}. \tag{2.2}$$

Because $\beta_n(\mathbf{x})$ describes the distribution of real quantities in the Universe, they must be real functions themselves. This translates into the requirement that imaginary Fourier modes $\beta_n(\mathbf{k})$ must satisfy the reality condition $\beta_n(-\mathbf{k}) = \beta_n^*(\mathbf{k})$. We note as well, that if the random field β is statistically isotropic, then $\beta_n(\mathbf{k})$ does not depend on the direction of the wave-vector $\mathbf{k}$, only on it's modulus k, i.e. $\beta_n(\mathbf{k}) = \beta_n(k)$, where $k \equiv |\mathbf{k}|$.

If the random field is invariant under spatial translations, i.e. if it is statistically homogeneous, then the Fourier transform of the two-point correlator in Eq. (2.1) is determined by the reality condition

$$\langle \beta_n(\mathbf{k}_1)\, \beta_n^*(\mathbf{k}_2) \rangle = (2\pi)^3\, \delta(\mathbf{k}_1 - \mathbf{k}_2)\, P_\beta(\mathbf{k})\,, \tag{2.3}$$

where $P_\beta(\mathbf{k}) \equiv \left\langle |\beta_n(\mathbf{k})|^2 \right\rangle$ is called the power spectrum (remember that $\langle \ldots \rangle$ means the ensemble average). Note that the presence of the delta function in this expression is the result of statistical homogeneity of the random field. This relation can be rewritten using $\beta_n(-\mathbf{k}) = \beta_n^*(\mathbf{k})$ as

$$\langle \beta_n(\mathbf{k}_1)\, \beta_n(\mathbf{k}_2) \rangle = (2\pi)^3\, \delta(\mathbf{k}_1 + \mathbf{k}_2)\, P_\beta(\mathbf{k}_1)\,. \tag{2.4}$$

The power spectrum $P_\beta(\mathbf{k})$ is related to the two-point correlation function in the position space by the Wiener-Khinchin theorem. This theorem states that $P_\beta(\mathbf{k})$ is the Fourier transform of the latter

$$P_\beta(\mathbf{k}) = \int \langle \beta_n(\mathbf{x})\, \beta_n(\mathbf{x}+\mathbf{r}) \rangle\, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}} \mathrm{d}\mathbf{r}. \tag{2.5}$$

It is often convenient to use another definition of the power spectrum which differs from the first one just by normalization

$$\mathcal{P}_\beta(\mathbf{k}) \equiv \frac{k^3}{2\pi^2} P_\beta(\mathbf{k})\,. \tag{2.6}$$

Both of these definitions have to satisfy the reality condition, i.e. $\mathcal{P}_\beta(-\mathbf{k}) = \mathcal{P}_\beta(\mathbf{k})$. For the future convenience we will parametrize the directional dependence of the power spectrum as [36]

$$\mathcal{P}_\beta(\mathbf{k}) = \mathcal{P}_\beta^{\mathrm{iso}}(k) \left[1 + g \left(\hat{\mathbf{d}} \cdot \hat{\mathbf{k}} \right)^2 + \ldots \right], \tag{2.7}$$

where $\mathcal{P}_\beta^{\mathrm{iso}}$ is the average over all directions, $\hat{\mathbf{d}}$ is some unit vector, $\hat{\mathbf{k}}$ is the unit vector along $\mathbf{k}$ and $k \equiv |\mathbf{k}|$ is the modulus of $\mathbf{k}$.

The meaning of the power spectrum $\mathcal{P}_\beta$ can be easily understood in case of statistically isotropic perturbations, i.e. when $\mathcal{P}_\beta(\mathbf{k}) = \mathcal{P}_\beta(k)$. Then from the inverse of Eq. (2.5) we find that the variance of the random field β is equal to

$$\sigma_\beta^2(\mathbf{x}) \equiv \left\langle \beta^2(\mathbf{x}) \right\rangle = \frac{1}{(2\pi)^3} \int_0^\infty P_\beta(k)\; \mathrm{d}^3 k. \tag{2.8}$$

Since for statistically isotropic perturbations $P_\beta(k)$ depends only on the modulus of $\mathbf{k}$ it is convenient to express this integral in spherical coordinates. Then the definition in Eq. (2.6) can be rewritten as

$$\sigma_\beta^2 = \int_0^\infty \mathcal{P}_\beta(k)\ \mathrm{d}\ln k. \tag{2.9}$$

Therefore, $\mathcal{P}_\beta(k)$ corresponds to the contribution to the variance σ_β^2 per logarithmic interval in k. And because we assumed statistical homogeneity of β, the variance does not depend on position.

If the power spectrum $\mathcal{P}_\beta$ is scale independent then the integral in Eq. (2.9) is logarithmically divergent. Divergences for large and small k in this integral are avoided by introducing cutoff scales. For large k the cutoff scale R_s corresponds to the smoothing scale and for small k (large spatial distances) R_box corresponds to the maximum size of the box in which we perform calculations

$$\sigma_\beta^2 = \int_{R_\mathrm{box}^{-1}}^{R_\mathrm{s}^{-1}} \mathcal{P}_\beta\, \mathrm{d}\ln k = \mathcal{P}_\beta \ln\frac{R_\mathrm{box}}{R_\mathrm{s}}. \tag{2.10}$$

With the minimal box size, such that $\ln(R_\mathrm{box}/R_\mathrm{s})$ is of order one, the mean-square is roughly of the order of the spectrum.

If Eq. (2.5) is to be applied in the cosmological perturbation theory it requires an additional assumption. In practice we can observe and make measurements only of one Universe. Hence, the ensemble average over one Universe does not make sense and we cannot use this equation directly. To connect theoretical predictions with observations we have to assume the validity of ergodicity for our Universe. This assumption states that the average over the whole ensemble of universes is equivalent to the spatial average over one universe. To see what this means in mathematical language let us write the spatial average of the product of two points over the universe of realization β_n

$$\overline{\beta_n(\mathbf{x})\,\beta_n(\mathbf{x}+\mathbf{r})} = L^{-3}\int \beta_n(\mathbf{x})\,\beta_n(\mathbf{x}+\mathbf{r})\,\mathrm{d}\mathbf{x}, \tag{2.11}$$

where L^{-3} is the box over which the averaging is performed. Then ergodic assumption states that in the limit $L\to\infty$

$$\langle\beta(\mathbf{x})\,\beta(\mathbf{x}+\mathbf{r})\rangle = \overline{\beta_n(\mathbf{x})\,\beta_n(\mathbf{x}+\mathbf{r})}. \tag{2.12}$$

As one can see, this assumption relates averages over the all ensemble of universes, which cannot be measured, to the average over one universe, which can be measured. For Eq. (2.12) to be strictly valid we required an infinite box over which the measurement is performed. Of course this cannot be realized practically. The effect of the finite box introduces the so called 'cosmic variance' - when the separation between points in the correlators approaches the size of the box, the probability that the spatial average differs from the ensemble average increases.

Until now we have considered only the two-point correlation function of Eq. (2.1) which is demanded by the reality condition. If the random field β is Gaussian, this correlator specifies PDF completely. Which means that the three-point correlator vanishes, while the four-point correlator can be expressed as the sum of two-point correlator products and so on. In the non-Gaussian case, the random field has a non-vanishing three-point correlator and the four-point correlator has additional terms which cannot be reduced to the product of two-point correlators. Let us limit ourselves only up to the three-point correlator. Although in cosmological context for some models higher order correlators might be as important as the three-point correlator, for the scope of this thesis the three-point correlator will be sufficient. It can parametrized similarly to Eq. (2.4) as

$$\langle \beta\left(\mathbf{k}_1\right)\beta\left(\mathbf{k}_2\right)\beta\left(\mathbf{k}_3\right)\rangle = \left(2\pi\right)^3\delta\left(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3\right)B_\beta\left(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\right), \tag{2.13}$$

where B_β is called the bispectrum.

2.1.2. The Curvature Perturbation and Observational Constraints

In the previous section we discussed random fields in general. Let us now turn to the discussion of the curvature perturbation ζ which will be the main topic for the rest of this thesis. As was explained in section 1.5 the largest achievement of the inflationary paradigm is that it predicts the statistical properties of the curvature perturbation which can be compared with observations.

Usually observational constraints on the statistical properties of ζ are obtained with the assumption of statistical isotropy. However, one would expect that the presence of anisotropy at 10% level or so would not alter the results significantly. The strongest constraints on ζ comes from the measurements of the CMB and large scale structure which probe the range $\Delta\ln k\sim 10$ [17]. The largest probable scale corresponds to the size of the observable Universe, $k^{-1}\sim H_0^{-1}$.

2.1.2.1. The Power Spectrum

The shape of the power spectrum $\mathcal{P}_\zeta$ is the primary tool to contrast predictions of the inflationary models with observations. To quantify this shape the power spectrum is parametrized as

$$\mathcal{P}_\zeta(k) = \mathcal{P}_\zeta(k_0)\left(\frac{k}{k_0}\right)^{n(k_0)-1+\frac{1}{2}n'}, \tag{2.14}$$

where $k_0 \equiv 0.002\,\mathrm{Mpc}^{-1}$ is the pivot scale, n is called the spectral index and parametrizes the scale dependence of the power spectrum and $n' \equiv \mathrm{d}n/\mathrm{d}\ln k$ is the running of the spectral index. Such parametrization is sufficient because according to observations $n' \ll n$, thus higher derivatives are even smaller and can be neglected. Of course with such simple parametrization one looses sensitivity to the sharp features of the power spectrum. But according to some investigations (e.g. Ref. [37]) such features are not detected. The normalization of the power spectrum $\mathcal{P}_\zeta(k_0)$ was first measured by the COBE satellite and most recently by the WMAP [17]. The present value is

$$\mathcal{P}_\zeta(k_0) = (2.445 \pm 0.096) \times 10^{-9}, \tag{2.15}$$

where this and later intervals are given at 68% CL.

For the simplest, scale invariant case, called Harrison-Zel'dovich or flat power spectrum, $n = 1$ and $n' = 0$. However, according to current observations the spectral index is 3.1 standard deviations away from the Harrison-Zel'dovich one. Indeed, n is smaller than 1. Such power spectrum is called red. With the assumption of negligible running, $n' = 0$, and no gravitational waves the spectral index is determined as

$$n = 0.960 \pm 0.013. \tag{2.16}$$

If the running of the spectral index is allowed then this constraint is relaxed

$$n = 1.017 \pm 0.043 \quad \text{and} \quad n' = -0.028 \pm 0.020, \tag{2.17}$$

where gravitational wave contribution is still neglected. Letting non-negligible contribution from the gravitational waves relaxes these bound even further. However, gravitational waves are not observed yet, and as was mentioned earlier, in this thesis I will assume that their contribution is negligible.

There are several reports of the detection of the angular modulation of the power spectrum in Refs. [38, 39]. Ref. [38] determined the modulation amplitude g defined in

Eq. (2.7) as

$$g = 0.29 \pm 0.031, \tag{2.18}$$

at 9σ confidence level. This is a definite proof of the existence of the preferred direction in the power spectrum. However, these two works also show that this direction is very close to the ecliptic poles, with the galactic coordinates $(l, b) = (96, 30)$. This is a very strong indication that the origin of the observed anisotropy is not cosmological but most probably caused by some systematic effects or comes from within the solar system. Although Ref. [38] have investigated the known systematic effects, including the Zodiacal light, but they could not find any which reproduces the observed signal.

Given the above value of g we may place an upper limit on the anisotropy in the power spectrum of the cosmological origin. In this thesis we will assume that the upper bound on g in the primordial power spectrum is

$$g \lesssim 0.3. \tag{2.19}$$

2.1.2.2. The Bispectrum

Although the shape of the two-point correlator or it's counterpart in the Fourier space, the power spectrum, provides a very valuable information in discriminating inflationary scenarios and constraining physics of the early Universe, it has a limited potential. There are plenty of different inflationary models which predict similar power spectrum. Very powerful additional tools for distinguishing these models are higher order correlators. The Fourier transform of the three-point correlator is called the bispectrum and was defined in Eq. (2.13). While only two points are cross-correlated to obtain the power spectrum an infinitely more configurations are possible by cross-correlating three points. Therefore, the amount of information stored in the bispectrum is immensely richer than in the power spectrum, provided the curvature perturbation is non-Gaussian.

However, as will be seen in section 2.4.1 single field, slow-roll inflationary models predict negligible non-Gaussianity of the curvature perturbation. Observationally interesting non-Gaussianity can be generated only if any of the single field slow-roll assumptions or some combination of them are violated. These can be classified into four classes [40]: 1) single free field, 2) canonical kinetic energy, 3) slow roll and 4) initial Bunch-Davies vacuum. In the first case large non-Gaussianity can be present if the curvature perturbation is generated by the different field from the one which drives inflation (two of such mechanisms are discussed in sections 2.4.2 and 2.4.3) or in the multifield inflation where the curvature perturbation is generated by many fields which drives inflation. In addition if the inflaton cannot be considered as a free field, interaction terms can produce large non-Gaussianity as well. The second condition is violated for example in the k-inflation models [41]. In these class of models the speed of sound is different from the speed of light. The third condition might be violated if, for example, the inflaton potential has some sharp features which result in temporally violation of slow-roll conditions. The fourth assumption considers the initial fluctuations of the field. Usually it is assumed that initial quantum fluctuations correspond to the Bunch-Davies vacuum (see section 2.2.2.4) which results in Gaussian statistics. If, due to some quantum gravitational effects, the initial quantum state does not correspond to the Bunch-Davies vacuum, field perturbations may be non-Gaussian and this non-Gaussianity will be translated into the statistical properties of the curvature perturbation ζ.

Usually non-vanishing three point correlator of the curvature perturbation is parametrized by the non-linearity parameter $f_{\rm NL}$. There are several definitions of $f_{\rm NL}$ in the literature. I will use the one which coincides with the definition used by WMAP team

$$\frac{6}{5} f_{\rm NL} \equiv \frac{B_\zeta\left(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\right)}{P_\zeta\left(\mathbf{k}_1\right) P_\zeta\left(\mathbf{k}_2\right) + \text{c.p.}}, \tag{2.20}$$

where 'c.p.' stands for 'cyclic permutations'.[2]

The simplest form of non-Gaussianity is of the local type which can be written as

$$\begin{aligned} \zeta\left(\mathbf{x}\right) &= \zeta_{\rm g}\left(\mathbf{x}\right) + \zeta_{\rm ng}\left(\mathbf{x}\right) \\ &= \zeta_g\left(\mathbf{x}\right) + \frac{3}{5} f_{\rm NL}\left(\zeta_{\rm g}^2\left(\mathbf{x}\right) - \left\langle\zeta_{\rm g}^2\right\rangle\right), \end{aligned} \tag{2.21}$$

where $\zeta_{\rm g}$ is the Gaussian part with zero mean, $\langle\zeta_g\rangle = 0$.

The strongest constraints on $f_{\rm NL}$ comes from the measurements of the CMB sky. If the non-Gaussianity is of the local type in Eq. (2.21), then from WMAP5 data the constraint with 95% confidence level (CL) is (Ref. [17])

$$-9 < f_{\rm NL}^{\rm local} < 111. \tag{2.22}$$

In this expression 'local' means the 'squeezed' configuration where one momentum is much smaller that the other two, $k_1 \simeq k_2 \gg k_3$. In the equilateral configuration with all three momenta of the same size, $k_1 = k_2 = k_3$, the constraint on $f_{\rm NL}$ is weaker

$$-151 < f_{\rm NL}^{\rm equil} < 253 \tag{2.23}$$

[2]The factor 6/5 comes from the fact that during matter domination, which is the case at the era of decoupling, the Newtonian potential Φ is related to the curvature perturbation by $\Phi = \frac{3}{5}\zeta$.

at the same CL.

The bounds on the magnitude of $f_{\rm NL}$ will improve substantially in the very near future. If it is not detected by the Planck satellite, the constraints will reduce to $|f_{\rm NL}| \lesssim 5$ at 95% CL, which is very close to the limit of an ideal experiment of $|f_{\rm NL}| \approx 3$ at 95% CL, limited by the cosmic variance [42]. The above bounds are given with the assumption that $f_{\rm NL}$ is isotropic. Ref. [43] analyzed the WMAP5 data for the angular modulation of $f_{\rm NL}^{\rm local}$. However, due to the large measurement errors no conclusive statement can be made.

2.2. Scalar Field Quantization

In this section we discuss the quantization procedure of quantum field theory (QFT) in flat space-time (FST) and then generalize this formalism to curved space-time (CST). The discussion is solely about quantization of scalar fields, because they are the most simple ones and help to highlight the underlying principles. The extension to vector fields will be given in Chapter 3.

Quantum mechanics was firstly formulated in the so called Schrödinger picture in which operators are time independent and state vectors evolve according to the Schrödinger equation. Equally well one can formulate this theory in the Heisenberg picture, where state vectors are constant but operators are changing with time. Quantum field theory can be formulated in both of these pictures as well, but this is much easier done in the Heisenberg picture, where operators are time dependent and satisfying field equations. Hence, the name quantum *field* theory.

2.2.1. Quantization in Flat Space-Time

Field quantization in FST may be presented in two ways [44]. In the first one we consider a classical field theory. Expand the field in Fourier modes and find that Fourier coefficients obey the equation of harmonic oscillator. With every harmonic oscillator we associate a position variable and the conjugate momentum in field space. The classical harmonic oscillator is then first quantized. This is done by substituting c-numbers (classical numbers) of the position and momentum to the q-numbers (quantum numbers) and imposing commutation relations which are the result of the Heisenberg uncertainty principle. Then one finds that the Fourier coefficients (which are now operators) correspond to the raising and lowering operators in the Fock space, which are commonly called creation and annihilation operators respectively.

Another approach is to quantize the degrees of freedom of the classical field directly. In this approach one identifies the degrees of freedom of the field, finds their conjugate pairs of variables, changes them into q-numbers and imposes the same commutation relations as in the previous case. Only after quantization do we resort to the Fourier series. Again, we find that Fourier coefficients correspond to creation and annihilation operators.

Results of both methods are the same. Although the first method is more intuitive and easier interpretable, the second method is more directly generalizable to CST. Hence, in this section we will take a standpoint of the second method in order to present the FST formalism in a way which is directly generalizable to the CST case.

In the classical field theory equations of motion (EoM) for fields are obtained using the least action principle. Forming the action as

$$S(\phi) = \int \mathcal{L}\left(\phi_I, \partial_\mu \phi_I\right) \mathrm{d}^4 x, \tag{2.24}$$

the classical field equations are calculated by requirement that the variation of the action should vanish

$$\frac{\delta S}{\delta \phi_I(x)} = 0, \tag{2.25}$$

where $x = (t, \mathbf{x})$ and $\phi_I(x)$ are classical fields. In principle these fields could be complex and after quantization we would find that they describe pairs of particles and antiparticles, i.e. the field would have a charge. But in context of producing the curvature perturbation in the Universe we are interested only in the neutral particles, which agrees with observations of the neutrality of the Universe. Therefore we will be interested only in real fields $\phi_I(x)$.

Let us start with a free, massive, real scalar field. The relativistically invariant Lagrangian for such field is written as

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2, \tag{2.26}$$

where m is the mass term. The variation of the action in Eq. (2.24) with this Lagrangian gives the familiar Klein-Gordon field equation for the relativistic field

$$\left[\partial_\mu \partial^\mu + m^2\right] \phi = 0. \tag{2.27}$$

The general solution for this equation can be written as the superposition of the complete set of orthonormal solutions, $\{u_\alpha(x)\}$. Where orthonormality is defined through the scalar product. For the Klein-Gordon equation in FST the scalar product of two wave functions is

$$(u_m, u_n) = i \int u_n^* \overleftrightarrow{\partial_0} u_m \mathrm{d}\mathbf{x} \equiv i \int \left(u_m^* \partial_0 u_n - u_n \partial_0 u_m^*\right) \mathrm{d}\mathbf{x}. \tag{2.28}$$

Then the complete set of orthonormal solutions $\{u_\alpha(x)\}$ must satisfy

$$(u_m, u_n) = \delta_{mn}, \quad (u_m^*, u_n^*) = -\delta_{mn} \quad \text{and} \quad (u_m, u_n^*) = (u_m^*, u_n) = 0. \tag{2.29}$$

Indices m and n can be discrete or continuous. In the latter case Kronecker symbols δ_{mn} should be replaced by Dirac delta functions. In these expressions $\{u_m(x)\}$ and $\{u_m^*(x)\}$ denotes a complete set of positive and negative frequency solutions respectively. Using these sets of solutions the general solution of Eq. (2.27) may be written as the sum of $\{u_m, u_m^*\}$:

$$\phi(x) = \sum_m \left[a_m u_m(x) + a_m^\dagger u_m^*(x)\right], \tag{2.30}$$

where coefficients a_m are given by

$$a_m = (\phi, u_m) \tag{2.31}$$

In the classical field theory $\phi(x)$ actually describes an infinite number of degrees of freedom at each space point $\mathbf{x}$. One can find a conjugate momentum for each of these degrees of freedom by using equation

$$\pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = \dot{\phi}. \tag{2.32}$$

In this way for each spatial point $\mathbf{x}$ we prescribe a generalized coordinate variable, $\phi(x)$, and a conjugate momentum, $\pi(x)$. Field quantization proceeds by analogy with quantum mechanics, which is changing c-numbers $\phi(x)$ and $\pi(x)$ into q-numbers $\hat{\phi}(x)$ and $\hat{\pi}(x)$ and imposing commutation relations for them

$$\left[\hat{\phi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{x}')\right] = i\delta^3(\mathbf{x} - \mathbf{x}'), \ \left[\hat{\phi}(t, \mathbf{x}), \hat{\phi}(t, \mathbf{x}')\right] = \left[\hat{\pi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{x}')\right] = 0. \tag{2.33}$$

Because the field variable $\phi(x)$ was promoted into the operator $\hat{\phi}(x)$, the expansion coefficients in Eq. (2.30) have to be operators as well, i.e. the substitution $a_m \to \hat{a}_m$ must be made. And commutation relations for these coefficients may be calculated from

Eqs. (2.33) [45, 46, 47]:

$$\left[\hat{a}_m, \hat{a}_n^\dagger\right] = \delta_{mn} \quad \text{and} \quad [\hat{a}_m, \hat{a}_n] = \left[\hat{a}_m^\dagger, \hat{a}_n^\dagger\right] = 0. \tag{2.34}$$

Operators $\hat{a}_m$ and $\hat{a}_m^\dagger$ are interpreted as the rising and lowering operators in Fock space, or creation and annihilation operators respectively,

$$\hat{a}_m^\dagger \left|n_m, \{u\}\right\rangle = \sqrt{n_m + 1} \left|n_m + 1, \{u\}\right\rangle \quad \text{and} \quad \hat{a}_m \left|n_m, \{u\}\right\rangle = \sqrt{n_m} \left|n_m - 1, \{u\}\right\rangle, \tag{2.35}$$

where n_m is the number of particles in a state m, notation of $\{u\}$ in the ket reminds us that the definition is for particular complete set of orthonormal mode functions $\{u_m\}$. In these equations coefficients $\sqrt{n_m + 1}$ and $\sqrt{n_m}$ are chosen for the correct normalization of the vacuum state $\langle 0, \{u\} | \, 0, \{u\} \rangle = 1$, where the vacuum of this Fock space is defined as

$$\hat{a}_m \left|0, \{u\}\right\rangle = 0. \tag{2.36}$$

For the following discussion it will be useful to introduce an operator $\hat{N}$ such that

$$\hat{N}_m \equiv \hat{a}_m^\dagger \hat{a}_m. \tag{2.37}$$

The meaning of this operator becomes clear when we take the expectation value $\langle n_m | \, \hat{N}_m \, | n_m \rangle$,

$$\langle n_m | \, \hat{N}_m \, | n_m \rangle = \langle n_m | \, \hat{a}_m^\dagger \hat{a}_m \, | n_m \rangle = n_m. \tag{2.38}$$

Hence $\hat{N}_m$ can be interpreted as the number operator of m particles.

2.2.1.1. Interpretation of $\hat{a}_m$ and $\hat{a}_m^\dagger$

We have mentioned that operators $\hat{a}_m$ and $\hat{a}_m^\dagger$ are interpreted as creation and annihilation operators. What justifies such interpretation? To show this let us find mode functions $\{u_m\}$ explicitly.

First of all $\{u_m\}$ must represent particles with positive energy, therefore these functions must be positive frequency solutions of Eq. (2.27), where positive frequency is defined along some time-like Killing vector satisfying Lie equation

$$\pounds_\xi u_m = -i\omega_m u_m, \quad \omega_m > 0. \tag{2.39}$$

To find such solution in the case of FST is a straightforward task related to the fact that the Poincaré group is the symmetry group of Minkowski space-time. Therefore, FST possesses the global time-like Killing vector $\xi^\mu = (1, 0, 0, 0)$. With this Killing vector Eq. (2.39) becomes

$$\frac{\partial u_m}{\partial t} = -i\omega_m u_m, \tag{2.40}$$

from which it is clear that functions u_m must be proportional to

$$u_m \propto \mathrm{e}^{-i\omega_m t}. \tag{2.41}$$

Let us conjecture that the full set of orthonormal mode functions have the form of plane waves

$$u_{\mathbf{k}} = \frac{A_{\mathbf{k}}}{(2\pi)^{3/2}} \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)}, \tag{2.42}$$

where instead of the discrete indices m now we have continuous indices $\mathbf{k}$, corresponding to the wave number of the plane wave. At the moment $\mathbf{k}$ is just a parameter of the mode function not related to the momentum. $A_{\mathbf{k}}$ is the normalization constant which will be fixed later. We can easily check that these mode functions satisfy the orthonormality conditions in Eqs. (2.29). The frequency $\omega_{\mathbf{k}}$ is defined using the Klein-Gordon equation (2.27) to be

$$\omega_{\mathbf{k}} = +\sqrt{\mathbf{k}^2 + m^2}. \tag{2.43}$$

This equation is called relativistic dispersion relation. Note that we have chosen $\omega_{\mathbf{k}} > 0$ in accordance with Eq. (2.39).

With these mode functions the expansion of the field operator in Eq. (2.30) becomes

$$\hat{\phi}(x) = \int A_{\mathbf{k}} \left[\hat{a}_{\mathbf{k}} \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)} + \hat{a}^\dagger_{\mathbf{k}} \mathrm{e}^{-i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)} \right] \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}}. \tag{2.44}$$

And because $\mathbf{k}$ is the continuous index, the Kronecker delta in commutation relations of Eq. (2.34) must be changed into Dirac delta

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'} \right] = \delta^3 \left(\mathbf{k} - \mathbf{k}' \right) \tag{2.45}$$

with other commutators being zero. Using these conditions and commutation relations for the field operator in Eq. (2.33) we find

$$\left[\hat{\phi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{x}') \right] = i \int A^2_{\mathbf{k}} \omega_{\mathbf{k}} \left(\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} + \mathrm{e}^{-i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \right) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} = i\delta^3 \left(\mathbf{x} - \mathbf{x}' \right), \tag{2.46}$$

which fixes the normalization constant

$$A_{\mathbf{k}} = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}. \tag{2.47}$$

A more physical motivation for the normalization constant being proportional to $A_{\mathbf{k}} \propto \omega_{\mathbf{k}}^{-1/2}$ is that the expansion of the operator $\hat{\phi}$ in Eq. (2.44) with this normalization becomes relativistically invariant.

To motivate the interpretation of $\hat{a}_{\mathbf{k}}$ and $\hat{a}^{\dagger}_{\mathbf{k}}$ as annihilation and creation operators let us exploit the space translational symmetry of the FST due to which the following relation must hold:

$$\hat{\phi}\left(t, \mathbf{x}\right) = \hat{\phi}\left(t, \mathbf{x} + \delta\mathbf{x}\right), \tag{2.48}$$

where $\delta\mathbf{x}$ is the infinitesimal translation vector. The process of spatial translation of the system may be described using a unitary transformation

$$\hat{U} = \mathrm{e}^{i\hat{\mathbf{P}}\cdot\mathbf{l}}, \tag{2.49}$$

where $\mathbf{l}$ is the finite translation vector. Hence, we can write

$$\hat{\phi}\left(t, \mathbf{x} + \mathbf{l}\right) = \hat{U}^{-1}\hat{\phi}\left(t, \mathbf{x}\right)\hat{U}. \tag{2.50}$$

For the infinitesimal translation $\mathbf{l} = \delta\mathbf{x}$ the exponent in Eq. (2.49) may be expanded to the first order as $\exp\left(i\hat{\mathbf{P}}\cdot\delta\mathbf{x}\right) = \hat{I} + i\hat{\mathbf{P}}\cdot\delta\mathbf{x}$ and the translational transformation Eq. (2.50) becomes

$$\hat{\phi}\left(t, \mathbf{x} + \delta\mathbf{x}\right) = \left(\hat{I} - i\hat{\mathbf{P}}\cdot\delta\mathbf{x}\right)\hat{\phi}\left(t, \mathbf{x}\right)\left(\hat{I} + i\hat{\mathbf{P}}\cdot\delta\mathbf{x}\right) = \hat{\phi}\left(t, \mathbf{x}\right) + i\left[\hat{\phi}\left(t, \mathbf{x}\right), \hat{\mathbf{P}}\right]\delta\mathbf{x}. \tag{2.51}$$

On the other hand for infinitesimal $\delta\mathbf{x}$ we can also write

$$\hat{\phi}\left(t, \mathbf{x} + \delta\mathbf{x}\right) = \hat{\phi}\left(t, \mathbf{x}\right) + \nabla_{\mathbf{x}}\hat{\phi}\left(t, \mathbf{x}\right)\cdot\delta\mathbf{x}. \tag{2.52}$$

Combining these two equations we find that the commutator of $\hat{\phi}$ and $\hat{\mathbf{P}}$ defines the gradient of the quantum field $\hat{\phi}$:

$$\left[\hat{\phi}\left(t, \mathbf{x}\right), \hat{\mathbf{P}}\right] = -i\nabla_{\mathbf{x}}\hat{\phi}\left(t, \mathbf{x}\right). \tag{2.53}$$

Let's concentrate now only on the positive frequency part of Eq. (2.44)

$$\hat{\phi}^{+}(x) \equiv \int \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}\hat{a}_{\mathbf{k}}\mathrm{d}\tilde{\mathbf{k}}, \tag{2.54}$$

where

$$\mathrm{d}\tilde{\mathbf{k}} \equiv \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}}. \tag{2.55}$$

Inserting this expression into Eq. (2.53) we find

$$\int \mathbf{k}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\hat{a}_{\mathbf{k}}\mathrm{d}\tilde{\mathbf{k}} = \left[\hat{\phi}^{+}(x), \hat{\mathbf{P}}\right]. \tag{2.56}$$

Using commutation relations in Eq. (2.45) one can show that this equation is satisfied if the operator $\hat{\mathbf{P}}$ is equal to

$$\hat{\mathbf{P}} = \int \mathbf{k}\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\mathrm{d}\tilde{\mathbf{k}}. \tag{2.57}$$

Acting with $\hat{\mathbf{P}}$ on the state $\hat{a}^{\dagger}_{\mathbf{k}}|0\rangle$ gives

$$\hat{\mathbf{P}}\hat{a}^{\dagger}_{\mathbf{k}}|0,\{u\}\rangle = \mathbf{k}\hat{a}^{\dagger}_{\mathbf{k}}|0,\{u\}\rangle. \tag{2.58}$$

What does this relation mean? From Nöther's theorem we know that translational invariance corresponds to the conservation of momentum. From which follows that the generator of the infinitesimal spatial translation is the operator for the total momentum, i.e. the operator $\hat{\mathbf{P}}$. From Eq. (2.58) it is clear that the state $|\mathbf{k},\{u\}\rangle = \hat{a}^{\dagger}_{\mathbf{k}}|0,\{u\}\rangle$ is the eigenstate of the total momentum $\hat{\mathbf{P}}$ with the eigenvalue $\mathbf{k}$. Remember, that until know $\mathbf{k}$ was just the index for the mode function. From the last relation $\mathbf{k}$ can be interpreted as the momentum and $\hat{a}^{\dagger}_{\mathbf{k}}$ acts as the momentum rising operator.

If instead of using spatial translation symmetry we would have used time translational symmetry of the FST with the corresponding unitary operator

$$\hat{T} = \mathrm{e}^{-i\hat{H}t}, \tag{2.59}$$

we would have found that infinitesimal time translation gives

$$\hat{H} = \int \left(\omega_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} + c\hat{I}\right)\mathrm{d}\tilde{\mathbf{k}}, \tag{2.60}$$

where c is an arbitrary constant. The divergent constant term in the above equation in FST can be subtracted by appropriate procedures, but at the moment it must not concern us. The analogous arguments which relates $\hat{\mathbf{P}}$ with the total momentum operator, leads to $\hat{H}$ interpretation as the energy operator. Acting on the $\hat{a}^{\dagger}_{\mathbf{k}}\,|0,\{u\}\rangle$ state we would find that the operator $\hat{a}^{\dagger}_{\mathbf{k}}$ raises the energy of the state by one unit and that $\omega_{\mathbf{k}}$ can be interpreted as the total energy of that unit or quantum.

2.2.2. Quantization in Curved Space-Time

In the previous section we have described how fields are quantized in FST. This procedure is sufficient for the particle physics models, which studies only three fundamental forces of nature: electromagnetic, weak and strong. But to give a complete description of the Universe we need to study how all four fundamental forces, including gravity, shape and influence each other as well as the structure of the Universe. This requires a theory which puts all four forces on the same footing. Unfortunately such theory is still absent - the gravitational force resists the unification with the other three. In the presence of such resistance the only hope is to use a semiclassical description of the Nature, where we treat a classical gravitational background on which other quantized fields live.

This approximation can be justified by noting that the Planck scale is the only scale of GR. If we consider small perturbations of the gravitational field and try to quantize them, then m_{Pl}^2 plays the role of the coupling constant. Hence, perturbation theory should be a good approximations for the energies much smaller than m_{Pl}.

We gain confidence in this approach from the early development stages of the quantum electrodynamics (QED) theory, where the electromagnetic field was considered as a classical background on which fully quantized matter lives. And this method is fully consistent with a complete QED theory.

2.2.2.1. From FST to CST

The quantization of the field living in CST proceeds in the same line as the quantization in FST. First we write the action for the field. In CST the analog of Eq. (2.24) would be

$$S=\int\sqrt{-\mathcal{D}_g}\,\mathcal{L}\left(\phi_I,\nabla_\mu\phi_I\right)\mathrm{d}^4x, \tag{2.61}$$

where $\mathcal{D}_g\equiv\det\left(g_{\mu\nu}\right)$ is the determinant of the metric and ∇_μ is the covariant derivative. With the massive free scalar field Lagrangian, which is written in Eq. (2.26), this action becomes

$$S=\int\sqrt{-\mathcal{D}_g}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2\right)\mathrm{d}^4x. \tag{2.62}$$

Taking the variation with respect to the scalar field, $\delta S/\delta\phi = 0$, we arrive at the field equation (cf. Eq. (2.27))

$$\left(\Box + m^2\right)\phi = 0, \tag{2.63}$$

where the $\Box$ operator is defined by

$$\Box\phi \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu = \frac{1}{\sqrt{-\mathcal{D}_g}}\partial_\mu\left(\sqrt{-\mathcal{D}_g}g^{\mu\nu}\partial_\nu\phi\right). \tag{2.64}$$

The scalar product of Eq. (2.28) for the Klein-Gordon equation in CST (Eq. (2.63)) must be generalized as well (e.g. Ref. [46])

$$(u_m, u_n) = i\int_\Sigma u_n^* \overleftrightarrow{\partial}_\mu u_m \mathrm{d}\Sigma^\mu, \tag{2.65}$$

where $\mathrm{d}\Sigma^\mu \equiv n^\mu \mathrm{d}\Sigma$ and $\mathrm{d}\Sigma$ is the volume element in a given space-like hypersurface while n^μ is the orthogonal to this hypersurface time-like unit vector. It may be shown that the value of (u_m, u_n) is independent on the choice of the space-like hypersurface Σ, i.e. $(u_m, u_n)_{\Sigma_1} = (u_m, u_n)_{\Sigma_2}$.

As in the FST, functions u_m must satisfy the orthonormality conditions in Eq. (2.29) and then we can write a general solution of Eq. (2.63) as the superposition of a complete set of positive frequency $\{u_m\}$ and negative frequency $\{u_m^*\}$ solutions

$$\phi(x) = \sum_m \left[a_m u_m(x) + a_m^\dagger u_m^*(x)\right]. \tag{2.66}$$

The quantization of the field proceeds exactly as in the FST: change c-numbers ϕ and π into q-numbers $\hat{\phi}$ and $\hat{\pi}$ and impose canonical commutation relations of Eq. (2.33). Then operators $\hat{a}_m$ and $\hat{a}_m^\dagger$ are interpreted as lowering and rising operators in the Fock space

$$\hat{a}_m^\dagger \left|n_m, \{u\}\right\rangle = \sqrt{n_m + 1}\left|n_m + 1, \{u\}\right\rangle, \tag{2.67}$$

with the vacuum defined as

$$\hat{a}_m \left|0, \{u\}\right\rangle = 0. \tag{2.68}$$

As in the previous section $\{u\}$ inside the ket reminds us that we are dealing with the vacuum defined by the complete set of orthonormal mode functions $\{u_m\}$. This emphasis on the choice of mode functions becomes very important in CST as will be seen in a moment.

2.2.2.2. Bogolubov Transformations

In section 2.2.1.1 it was shown that in FST a natural choice for the complete, orthonormal set of mode functions exist, which are plane waves of Eq. (2.42). And it was emphasized that this happens because the Poincaré group is the symmetry group of the Minkowski space-time. Hence, using a global time-like Killing vector, $\partial/\partial t$, of this symmetry group we could pick-out positive frequency solutions $u_{\mathbf{k}} \propto \exp\left(-i\omega_{\mathbf{k}}t\right)$. And in all Lorentz frames, where t is the time coordinate, these mode functions define the same vacuum state. But in CST the Poincaré group is no longer a symmetry group and in general there will be no global time-like Killing vectors in respect to which one could define positive frequency solutions. Therefore, the field expansion in mode functions $\{u_m\}$ in Eq. (2.66) is as good as in any other complete set of orthonormal functions:

$$\phi\left(x\right) = \sum_m \left[b_m v_m\left(x\right) + b_m^{\dagger} v_m^*\left(x\right)\right]. \tag{2.69}$$

After quantization the vacuum state for this expansion is defined by

$$\hat{b}_m \left|0, \{v\}\right\rangle = 0. \tag{2.70}$$

The definition of the vacuum state in Eq. (2.68) with mode functions $\{u_m\}$ at least formally differs from the definition with the mode functions $\{v_m\}$ in Eq. (2.70). Shortly it will be clear that this difference is not only formal but indeed both states $|0,\{u\}\rangle$ and $|0,\{v\}\rangle$ correspond to a different physical vacuum. Which means that there is no way to define uniquely a state without particles: what for one is a vacuum state, for the other this state contains particles. In such situation the notion of "the physical particle" becomes ambiguous.

Since both sets of mode functions are complete orthonormal sets of solutions, each function in one set can be expanded in terms of the another set, i.e.

$$v_n = \sum_m \left(\alpha_{nm} u_m + \beta_{nm} u_m^*\right) \quad \text{or} \quad u_m = \sum_n \left(\alpha_{nm}^* v_n + \beta_{nm} v_n^*\right). \tag{2.71}$$

These are the so called Bogolubov transformations, and matrices α_{nm} and β_{nm} are called Bogolubov coefficients. It can be easily checked that these coefficients satisfy the relations

$$\sum_l \left(\alpha_{nl}\alpha_{ml}^* - \beta_{ml}\beta_{nl}^*\right) = \delta_{nm}, \tag{2.72}$$

$$\sum_l \left(\alpha_{ml}\beta_{nl} - \beta_{ml}\alpha_{nl}\right) = 0. \tag{2.73}$$

Comparing Eqs. (2.66) and (2.69) and using Bogolubov transformations in Eq. (2.71) we find the relation between creation and annihilation operators of one set of mode functions and the other:

$$\hat{a}_m = \sum_n \left(\alpha_{nm}\hat{b}_n + \beta^*_{nm}\hat{b}^\dagger_n\right) \quad \text{or} \quad \hat{b}_n = \sum_m \left(\alpha^*_{nm}\hat{a}_m - \beta^*_{nm}\hat{a}^\dagger_m\right). \tag{2.74}$$

Using these relations we can calculate the expectation value of the number operator defined by $\hat{N}^{\{u\}}_m \equiv \hat{a}^\dagger_m\hat{a}_m$ (cf. Eq. (2.37)). Acting with $\hat{N}^{\{u\}}_m$ on the vacuum defined by the mode functions $\{v_m\}$ we find

$$\begin{aligned} \langle 0,\{v\}|\,\hat{N}^{\{u\}}_m\,|0,\{v\}\rangle &= \langle 0,\{v\}|\sum_{n,n'}\left(\alpha^*_{nm}\hat{b}^\dagger_n + \beta_{nm}\hat{b}_n\right)\left(\alpha_{n'm}\hat{b}_{n'} + \beta^*_{n'm}\hat{b}^\dagger_{n'}\right)|0,\{v\}\rangle \\ &= \langle 0,\{v\}|\sum_{n,n'}\beta_{nm}\beta^*_{n'm}\hat{b}_n\hat{b}^\dagger_{n'}\,|0,\{v\}\rangle \\ &= \sum_n |\beta_{mn}|^2 . \end{aligned} \tag{2.75}$$

This shows that the vacuum defined by the complete set $\{v_m\}$ contains particles of the mode functions $\{u_m\}$.

The freedom of the choice of mode functions and the related ambiguity of the vacuum state constitutes the main problem of quantum field theory in curved space-time. One is naturally led to ask, which is "the physical vacuum" and what are the observables of such theory. In general, there is no way to pick out one particular set of mode functions. But in some space-times, which have a high degree of symmetry, this might be possible. As will be seen in the following subsection, this for example happens in a space-time with maximal spatial symmetry such as FRW and (quasi) de Sitter universes. The phenomena described in Eq. (2.75) are very important in inflationary particle creation.

2.2.2.3. Quantization in Spatially Homogeneous and Isotropic Backgrounds

In this section we describe the process of the scalar field particle creation in the exponentially expanding Universe. Let us consider spatially homogeneous and isotropic FRW metric of Eq. (1.2). But instead of using the cosmic time t we rewrite this metric in terms of the conformal time τ defined as

$$\tau(t) \equiv \int^t \frac{\mathrm{d}t'}{a(t')}. \tag{2.76}$$

Then the line element with FRW metric becomes manifestly conformal to Minkowski space-time

$$\mathrm{d}s^2 = a^2(\tau)\left(\mathrm{d}\tau^2 - \mathrm{d}\mathbf{x}^2\right). \tag{2.77}$$

In the FRW metric the action of the free massive scalar field written Eq. (2.62) becomes

$$S = \frac{1}{2}\int a^2\left(\phi'^2 - (\nabla\phi)^2 - a^2m^2\phi^2\right)\mathrm{d}\mathbf{x}\mathrm{d}\tau, \tag{2.78}$$

where the prime denotes the derivative with respect to the conformal time, $' \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}$ and $\nabla \equiv \partial_i$ is the spatial gradient. As was already performed several times, taking the variation of this action gives the field equation

$$\phi'' + 2\frac{a'}{a}\phi' - \nabla^2\phi + a^2m^2\phi = 0. \tag{2.79}$$

This equation is very similar to the Klein-Gordon equation in FST given in Eq. (2.27), except that it has a friction term $2a'/a \cdot \phi'$. Let us transform this equation in such a way that it does become like the Klein-Gordon equation in FST. This can be achieved using the following mathematical trick, which brings any second order linear differential equation to it's normal form. If the equation is given as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)\,y = 0, \tag{2.80}$$

then the transformation

$$u = y\mathrm{e}^{\frac{1}{2}\int^x P(x')\mathrm{d}x'} \tag{2.81}$$

brings it into the form of the harmonic oscillator

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \left(Q - \frac{1}{2}\frac{\mathrm{d}P}{\mathrm{d}x} - \frac{1}{4}P^2\right)u = 0. \tag{2.82}$$

For the equation (2.79) the analogous transformation would be

$$\chi \equiv \phi\mathrm{e}^{\frac{1}{2}\int 2\frac{a'}{a}\mathrm{d}\tau} = a(\tau)\,\phi, \tag{2.83}$$

which transforms Eq. (2.79) into the form

$$\chi'' - \nabla^2\chi + \left(a^2m^2 - \frac{a''}{a}\right)\chi = 0. \tag{2.84}$$

This equation does look like the Klein-Gordon one in FST except the time varying mass.

The quantization of the scalar field χ again proceeds as in the previous section: find the conjugate momentum of the field, which in conformal FRW space-time is $\pi \equiv \delta\mathcal{L}/\delta\chi' = \chi' - \frac{a'}{a}\chi$, make changes of c-numbers into q-numbers, i.e. $\chi \to \hat{\chi}$ and $\pi \to \hat{\pi}$ and impose canonical commutation relations (cf. Eq. (2.33))

$$\left[\hat{\chi}\left(\tau,\mathbf{x}\right),\hat{\pi}\left(\tau,\mathbf{x}'\right)\right] = i\delta\left(\mathbf{x}-\mathbf{x}'\right),\ \left[\hat{\chi}\left(\tau,\mathbf{x}\right),\hat{\chi}\left(\tau,\mathbf{x}'\right)\right] = \left[\hat{\pi}\left(\tau,\mathbf{x}\right),\hat{\pi}\left(\tau,\mathbf{x}'\right)\right] = 0. \quad (2.85)$$

The field operator $\hat{\chi}$ expanded into the creation and annihilation operators is written as (c.f. Eqs. (2.66) or (2.69))

$$\hat{\chi}\left(x\right) = \sum_m \left[\hat{a}_m \chi_m\left(x\right) + \hat{a}_m^\dagger \chi_m^*\left(x\right)\right], \quad (2.86)$$

where $x = (\tau, \mathbf{x})$ from the metric in Eq. (2.77).

Mode functions $\chi_m\left(x\right)$ must satisfy the orthonormality conditions in Eq. (2.29). The general scalar product of Eq. (2.65) in the FRW metric becomes

$$\left(\chi_m, \chi_n\right) = i\int \left(\chi_n^*\chi_m' - \chi_m\chi_n^{*\prime}\right)\mathrm{d}\mathbf{x}. \quad (2.87)$$

Because the $(0,i)$ and $(i,0)$ components of the FRW metric in Eq. (2.77) are zero, the mode functions $\chi_m\left(x\right)$ can be chosen in such a way that the temporal and spatial parts are separated

$$\chi_m\left(\tau,\mathbf{x}\right) \equiv \left(2\pi\right)^{-3/2}\chi_{\mathbf{k}}\left(\tau\right)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}, \quad (2.88)$$

where $\mathbf{k}$ is now the continuous expansion coefficient and the factor $\left(2\pi\right)^{-3/2}$ is pulled out in order for the normalization of $\chi_{\mathbf{k}}\left(\tau\right)$ (see Eq. (2.29)) to give the Wronskian of the form

$$\chi_{\mathbf{k}}\chi_{\mathbf{k}}^{*\prime} - \chi_{\mathbf{k}}^*\chi_{\mathbf{k}}' = i, \quad (2.89)$$

which is required from the orthonormality condition and is obtained using the scalar product in Eq. (2.87).

With the ansatz in Eq. (2.88) the expansion of the operator $\hat{\chi}$ in Eq. (2.86) becomes

$$\hat{\chi}\left(x\right) = \int \left(\hat{a}_{\mathbf{k}}\chi_{\mathbf{k}}\left(\tau\right)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_{\mathbf{k}}^\dagger\chi_{\mathbf{k}}^*\left(\tau\right)\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\right)\frac{\mathrm{d}\mathbf{k}}{\left(2\pi\right)^{3/2}}. \quad (2.90)$$

Substituting $\hat{\chi}\left(x\right)$ into Eq. (2.84) we find that functions $\chi_{\mathbf{k}}\left(\tau\right)$ must satisfy the equation of motion

$$\chi_{\mathbf{k}}(\tau)'' + (aH)^2 \left[\left(\frac{m}{H}\right)^2 + \left(\frac{k}{aH}\right)^2 - \frac{\dot{H}}{H^2} - 2 \right] \chi_{\mathbf{k}}(\tau) = 0, \tag{2.91}$$

where we used $a''/a^3 = \dot{H} + 2H^2$. One can see that the equation of motion for $\chi_{\mathbf{k}}$ is formally the same as of the harmonic oscillator with the time dependent frequency

$$\omega_{\mathbf{k}}^2(\tau) \equiv a^2 m^2 + k^2 - \frac{a''}{a} = (aH)^2 \left[\left(\frac{m}{H}\right)^2 + \left(\frac{k}{aH}\right)^2 - \frac{\dot{H}}{H^2} - 2 \right]. \tag{2.92}$$

Although Eq. (2.91) constraints the time dependence of functions $\chi_{\mathbf{k}}(\tau)$ it does not determine the function uniquely. In fact, any function which satisfies Eq. (2.89) will be as good a choice as $\chi_{\mathbf{k}}(\tau)$. As was explained in the previous subsection, this fact deprive us of possibility to determine a vacuum state, defined as $\hat{a}_{\mathbf{k}} |0, \{\chi_{\mathbf{k}}\}\rangle$, which would be seen as absent of particles by any observer. On the other hand, the quantum field theory in FST is a very successful theory although we do live in the expanding Universe, i.e. curved space-time. Therefore, we can expect that it is possible to pick out some special definition of the vacuum which would give correct predictions for laboratory experiments. The main reason why flat space-time QFT is so successful from this point of view is that it describes phenomena which take place in a very weak gravitational field, or in other words, very weakly curved space-time, which may be neglected.

This can be easily seen from Eq. (2.91). If we neglect the effect of gravity, which corresponds to taking $a = 1$ and therefore $H = 0$, the frequency term in Eq. (2.92) becomes constant, $\omega_{\mathbf{k}}^2 = m^2 + k^2 = \text{const}$, the same as in FST in Eq. (2.43).[3] With a constant frequency, functions $\chi_{\mathbf{k}}(\tau)$ (or functions $u_m(t)$ in Eq. (2.41)) have the time dependent part $\exp(-i\omega_{\mathbf{k}}\tau)$. The vacuum defined in this way will be the same for all inertial observers at all times. But the frequency term in the expanding Universe in Eq. (2.91) is time dependent. Hence, the vacuum defined at time τ_i will contain particles as seen by the observer at some later time. This is the main reason why particles get produced during inflation, but let us postpone this discussion until a bit later. At the moment the important thing is the choice of initial conditions which would fix the form of mode functions and therefore the initial vacuum state.

[3]This motivates us to interpret the expansion coefficient $\mathbf{k}$ as the comoving momentum of the particle (cf. section 2.2.1.1) and $\mathbf{k}/a$ as the physical momentum.

2.2.2.4. The Vacuum State in FRW Background

Although, as was mentioned earlier, in general, the particle concept in CST is ambiguous, in some special cases it is possible to define an approximate particle concept which would be as close as possible to the one known from QFT in FST. This is the case, for example, in space-times described by the FRW metric or anisotropic Bianchi universes. In the case of the present interest we may look for the solution of Eq. (2.91) with the ansatz

$$\chi_{\mathbf{k}}(\tau) = \frac{1}{\sqrt{2W_{\mathbf{k}}(\tau)}} \mathrm{e}^{-i\int_{\eta_i}^{\eta} W_{\mathbf{k}}(\tau')\mathrm{d}\tau'}. \tag{2.93}$$

The factor $1/\sqrt{2}$ is chosen so that $\chi_{\mathbf{k}}(\tau)$ would satisfy Eq. (2.89) and the function $W_{\mathbf{k}}(\tau)$ satisfies

$$W_{\mathbf{k}}^2(\tau) = \omega_{\mathbf{k}}^2(\tau) - \left[\frac{1}{2}\frac{W_{\mathbf{k}}''}{W_{\mathbf{k}}} - \frac{3}{4}\left(\frac{W_{\mathbf{k}}'}{W_{\mathbf{k}}}\right)^2\right], \tag{2.94}$$

which can be found by substituting the ansatz in Eq. (2.93) into Eq. (2.91). If the time variation of $\omega_{\mathbf{k}}(\tau)$ is very slow, it is said that it satisfies the adiabatic condition and the vacuum defined when this condition is valid is called the adiabatic vacuum. By "slow" we mean that $\omega_{\mathbf{k}}(\tau)$ and all its derivatives change substantially, $\Delta\omega_{\mathbf{k}}/\omega_{\mathbf{k}} \sim \mathcal{O}(1)$, only during the time interval $T \gg \omega_{\mathbf{k}}^{-1}$ (Ref. [48]). In the adiabatic case, derivative terms in Eq. (2.94) will be small and this equation can be solved using the recursive method. For example, to the zeroth order we can take

$$W_{\mathbf{k}}^{(0)}(\tau) = \omega_{\mathbf{k}}(\tau). \tag{2.95}$$

Note that for the constant frequency, $\omega_{\mathbf{k}} = \mathrm{const}$, the mode functions in CST (Eq. (2.88)) with $\chi_{\mathbf{k}}(\tau)$ given by Eq. (2.93) reduce to the mode functions in FST (cf. Eqs. (2.42) and (2.47)).

In Eq. (2.91) the adiabatic vacuum can be defined for light particles ($m/H \ll 1$) whose Compton wavelength is much smaller than the curvature scale H^{-1}, or in other words whose physical momentum is much grater than the Hubble expansion rate, $k/a \gg H$.[4] We may say that such particles do not "feel" the gravitational field. Therefore, by substituting $W_{\mathbf{k}}^{(0)}(\tau)$ into Eq. (2.93) and taking that

$$\omega_{\mathbf{k}} \approx k, \tag{2.96}$$

[4] For the quasi de Sitter expansion $\dot{H}/H^2 \ll 1$ (cf. Eq. (1.42)) and for the FRW Universe $\dot{H}/H^2 \sim \mathcal{O}(1)$, so that, when $k/a \gg H$, terms of order one or less are subdominant in Eq. (2.92).

we find the initial condition for the mode function $\chi_{\mathbf{k}}(\tau)$

$$\chi_{\mathbf{k}}(\tau) = \frac{1}{\sqrt{2k}} \mathrm{e}^{-ik\tau}, \tag{2.97}$$

which is the same as that of the massless field in FST (cf. Eqs. (2.42) and (2.47)). This vacuum state is often called the Bunch-Davies vacuum.

The result of Eq. (2.97) was obtained by assuming that the space-time can be considered flat at the zero order approximation for subhorizon modes. This must be always valid for the Einstein gravity in accordance to the equivalence principle. But in many models the inflationary energy scale is just couple of orders of magnitude below the Planck scale. At such energy scales it might be that Einstein's theory of gravity is not precise enough to describe Nature. In this case, one may expect that the equivalence principle does not hold anymore. But this failure might be only at the level of $(H/m_{\mathrm{Pl}})^2 \lesssim 10^{-10}$, where H is the inflationary Hubble parameter [5].

2.2.2.5. The Field Perturbation in the Inflationary Universe

It was already mentioned in section 1.5 that inflation provides a natural mechanism to explain the origin of the curvature perturbation in the early Universe. Upon entering the horizon this perturbation sets the initial conditions for the tiny density inhomogeneities which seeded the subsequent growth of large scale structure such as galaxies and galaxy clusters. In this subsection we describe how field perturbations are generated in the inflationary Universe and in the next section how they are transformed into the curvature perturbation.

The generation of the field perturbation can be computed from Eq. (2.91) with appropriate initial conditions and assumptions relevant for the inflationary expansion. During inflation the Universe undergoes quasi de Sitter expansion for which the condition $\left|\dot{H}\right|/H^2 \ll 1$ is satisfied (see Eq. (1.42)). But for our purpose in this section and for later discussions in Chapter 3 it is enough to take the approximation of a quasi de Sitter Universe. Hence, we will set $\dot{H} = 0$ which is equivalent to considering exact de Sitter space-time.[5]

Another assumption we make is that initially the state corresponds to its vacuum, i.e. the average occupation number of particles with momentum k is much less than 1, $\bar{n}_k \ll 1$. This assumption is easily justified if enough amount of inflation occurred before the horizon exit of the scales of interest (see e.g. Ref. [5]). In this case the

[5]More precisely only a part of de Sitter space-time is considered because inflation lasts only for a finite time.

initial conditions for the mode functions is determined by the Bunch-Davies vacuum in Eq. (2.97).

Perturbations of scalar fields with the mass comparable to the Hubble parameter do not grow significantly as will be seen in Eq. (2.105). While in section 2.2.2.6 it will be shown that perturbations of heavy fields, with the mass $m > \frac{3}{2}H$, do not become classical after horizon crossing. Therefore, only light scalar fields are considered for the generation of the curvature perturbation. Assuming de Sitter expansion the general solution of Eq. (2.91) becomes

$$\chi_{\mathbf{k}}(\tau) = \sqrt{\frac{-\tau\pi}{2}} \mathrm{e}^{i\frac{\pi}{4}(2\nu+1)} H_{\nu}^{(1)}(-k\tau), \tag{2.98}$$

where the initial state was matched to the Bunch-Davies vacuum in Eq. (2.97), and $H_{\nu}^{(1)}(-k\tau)$ denotes the Hankel function and we used in the de Sitter space-time $\tau = -(aH)^{-1}$. The order of $H_{\nu}^{(1)}$ is defined as

$$\nu \equiv \sqrt{\frac{9}{4} - \left(\frac{m}{H}\right)^2}. \tag{2.99}$$

Well after horizon exit, when $|k\tau| \ll 1$, this solution approaches to

$$\chi_{\mathbf{k}}(\tau) \simeq \frac{\mathrm{e}^{i\frac{\pi}{2}\left(\nu-\frac{1}{2}\right)}}{\sqrt{2k}} \frac{\Gamma(\nu)}{\sqrt{\pi}} \left(\frac{-k\tau}{2}\right)^{\frac{1}{2}-\nu}. \tag{2.100}$$

For the light field $m < \frac{2}{3}H$ the parameter ν is real and this solution is not oscillatory, therefore interpretation of corresponding states $|\psi\rangle$ as physical particle states in the Fock space is problematic. The reason for this, as can be seen from Eq. (2.92), is that the dispersion relation for a light field, with $m \ll H$, becomes imaginary and the mode function does not oscillate.

However, the amplitude of quantum fluctuation in the state $|\psi\rangle$ is always well defined and we can calculate the expectation value for the vacuum state $|0\rangle$ as

$$\langle 0| \hat{\chi}(\tau,\mathbf{x}) \hat{\chi}(\tau,\mathbf{y}) |0\rangle = \frac{1}{2\pi^2} \int_0^\infty k^3 |\chi_{\mathbf{k}}(\tau)|^2 \frac{\sin kL}{kL} \frac{\mathrm{d}k}{k} \equiv \int_0^\infty \mathcal{P}_\chi(k) \frac{\sin kL}{kL} \frac{\mathrm{d}k}{k}, \tag{2.101}$$

where $L \equiv |\mathbf{x}-\mathbf{y}|$ and $\mathcal{P}_\chi(k) \equiv \left(k^3/2\pi^2\right) |\chi_{\mathbf{k}}|^2$ is the power spectrum.

The power spectrum of the superhorizon massive scalar field perturbations can be easily calculated using Eq. (2.100)

$$\mathcal{P}_\chi = \frac{4\Gamma^2(\nu)}{\pi}\left(\frac{aH}{2\pi}\right)^2\left(\frac{k}{2aH}\right)^{3-2\nu}. \tag{2.102}$$

However, this expression is derived for the comoving field χ. Going back to the physical field $\phi = \chi/a$ (see Eq. (2.83)) and considering the massless limit $\nu = 3/2$, this expression reduces to

$$\mathcal{P}_\phi = \left(\frac{H}{2\pi}\right)^2. \tag{2.103}$$

Or more generally, for a light field, $m \lesssim \frac{3}{2}H$, we can express Eq. (2.99) as

$$\nu \simeq \frac{3}{2} - \frac{m^2}{3H^2}, \tag{2.104}$$

and the power spectrum becomes

$$\mathcal{P}_\phi \simeq \left(\frac{H}{2\pi}\right)^2\left(\frac{k}{2aH}\right)^{\frac{2}{3}\left(\frac{m}{H}\right)^2}. \tag{2.105}$$

Although Eq. (2.103) was calculated for the exact de Sitter expansion, one can easily include a very slow variation of the Hubble parameter, $\dot{H} \neq 0$. Due to this small variation, the horizon size changes very slightly during inflation and therefore each mode k exits a horizon of slightly different size. But from Eq. (2.103) it is clear that the amplitude of the field perturbation is proportional to the horizon size. Therefore, different modes will have slightly different amplitudes. This mild dependence of the power spectrum on k may be accounted for in Eq. (2.103) by writing

$$\mathcal{P}_\phi = \left(\frac{H_k}{2\pi}\right)^2, \tag{2.106}$$

where the Hubble parameter H_k in this expression has to be evaluated at the horizon exit for each mode, i.e. when $aH_k = k$, and H_k is slowly varying with k.

It is important to note that because we have assumed the initial state for each mode to start in the Bunch-Davies vacuum and considered a free field, perturbations of the field are Gaussian. This is the result of the equivalence principle valid for Einstein's gravity. But if at inflationary energies the equivalence principle does not hold, for example due to modified gravity, the perturbations of the field may be significantly non-Gaussian. In addition, because we were concerned in this section about scalar fields which are rotationally invariant, the perturbations of the field are statistically isotropic. This will not be the case in Chapter 3 where we discuss perturbations of vector fields.

2.2.2.6. Quantum to Classical Transition

As it is clear from the discussion so far, the origin of the field perturbation is quantum mechanical. But as was claimed in section 1.5.1 the greatest success of inflationary paradigm is that it can explain how these quantum mechanical perturbations give rise to the initial density perturbations in the Universe which are observed as CMB temperature anisotropies and which seed the formation of galaxies. But CMB temperature anisotropies and galaxies are not quantum but classical objects. Hence, there must be some transition period where quantum mechanical perturbations are transformed into classical ones. This process is analogous to the decoherence in quantum mechanics. The result of it is that the quantum mechanical superposition principle is violated and the wavefunction collapses to a particular state obeying the classical evolution. The coherence between different states is lost, since after the collapse only one state can be observed, although quantum mechanically all states should be allowed. In usual applications of quantum mechanics this happens due to the wavefunction interaction with the degrees of freedom of the environment. But in cosmological context the transition from quantum-to-classical does not require environment, therefore in Ref. [49] it was named "decoherence without decoherence". In the exposition of quantum-to-classical transition below we will follow Refs. [49, 50, 51, 52].

What does it mean, that perturbations in the Universe are classical? As was discussed in section 2.1 these perturbations can be described by random fields β. The classicality of perturbations means that β is described by classical stochastic variables. But to make the discussion easier, instead of treating some general variable β let us specialize to a field perturbation $\chi(\mathbf{x})$ defined in Eq. (2.83). Then $\chi(\mathbf{x})$ will be classical if it is described as classical stochastic variable with the probability distribution function $p(|\chi|, |\pi|)$, where π is the canonical conjugate of χ.

In quantum mechanics a classical limit is achieved when the state collapses to the definite numerical value. But in the cosmological context we cannot assign the definite numerical value to a collapsed state. So we say that the quantum state becomes classical if the field modes become equivalent to the classical stochastic functions with the probability distribution $p(|\chi|,|\pi|)$ [49]. This can be written as

$$\begin{aligned}\langle 0| G\left(\hat{X}_m, \hat{\pi}_m\right) G^{\dagger}\left(\hat{X}_m, \hat{\pi}_m\right)|0\rangle = \\ = \int \mathrm{d}X_{m1}\mathrm{d}X_{m2}\mathrm{d}\pi_{m1}\mathrm{d}\pi_{m2}\, p\left(|X_m|, |\pi_m|\right)\left|G\left(X_m, \pi_m\right)\right|^2,\end{aligned} \tag{2.107}$$

where from Eq. (2.86)

$$\hat{\chi}(x) = \int \hat{X}_m(x)\,\mathrm{d}m \equiv \int \left[\hat{a}_m \chi_m(x) + \hat{a}_m^\dagger \chi_m^*(x)\right]\mathrm{d}m, \tag{2.108}$$

m is a continuous index and $X_{m1} \equiv \mathrm{Re}(X_m)$, $X_{m2} \equiv \mathrm{Im}(X_m)$. Note that in this equation operators are denoted by hats and classical fields without hats.

Of course Eq. (2.107) is not valid in general. But this equality is valid when quantum fields can be treated as classical, i.e. when conjugate variables commute. To show this let us use a quantum field $\hat{\chi}$. For the moment restoring physical units, the non-zero commutation relation for this operator and its conjugate pair in Eq. (2.85) becomes

$$\left[\hat{\chi}(\tau, \mathbf{x}), \hat{\pi}\left(\tau, \mathbf{x}'\right)\right] = i\hbar\delta\left(\mathbf{x} - \mathbf{x}'\right). \tag{2.109}$$

The classical limit of a quantum description must be achieved when the Planck constant becomes negligibly small, $\hbar \to 0$. In this limit the commutator in Eq. (2.109) becomes zero. If the operator $\hat{\chi}$ is expanded into the complete set of orthonormal mode functions as in Eq. (2.86), then in the limit $\hbar \to 0$ the orthonormality condition in Eq. (2.29) for the mode functions $\{\chi_m\}$ becomes $(\chi_m, \chi_m) \to 0$. Using Eq. (2.28) we find that this condition results in the mode function χ_m and its complex conjugate χ_m^* being different only by the *time independent* phase factor

$$\chi_m^* = c_m \chi_m. \tag{2.110}$$

But the phase of χ_m is completely arbitrary. Therefore we are free to choose it in a way that makes χ_m real. And because c_m is time independent, χ_m is real at all times and $\hat{X}_m$ in Eq. (2.108) can be rewritten as

$$\hat{X}_m(x) = \chi_m(x)\left[\hat{a}_m + \hat{a}_m^\dagger\right]. \tag{2.111}$$

One can easily check that with this expression the commutator in Eq. (2.109) is zero. In Refs. [49, 50] it was calculated explicitly that an operator of the form in Eq. (2.111) satisfies the equivalence equation (2.107) for the quantum field and stochastic classical field. Which shows as well that the classical stochastic field can be expressed as

$$X_m(x) = \chi_m(x)\, e_m, \tag{2.112}$$

where e_m are time independent, complex, stochastic c-number functions with zero average and unit dispersion: $\langle e_m \rangle = 0$ and $\langle e_m, e_n^* \rangle = \delta(m - n)$. And e_m obeys the same statistics as $\hat{X}_m(\tau_0, \mathbf{x})$.

Note that the time dependent part $\chi_m(\tau, \mathbf{x})$ can be factored out from both: the quantum field $\hat{X}_m$ in Eq. (2.111) and from the classical field X_m in Eq. (2.112). As the result the evolution of a given mode function is completely deterministic after the realization of some stochastic amplitude have occurred. In other words, if we measure the amplitude of the field perturbation some time after the horizon exit, it will continue to have a definite value.

To show how a quantum field becomes of the form in Eq. (2.111) in the accelerating Universe, let us consider such a field in the de Sitter background. In subsection 2.2.2.5 it was shown that if we choose mode functions $\chi_m(x)$ to be Fourier modes (see Eq. (2.88)) on the superhorizon scales they will have the solution

$$\chi_m(\tau, \mathbf{x}) \equiv \sqrt{2}\frac{\mathrm{e}^{i\frac{\pi}{2}\left(\nu-\frac{5}{2}\right)}}{\sqrt{2k}}\frac{\Gamma(\nu)}{4\pi^2}\left(\frac{-k\tau}{2}\right)^{\frac{1}{2}-\nu}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.113}$$

where we have used Eq. (2.100) and ν is defined in Eq. (2.99). If ν is not imaginary, corresponding to $m^2 < 9H^2/4$, this mode function does not have a time dependent phase and can be made real by choosing a time independent phase rotation $c_\mathbf{k}$ (defined in Eq. (2.110)). Note that this transformation makes χ_m real at all times on the superhorizon scales. Therefore, quantum mechanical operator $\hat{X}_m$ satisfies Eq. (2.111) and consequently is equivalent to the classical stochastic field X_m.

The process of quantum-to-classical transition described above suffers from the usual interpretational problem of measurement in quantum mechanics. The first question is why did Nature choose this particular value for the realization of the field amplitude when other infinite possibilities were available? Another is a cosmological variant of the Schrödinger's Cat problem related to the question of when the state collapsed into its observed value. According to the usual Copenhagen interpretation, this happens at the time of measurement. But does that mean that CMB perturbation pattern did not exist before we have measured it for the first time?

2.3. The Primordial Curvature Perturbation

Quantum field perturbations described in the last section on superhorizon scales give rise to the classical cosmological perturbations. These perturbations are most conveniently described by the intrinsic spatial curvature ζ, which is commonly called the curvature perturbation. It is defined on the hypersurfaces of constant energy density. We will postpone the discussion how quantum field perturbations may generate ζ until section 2.4. In this section we discuss the properties of ζ and the formalism which relates perturbations of quantum fields with the curvature perturbation.

2.3.1. Gauge Freedom in General Relativity

When discussing the FRW Universe in section 1.1 we implicitly chose the coordinate system in which the metric in Eq. (1.2) attained it's elegant form. Although the physical results in GR should not depend on the coordinate system, the homogeneity and isotropy of the Universe singles out a preferred reference frame in which equations reduces to their simplest form. However, in space-times without symmetries such preferred coordinate system does not exist. Therefore, the choice of coordinates is purely arbitrary and may be selected depending on the problem at hand.

Fixing the coordinate system in GR specifies how space-time is threaded by the lines of constant spatial coordinate $\mathbf{x}$ (*threading*) and how it is divided into the hypersurfaces of constant coordinate time t (*slicing*). And because the coordinate system is arbitrary, so are the threading and slicing.

The arbitrariness of the coordinate system becomes especially problematic when describing tiny departures from homogeneity of the actual Universe. Since the real space-time with these departures does not posses any symmetry, it would be impossible to solve the exact relativistic evolution equations because GR is a non-linear theory. But luckily on large enough scales these departures from ideal homogeneity are very small, only of the order 10^{-5} (see the discussion on the cosmological principle in section 1.1). Therefore, a very good approximation to the actual Universe is to treat these inhomogeneities as tiny perturbations of the otherwise homogeneous and isotropic background. And using perturbation theory, non-linear equations of GR may be linearized.

However, separating physical quantities into background value and perturbations are not so trivial. Due to the freedom for the choice of the coordinate system this separation is not unique. Such complication is due to the fact that by separation we mean that for each space-time point in the background or reference manifold we associate a perturbation, corresponding to the actual or physical space-time. But because these are two different manifolds with different curvature we must specify how the mapping from one manifold to the other is performed. This may be done by choosing the specific threading and slicing of the space-time. Or in perturbation theory it is called by fixing the *gauge*.

By changing the gauge we must redefine what is the background value and what is the perturbation. Let us consider an infinitesimal gauge transformation in which new coordinates $\tilde{x}^\mu$ are related to the old ones by

$$\tilde{x}^\mu = x^\mu + \delta x^\mu (x), \tag{2.114}$$

and see how the perturbation of some scalar quantity $\delta f(x)$ changes by this transformation, where the perturbation is defined as the difference between the actual value $f(x)$ and the background value $f_0(x)$, $\delta f(x) \equiv f(x) - f_0(x)$. In the new gauge this perturbation will be $\widetilde{\delta f}(\tilde{x}) = \tilde{f}(\tilde{x}) - \tilde{f}_0(\tilde{x})$. But because the physical value of the quantity does not change under the gauge transformation, only the separation into the background and perturbation does, we can write $\tilde{f}(\tilde{x}) = f(\tilde{x})$. Further, because f is a scalar, it should be invariant under the coordinate change, thus $f(\tilde{x}) = f(x)$. On the other hand, we are keeping fixed the point in the background manifold and investigate the change in the mapping to the perturbed manifold, therefore $\tilde{f}_0(\tilde{x}) = f_0(\tilde{x})$. However, although it is the same point on the background manifold, in a new coordinate system it will have a different value. With the infinitesimal transformation in Eq. (2.114) this may be written as $f_0(\tilde{x}) = f_0(x) + (\partial f_0(x)/\partial x) \cdot \delta x$. Putting all this discussion together, we may write

$$\begin{aligned} \widetilde{\delta f}(\tilde{x}) &= \tilde{f}(\tilde{x}) - \tilde{f}_0(\tilde{x}) \\ &= f(x) - f_0(\tilde{x}) \\ &= \delta f(x) + f_0(x) - \frac{\partial f_0(x)}{\partial x^\mu}\delta x^\mu - f_0(x) \\ &= \delta f(x) - \frac{\partial f_0(x)}{\partial x^\mu}\delta x^\mu. \end{aligned} \tag{2.115}$$

In the later discussion, of special importance will be the transformation of the scalar quantity when only the slicing is changed, keeping the threading constant. This corresponds to a time shift, for which Eq. (2.114) reduces to

$$\tilde{t} = t + \delta t(x). \tag{2.116}$$

And Eq. (2.115) becomes

$$\delta\tilde{f} - \delta f = -\dot{f}_0 \delta t. \tag{2.117}$$

2.3.2. Smoothing and The Separate Universe Assumption

Calculations of the curvature perturbation in this thesis are performed using the separate Universe assumption. And the central concept for this assumption is that of smoothing.

Let us assume that we are interested in the perturbation of the energy density $\delta\rho$. Then the smoothed value $\delta\rho(t, \mathbf{x}, L)$ will correspond to the value of the energy density

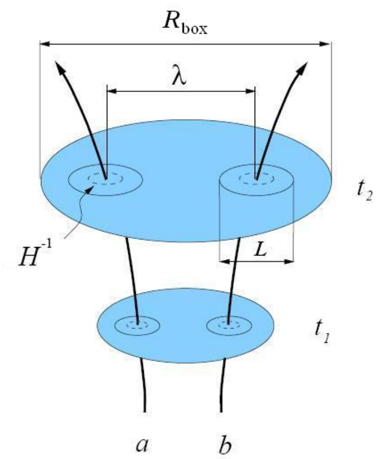

Figure 2.1.: The schematic illustration of the separate Universe assumption. According to this assumption all threadings coincide with the comoving one. Hence, curves (a) and (b) represent comoving trajectories of any of these threadings (adapted from Ref. [53])

at the space-time point $(t, \mathbf{x})$ after averaging over the sphere of comoving size L. If we expand $\delta\rho\,(t, \mathbf{x}, L)$ in a Fourier series, smoothing would correspond to dropping off all modes bigger than $k \sim L^{-1}$. The dynamics of this smoothed quantity is assumed to be determined by the averaged Einstein equations. However, GR is a non-linear theory which makes the issue of smoothing non trivial. For example, it is not clear how small scale fluctuations with $k > L^{-1}$ can influence the evolution of the quantity smoothed on scales L^{-1}. At present there is no satisfactory conclusion to this issue, but we will assume that such length scale does exist above which the smoothed Universe is a good approximation to the actual one.

The separate Universe approach assumes that each region, smoothed on scales larger than the horizon size, locally evolves as a separate unperturbed Universe. The basic idea is presented in Figure 2.1. In this figure L corresponds to the smoothing length scale which is larger than the horizon size H^{-1} but smaller than the largest box size $R_{\rm box}$, within which we perform our calculations (see Eq. 2.10). Ideally $R_{\rm box} \to \infty$, but, as was discussed in section 2.1.1, one should keep a box size finite in order to avoid unknown physics and keep calculations under control. The lines (a) and (b) represent two comoving worldlines for two different space points.

Another assumption made in the separate Universe approach is that all length scales introduced by the energy momentum tensor are much smaller than the smoothing length scale k^{-1}. Then k^{-1} is the only relevant superhorizon length scale and all spatial gradients of order k/a are negligible. When this assumption is satisfied, the locally measurable parts of the metric should reduce to those of the FRW [54]. In other words, every comoving location smoothed on superhorizon distances, evolves as the unperturbed Universe with the FRW metric of Eqs. (1.1) or (1.2).

2.3.3. Conservation of the Curvature Perturbation

Every smooth space-time metric can be decomposed into 3+1 components as [55]:

$$\mathrm{d}s^2 = \mathcal{N}^2\mathrm{d}t^2 - \gamma_{ij}\left(\mathrm{d}x^i + \beta^i\mathrm{d}t\right)\left(\mathrm{d}x^j + \beta^j\mathrm{d}t\right), \tag{2.118}$$

where $\mathcal{N}$ is the lapse function, β^i is the shift vector and $-\gamma_{ij}$ is the spatial three metric tensor.

With this decomposition one can define a unit time-like vector, n^μ, normal to the hypersurface of constant coordinate time t. The components of this vector are

$$n_\mu = \left(\mathcal{N}, \mathbf{0}\right); \quad n^\mu = \left(-\frac{1}{\mathcal{N}}, \frac{\beta^i}{\mathcal{N}}\right). \tag{2.119}$$

Then the volume expansion rate of the hypersurface along some integral curve $\gamma(\tau)$ of n^μ will be given as

$$\vartheta = \nabla_\mu n^\mu, \tag{2.120}$$

where ∇_μ is the covariant derivative and τ is the proper time, which can be found from Eq. (2.118), $\mathrm{d}\tau = \mathcal{N}\mathrm{d}t$. Along each of these integral curves we may define the number of e-folds of expansion

$$N\left(t_1, t_2; \mathbf{x}\right) \equiv \frac{1}{3}\int_{\gamma(\tau)} \vartheta\mathrm{d}\tau = \frac{1}{3}\int_{t_1}^{t_2} \vartheta\mathcal{N}\mathrm{d}t, \tag{2.121}$$

where the vector $\mathbf{x}$ is chosen to be the comoving spatial coordinate.

The spatial metric of Eq. (2.118) can be further decomposed as

$$\gamma_{ij} = a^2\left(t, \mathbf{x}\right)\tilde{\gamma}_{ij}. \tag{2.122}$$

With the requirement $\det\left(\tilde{\gamma}_{ij}\right) = 1$, $a\left(t, \mathbf{x}\right)$ becomes a local scale factor. Note, that t in this equation is not necessarily a proper time, it is just the coordinate time labeling the slices. Since we are interested in the non-homogeneity of the scale factor, we may further decompose $a\left(t, \mathbf{x}\right)$ into some global scale factor $a\left(t\right)$, which is independent of position, and the local deviation $\psi(t, \mathbf{x})$

$$a(t, \mathbf{x}) = a(t)\, \mathrm{e}^{\psi(t,\mathbf{x})}. \tag{2.123}$$

This decomposition into the global quantity and its perturbation is completely arbitrary. Hence, we may choose $a(t)$ in such a way that $\psi(t, \mathbf{x})$ vanishes somewhere inside the observable Universe. Then $\psi(t, \mathbf{x})$ becomes small everywhere inside this Universe [54].

A similar decomposition may be done for the $\tilde{\gamma}_{ij}$ part of the metric

$$\tilde{\gamma}_{ij} = \left(I \mathrm{e}^{h}\right)_{ij}, \tag{2.124}$$

where I is the unit matrix and h_{ij} must be a traceless matrix due to the requirement $\det(\tilde{\gamma}) = 1$. It can be shown that h_{ij} corresponds to the primordial tensor perturbation, i.e. gravitational waves. But, for the time being, we assume that GWs are negligible so that we can set $h = 0$.

According to the separate Universe assumption, if the metric is smoothed on super-horizon scales, at each space-time point we should be able to find such coordinates which reduce the metric into the form of FRW:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\, \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j. \tag{2.125}$$

This metric is chosen to be flat in agreement with observations, but as noted in Ref. [54] a small homogeneous curvature should not make much difference.

In accord with this assumption and with the appropriate coordinate choice, the metric in Eq. (2.118) should reduce to the form

$$\mathrm{d}s^2 = \mathcal{N}^2 \mathrm{d}t^2 - a^2(t)\, \mathrm{e}^{2\psi(t,\mathbf{x})} \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j. \tag{2.126}$$

The separate Universe assumption does not pose any constraints on $\mathcal{N}$ and ψ since they are not locally observable quantities. And in view of this assumption we have neglected all terms of order $\mathcal{O}(k/aH)$, which on superhorizon scales approach zero, $k/aH \to 0$. In particular, in Ref. [54] it was shown that

$$\beta_i = \mathcal{O}(k/aH) \qquad \text{and} \qquad \dot{\tilde{\gamma}}_{ij} = \mathcal{O}\left[(k/aH)^2\right]. \tag{2.127}$$

In the following discussions we will keep in mind that the separate Universe assumption is valid up to this order, but will omit terms $\mathcal{O}(k/aH)$ from equations. Note, however, that the smallness of β_i just corresponds to our choice of the coordinate system. The generalization of the formalism to a threading with non negligible β_i is straightforward [54].

With the line element of Eq. (2.126) the local expansion rate ϑ, defined in Eq. (2.120), takes the form

$$\vartheta = \frac{3}{\mathcal{N}}\left(\frac{\dot{a}(t)}{a(t)} + \dot{\psi}(t,\mathbf{x})\right). \tag{2.128}$$

For later convenience we define the local Hubble parameter $\tilde{H}(t,\mathbf{x}) \equiv \frac{1}{3}\vartheta$:

$$\tilde{H}(t,\mathbf{x}) = \frac{1}{\mathcal{N}}\left(\frac{\dot{a}(t)}{a(t)} + \dot{\psi}(t,\mathbf{x})\right). \tag{2.129}$$

In what follows, an important quantity is the number of e-folds of the local expansion, which is defined in Eq. (2.121). With the line element in Eq. (2.126) it becomes

$$N(t_1,t_2;\mathbf{x}) = \int_{\gamma(\tau)} \tilde{H}(t,\mathbf{x})\,\mathrm{d}\tau = \int_{t_1}^{t_2}\left(\frac{\dot{a}(t)}{a(t)} + \dot{\psi}(t,\mathbf{x})\right)\mathrm{d}t. \tag{2.130}$$

According to the separate Universe assumption each space point evolves as the unperturbed Universe with the locally defined expansion rate in Eq. (2.128) (or equivalently local Hubble parameter in Eq. (2.129)). Therefore, at each point we can write the energy-momentum conservation law, $\nabla_\nu T^{\mu\nu} = 0$, from which it follows

$$\frac{\mathrm{d}\rho(t,\mathbf{x})}{\mathrm{d}t} = -3\tilde{H}(t,\mathbf{x})\left[\rho(t,\mathbf{x}) + p(t,\mathbf{x})\right]. \tag{2.131}$$

It has the same form in the FRW Universe (c.f. Eq. (1.4)).

This equation is valid independently of the slicing. But let us specialize further to the slicing on which energy density is uniform, i.e. independent on space coordinate at each given time. Such slicing is called comoving or uniform density slicing and the value of ψ on this slicing is usually denoted by ζ. It determines the perturbation in the intrinsic curvature of the slices. Then, Eq. (2.131) can be rewritten as

$$\dot{\rho}(t) = -3\left[\frac{\dot{a}(t)}{a(t)} + \dot{\zeta}(t,\mathbf{x})\right]\left[\rho(t) + p(t,\mathbf{x})\right], \tag{2.132}$$

where we have used Eq. (2.129) as well.

Now let us limit ourselves to the case where pressure is adiabatic, which is equivalent to saying that pressure is a unique function of the energy density, i.e. $p = p(\rho)$. In this case, because ρ is independent of position, the pressure must be independent of position as well, $p = p(t)$ only. Therefore, the same must be true for $\dot{\zeta}$, i.e. $\dot{\zeta} = \dot{\zeta}(t)$ only.

On the other hand, the decomposition of the spatial part of the metric in Eq. (2.123) into the background value $a(t)$ and deviation from that value $\psi(t, \mathbf{x})$ was purely arbitrary. So we may choose the normalization of $a(t, \mathbf{x})$ such that $a(t)$ corresponds to the scale factor at our location (or any other location). In other words, we choose $a(t)$ in such a way that ζ vanishes at our location at all times. Hence, at this location

$$\dot{\zeta} = 0. \tag{2.133}$$

But because $\dot{\zeta}$ is independent on position (when $p = p(\rho)$) it must vanish everywhere.

In this way we have found a very important quantity, the curvature perturbation ζ, which determines the intrinsic curvature of constant time spatial hypersurfaces. As was shown above, on superhorizon scales ζ stays constant whenever pressure is the unique function of the energy density. In the history of the Universe this happens when the latter is dominated by radiation or matter. More generally, the pressure of the multicomponent fluid is adiabatic if each component of the fluid satisfies the relation $\rho_a = \rho_a(\rho)$, where ρ_a and ρ are the energy densities of each component and of the total fluid respectively. Thus, around the matter-radiation equality era, ζ is constant too if perturbations are adiabatic. By adiabatic perturbations we mean that on uniform total energy density slices, perturbations of each component are independent of position. A more rigorous proof of the constancy of the curvature perturbation ζ can be found for example in Refs. [54, 53]. In Ref. [53], the constancy of ζ was proved using perturbation theory, without the assumption of separate universes. As shown in these works, the change in the curvature perturbation to the first order is proportional to the non-adiabatic part of the pressure δp_{nad} as

$$\dot{\zeta} = -\frac{H}{\rho + p}\delta p_{\mathrm{nad}}, \tag{2.134}$$

where δp_{nad} is defined as the pressure perturbation on the uniform density slicing.

2.3.4. The δN Formalism

On superhorizon scales all threadings are equivalent to the comoving threading up to the order $\mathcal{O}\left[(k/aH)^2\right]$ [54]. Hence, only the slicing is arbitrary. In this section we show how using this fact and the separate Universe assumption one can relate the curvature perturbation ζ with the energy density perturbation $\delta\rho$ without invoking cosmological perturbation theory.

Let us consider one of the comoving threads drawn in Figure 2.1. Going from the coordinate time t_1 to t_2 along this thread we can calculate the change in the value of ψ

for some slicing using Eq. (2.130)

$$\psi\left(t_2,\mathbf{x}\right)-\psi\left(t_1,\mathbf{x}\right)=N\left(t_1,t_2;\mathbf{x}\right)-N_0\left(t_1,t_2\right), \tag{2.135}$$

where $N_0\left(t_1,t_2\right)\equiv\ln\left[a\left(t_2\right)/a\left(t_1\right)\right]$ is the number of e-folds for the background expansion. For the flat slicing we have $\psi_{\rm flat}=0$, and thus the number of e-folds for the local expansion coincides with the background one, $N_{\rm flat}\left(t_1,t_2;\mathbf{x}\right)=N_0\left(t_1,t_2\right)$.

Let us apply now Eq. (2.135) for two different choices $\psi_{\rm A}$ and $\psi_{\rm B}$ corresponding to two different slicings, which coincide at time t_1. Then at time t_2 the difference between $\psi_{\rm A}$ and $\psi_{\rm B}$ will be

$$\psi_{\rm A}\left(t_2,\mathbf{x}\right)-\psi_{\rm B}\left(t_2,\mathbf{x}\right)=N_{\rm A}\left(t_1,t_2;\mathbf{x}\right)-N_{\rm B}\left(t_1,t_2;\mathbf{x}\right). \tag{2.136}$$

Let us further specify slicings A and B in the following way. The slicing B will be the flat slicing, giving $N_{\rm B}=N_0$. And let the slicing A be such that at time t_1 it coincides with the flat slicing, $\psi_{\rm A}\left(t_1,\mathbf{x}\right)=0$, while at time t_2 it coincides with the uniform density slicing, $\psi_{\rm A}\left(t_2,\mathbf{x}\right)=\zeta\left(t_2,\mathbf{x}\right)$. Then Eq. (2.136) takes the form

$$\zeta\left(t_2,\mathbf{x}\right)=N_A\left(t_2;\mathbf{x}\right)-N_0\left(t_2\right). \tag{2.137}$$

Due to our choice of A such that $\psi_{\rm A}\left(t_1,\mathbf{x}\right)=0$, the number of e-folds of the local expansion on this slicing becomes $N_{\rm A}\left(t_2,\mathbf{x}\right)=\ln\left[a\left(t_2,\mathbf{x}\right)/a\left(t_1\right)\right]$. This means that Eq. (2.137) is independent on the initial time and this is why we omitted the notation of t_1. In other words, the calculation of ζ is independent on the initial epoch, because when going from one flat slice to the other the expansion is uniform.

From this equation it is clear that the curvature perturbation $\zeta\left(t,\mathbf{x}\right)$ specifies the perturbation in the number of e-folds of the local expansion starting from any flat slice and ending on the uniform density slice at time t:

$$\zeta\left(t,\mathbf{x}\right)=\delta N\left(t,\mathbf{x}\right). \tag{2.138}$$

Until now our discussion didn't require that perturbations should be small. Hence, they are valid to any order in the perturbation expansion. To relate $\delta N\left(t,\mathbf{x}\right)$ with the field perturbation (discussed in section 2.2.2) we will need to specialize further in small perturbations.

Let us assume that the local expansion of the Universe is determined solely by the value of a classical scalar field, $N\left(t,\mathbf{x}\right)=N\left(\phi\left(t,\mathbf{x}\right)\right)$. By this assumption we neglect the contribution, for example, by the kinetic term of the field, $\dot{\phi}(t, \mathbf{x})$. This is valid in most cosmologically interesting cases, for example during inflation with the slowly varying field. Then Eq. (2.137) can be written as

$$\zeta(t, \mathbf{x}) = N(\phi(t, \mathbf{x})) - N(\phi(t)). \tag{2.139}$$

Taking the field perturbation to be small $\delta\phi(t, \mathbf{x}) \ll \phi(t)$, where $\phi(t, \mathbf{x}) \equiv \phi(t) + \delta\phi(t, \mathbf{x})$, this equation becomes

$$\zeta(t, \mathbf{x}) = N_\phi \delta\phi + \frac{1}{2} N_{\phi\phi} (\delta\phi)^2 + \ldots, \tag{2.140}$$

where $N_\phi \equiv \partial N(\phi(t)) / \partial\phi$ and $N_{\phi\phi} \equiv \partial^2 N(\phi(t)) / \partial\phi^2$. Note that derivatives are taken of the unperturbed value of N, and the field perturbation is evaluated on the initial flat slice. This equation can be easily generalized to the many field case, when $N(t, \mathbf{x}) = N(\phi_1(t, \mathbf{x}), \phi_2(t, \mathbf{x}), \ldots)$, in which case Eq. (2.140) becomes

$$\zeta(t, \mathbf{x}) = \sum_I N_I \delta\phi_I + \frac{1}{2} \sum_{IJ} N_{IJ} \delta\phi_I \delta\phi_J + \ldots. \tag{2.141}$$

In the rest of this thesis it is sufficient to consider the curvature perturbation only to the second order in the field perturbations, i.e we will drop out terms denoted by the ellipsis.

2.3.5. The Power Spectrum and Non-Gaussianity of ζ

To calculate the power spectrum and bispectrum of the curvature perturbation, ζ must be transformed to the Fourier space by Eq. (2.2). Then Eq. (2.141) becomes

$$\zeta_k = N_\phi \delta\phi_k + \frac{1}{2} N_{\phi\phi} (\delta\phi_k)^2. \tag{2.142}$$

Note, that in this expression ζ_k is dependent only on the modulus of $k \equiv |\mathbf{k}|$. This is because perturbations of the scalar field are rotationally invariant. We will drop this assumption in Chapter 3 when discussing perturbations of vector fields.

The two point correlation function for the curvature perturbation in Eq. (2.142) is

$$\langle \zeta_k(t), \zeta_{k'}(t) \rangle = N_\phi^2 \langle \delta\phi_k, \delta\phi_{k'} \rangle + \frac{1}{4} N_{\phi\phi}^2 \left\langle (\delta\phi_k)^2 (\delta\phi_{k'})^2 \right\rangle. \tag{2.143}$$

Because we have assumed that field perturbations are Gaussian, the first term of this equation is the Gaussian contribution. The second term gives a non-Gaussian contribution and according to observations this contribution must be subdominant.

Taking only the dominant part in Eq. (2.143) we find that the power spectrum of the curvature perturbation is related to the power spectrum of the field perturbation by

$$\mathcal{P}_\zeta = N_\phi^2 \mathcal{P}_\phi, \tag{2.144}$$

where the power spectrum of the field perturbations $\mathcal{P}_\phi$ is the one in Eq. (2.103) for de Sitter inflation or in Eq. (2.106) for the slow-roll inflation.

As was discussed in section 2.1.1 if perturbations are Gaussian the two point correlator is the only non-zero correlator. The non-Gaussianity manifest itself in the non-vanishing higher order correlators. For the aim of the present thesis it is enough to consider only the three point correlator, although in some models it might be that, for example the four point correlator is even larger than the three point (see e.g. Ref. [56]). The bispectrum (defined in Eq. (2.13)) of the curvature perturbation usually is parametrized by the non-linearity (or non-Gaussianity) parameter f_{NL} defined in Eq. (2.20).

If the field perturbation $\delta\phi$ is Gaussian, then f_{NL} becomes practically independent of k. In Ref. [57] it was calculated that if the first term of Eq. (2.142) is dominant the f_{NL} parameter becomes

$$\frac{6}{5} f_{\mathrm{NL}} = -\frac{N_{\phi\phi}}{N_\phi^2}. \tag{2.145}$$

2.3.6. Density Perturbations

In the previous subsection we have shown how to calculate the curvature perturbation ζ, which is conserved on superhorizon scales whenever the pressure of the cosmic fluid is adiabatic. However, our primary interest is in small perturbations of the energy density, which upon horizon entry form the seeds for the subsequent structure formation in the Universe. To make a connection between the curvature perturbation ζ, which we calculated so far, and inhomogeneities of the energy density, we will use a limit of small perturbations up to the first order.

As was discussed earlier, on superhorizon scales the threading is defined uniquely, changing the slicing corresponds only to a shift in the coordinate time. So let us consider a change from the uniform density slicing to some generic one. At any given position this will correspond to a time change $\delta t\,(t, \mathbf{x})$, so that on a new slicing $\tilde{t} = t + \delta t\,(t, \mathbf{x})$. Then, the local scale factor on the new slicing can be found using Eq. (2.117) and considering that the background value is the same for both slicings

$$a\left(\tilde{t}, \mathbf{x}\right) = a\,(t, \mathbf{x}) - \dot{a}\,(t)\,\delta t. \tag{2.146}$$

Separating local scale factors into the background value and the perturbation as was done in Eq. (2.123) and considering that the perturbation is small, we find to first order

$$\psi = \zeta - H\delta t. \tag{2.147}$$

A similar reasoning applies to the energy density giving

$$\delta\rho_{\psi}(t, \mathbf{x}) = -\dot{\rho}(t)\,\delta t(t, \mathbf{x}), \tag{2.148}$$

where $\delta\rho = 0$ on the uniform density slicing. For the time being we use the index 'ψ' to remind ourselves that density perturbation is defined on an arbitrary slicing. While $\delta\rho$ without this index will correspond to the density perturbation in a flat slicing.

Combining the last two equations we arrive at

$$\zeta = \psi - H\frac{\delta\rho_{\psi}(t, \mathbf{x})}{\dot{\rho}(t)} = \psi + \frac{1}{3}\frac{\delta\rho_{\psi}(t, \mathbf{x})}{\rho + p}, \tag{2.149}$$

where in the last equation the continuity equation for the FRW Universe (Eq. (1.4)) was applied. Thus we derived the equation for the transformation going from the uniform density slicing to arbitrary slicing. An important choice of the latter is the flat slicing, $\psi = 0$, for which we get

$$\zeta = -H\frac{\delta\rho}{\dot{\rho}} = \frac{1}{3}\frac{\delta\rho}{\rho + p}. \tag{2.150}$$

To derive this equation the uniform density slicing was defined with respect to the total energy density of the cosmic fluid. But for the fluid with several components, we can equally well define the uniform density slicing for each component. Then if there is no total energy exchange between these components, Eq. (2.150) can be rewritten as

$$\zeta_n = -H\frac{\delta\rho_n}{\dot{\rho}_n} = \frac{1}{3}\frac{\delta\rho_n}{\rho_n + p_n}, \tag{2.151}$$

where n is the index for a particular component of the fluid and $\delta\rho_n$ is the energy density perturbation of that component on a flat slicing. Using $\delta\rho = \sum_n \delta\rho_n$ we can calculate the total curvature perturbation from Eq. (2.150)

$$\zeta = \frac{\sum_n (\rho_n + p_n)\,\zeta_n}{\rho + p}. \tag{2.152}$$

This equation will be important when we consider curvaton models where the primordial perturbation can be generated by several fluids.

2.4. Mechanisms for the Generation of the Curvature Perturbation

Sections 2.2.2.5 and 2.2.2.6 described how, in the inflationary Universe, quantum fluctuations are amplified and converted into classical field perturbations $\delta\phi$. Then in section 2.3 we have shown how to calculate the intrinsic curvature perturbation of the space-time which is measured after the horizon entry. In this section we will connect those two parts and show three mechanisms by which fluctuations of quantum fields during inflation can generate the curvature perturbation ζ. These three models of the generation of the curvature perturbation by no means are the only possible. However, only these three are necessary for our purpose when we discuss vector fields in Chapter 3.

2.4.1. Single Field Inflation

Let us assume in this section that the single field which drives the slow-roll inflation, as discussed in section 1.5, is the same field which is responsible for the total curvature perturbation in the Universe.

During single field inflation the value of the field $\phi\left(t,\mathbf{x}\right)$ at any given instant determines the energy density $\rho\left(t,\mathbf{x}\right)$. Therefore, we can calculate the curvature perturbation ζ directly from Eq. (2.150) by using the expression for the energy density and pressure of the scalar field in Eqs. (1.49) and (1.50). Imposing the slow-roll condition for which $\rho\simeq V\left(\phi\right)$ and $3H\dot{\phi}\simeq -V_{\phi}$ we find to the first order

$$\zeta=\frac{1}{3}\frac{V_{\phi}}{\dot{\phi}^{2}}\delta\phi=\frac{1}{m_{\mathrm{Pl}}^{2}}\frac{V}{V_{\phi}}\delta\phi. \tag{2.153}$$

Alternatively we can use the δN formula directly. This method renders the second order calculations more straightforward and comparison with other methods for generation of ζ easier.

In Eq. (2.140) N_{ϕ} and $N_{\phi\phi}$ are derivatives of the number of e-folds of expansion of the unperturbed Universe. During single field inflation the local evolution of the Universe is determined only by the value of a single scalar field ϕ. Therefore, the change of ϕ corresponds to the shift in time along the same unperturbed trajectory, N_{ϕ} and $N_{\phi\phi}$ are independent on the final epoch, which makes ζ independent on time[6]

$$\zeta\left(\mathbf{x}\right)=N_{\phi}\delta\phi\left(\mathbf{x}\right)+N_{\phi\phi}\left[\delta\phi\left(\mathbf{x}\right)\right]^{2}. \tag{2.154}$$

The definition of the number of unperturbed e-folds of expansion is given in Eq. (1.57), from which it follows that $\dot{N}=-H$, or alternatively

$$N_{\phi}=-\frac{H}{\dot{\phi}}=\frac{1}{m_{\mathrm{Pl}}^{2}}\frac{V}{V_{\phi}}, \tag{2.155}$$

where for the last equality the slow-roll equation of motion in Eq. (1.54) was used. Inserting this expression into Eq. (2.154) at the first order one recovers the same equation as in Eq. (2.153).

The field perturbations $\delta\phi_k$ was discussed in section 2.2.2.5, and the power spectrum for a light scalar field $\mathcal{P}_\phi$ was calculated in Eq. (2.106):

$$\mathcal{P}_{\phi}\left(k\right)\simeq\left(\frac{H_{k}}{2\pi}\right)^{2}. \tag{2.156}$$

A few Hubble times after horizon exit ζ becomes constant as shown in Eq. (2.133), hence it is enough to evaluate (2.144) at this time, which gives

$$\mathcal{P}_{\zeta}\left(k\right)=N_{\phi}^{2}\left(\frac{H_{k}}{2\pi}\right)^{2}. \tag{2.157}$$

Using this expression for the power spectrum of the scalar field and Eq. (2.155) with $3m_{\mathrm{Pl}}^2H_k^2\simeq V\left(\phi\right)|_k$ gives us the power spectrum of the curvature perturbation

$$\mathcal{P}_{\zeta}\left(k\right)=\frac{1}{24m_{\mathrm{Pl}}^{4}\pi^{2}}\left.\frac{V}{\epsilon}\right|_{k}, \tag{2.158}$$

where ϵ is a slow-roll parameter defined in Eq. (1.55) and the right hand side has to be evaluate at horizon exit, $aH_k=k$.

For exponential inflation, when $\dot{H}=0$, $\mathcal{P}_\zeta$ is scale independent. But in the slow-roll inflation H is only approximately constant, which makes the power spectrum slowly varying with k. This variation usually is approximated by a power law (c.f. Eq. (2.14)) and is parametrized by the spectral index n such that $\mathcal{P}_\zeta\left(k\right)\propto k^{n-1}$. One can take the

[6]The situation is different for the multifield inflation, where the change in the field space, $\vec{\phi}\equiv(\phi_1,\phi_2,\ldots)$, does not only correspond to the shift in time along the unperturbed trajectory, but also the rotation in this space. Then ζ becomes time dependent until the end of inflation or until the trajectories of fields $\phi_1,\phi_2,\ldots$ become straight lines.

definition of the spectral index to be

$$n - 1 \equiv \frac{\mathrm{d} \ln \mathcal{P}_\zeta}{\mathrm{d} \ln k}. \tag{2.159}$$

To evaluate this equation it is useful to derive the following relations

$$\mathrm{d} \ln k = \mathrm{d} \ln (a_k H_k) \simeq \mathrm{d} \ln a_k = H_k \mathrm{d}t, \tag{2.160}$$

from which we can write

$$\frac{\mathrm{d}}{\mathrm{d} \ln k} = \frac{\dot{\phi}}{H} \frac{\mathrm{d}}{\mathrm{d}\phi} = -\frac{1}{N_\phi} \frac{\mathrm{d}}{\mathrm{d}\phi}, \tag{2.161}$$

where Eq. (2.155) was used. Then one can readily calculate

$$\frac{\mathrm{d} \ln V}{\mathrm{d} \ln k} = -\frac{1}{N_\phi} \frac{V_\phi}{V} = -m_{\mathrm{Pl}}^2 \left(\frac{V_\phi}{V}\right)^2 = -2\epsilon, \tag{2.162}$$

where we have used Eq. (2.155) and the definition of the slow-roll parameter in Eq. (1.55). Now, we also have

$$\frac{\mathrm{d} \ln \epsilon}{\mathrm{d} \ln k} = 2 \left(\frac{V}{V_\phi}\right) \frac{\mathrm{d}}{\mathrm{d} \ln k} \left(\frac{V_\phi}{V}\right) = 2 m_{\mathrm{Pl}}^2 \left[\left(\frac{V_\phi}{V}\right)^2 - \frac{V_{\phi\phi}}{V}\right] = 4\epsilon - 2\eta. \tag{2.163}$$

Using the last two relations and the power spectrum in Eq. (2.158) from the definition of the spectral index in Eq. (2.159) we find that the spectral index for single field slow-roll inflation is

$$n - 1 = 2\eta - 6\epsilon. \tag{2.164}$$

From Eq. (2.155) we may write

$$\frac{1}{m_{\mathrm{Pl}}} \left|\frac{\mathrm{d}\phi}{\mathrm{d}N}\right| = m_{\mathrm{Pl}} \left|\frac{V_\phi}{V}\right| = \sqrt{2\epsilon}. \tag{2.165}$$

The number of e-folds of inflationary expansion when the observable cosmological scales leave the horizon correspond approximately to $N \sim 10$ [5]. Let us denote the change in the field value during this period by $\Delta\phi_{10}$. Considering that the low-roll parameter ϵ is almost constant during this period from Eq. (2.165) we find

$$\epsilon \sim 10^{-2} \left(\frac{\Delta\phi_{10}}{m_{\mathrm{Pl}}}\right)^2, \tag{2.166}$$

which is the value of ϵ when the cosmological scales leaves the horizon. Observational constraints on n where discussed in section 2.1.2.1 Eq. (2.16), i.e. $n-1 \approx 0.04$, which gives $2\eta - 6\epsilon \approx 0.04$. Therefore, from Eq. (2.166) we see that for small field inflationary models, for which $\Delta\phi_{10} \ll m_{\mathrm{Pl}}$, the slow-roll parameter ϵ is much smaller than η and Eq. (2.164) may be written as

$$n - 1 = 2\eta. \tag{2.167}$$

One can go further and consider the running of the spectral index as well, i.e. the scale dependence $n = n(k)$. The running is defined as $n' \equiv \mathrm{d}n/\mathrm{d}\ln k$, from which one finds

$$n' = -16\epsilon\eta + 24\epsilon^2 + 2\xi, \tag{2.168}$$

where

$$\xi \equiv m_{\mathrm{Pl}}^4 \frac{V_\phi V_{\phi\phi\phi}}{V^2}, \tag{2.169}$$

where $V_{\phi\phi\phi} \equiv \mathrm{d}^3V/\mathrm{d}\phi^3$.

So far we have discussed the two point correlation function of Eq. (2.143). If perturbations are exactly Gaussian all higher correlators must vanish. Indeed, the non-Gaussianity of the curvature perturbation generated by the inflaton field is very small in single field inflation and this can be seen by calculating the three point correlators. To show this let us calculate the second derivative of Eq. (2.155)

$$\frac{N_{\phi\phi}}{N_\phi^2} = 2\epsilon - \eta. \tag{2.170}$$

Then using Eq. (2.145) we find that non-Gaussianity is of order the slow roll parameters:

$$\frac{6}{5} f_{\mathrm{NL}} = \eta - 2\epsilon, \tag{2.171}$$

were we have used Eq. (2.170). From this relation it is clear that $|f_{\mathrm{NL}}| \ll 1$. The cosmic variance limits the detection of f_{NL} to the values $|f_{\mathrm{NL}}| > 3$ [42]. Therefore, non-Gaussianity produced by a single field inflation is too small to ever be observed. However, there are other scenarios of the generation of the curvature perturbation for which non-Gaussianity can be large enough to be observable in the near future.

One may consider even higher order correlators of the curvature perturbations. And in some models they can be large enough to be observable as well. But this is beyond the scope of this thesis.

2.4.2. At the End of Inflation

In the previous section we discussed the scenario when the inflation is driven by a single scalar field ϕ. In this scenario inflation ends when the inflaton field reaches a critical value ϕ_c, where the slow-roll conditions in Eqs. (1.55) and (1.56) are violated, which is solely determined by the inflaton field. Therefore, inflation ends on the uniform energy density slice.

In Ref. [58] a scenario was suggested where the critical value ϕ_c is modulated by some other scalar field σ, $\phi_c = \phi_c(\sigma)$. For single field inflation, the contribution from σ to the inflaton dynamics must be negligible. ϕ_c in this case must depend only on the perturbation $\delta\sigma(\mathbf{x})$. Then the hypersurface of constant ϕ_c does no longer coincide with the uniform density hypersurface.

For clarity let us assume for the moment that the perturbation $\delta\phi$ of the inflaton field generated during inflation is negligible. Then, if the inflaton is a free field, the slice of constant ϕ will coincide with the flat slice and with the constant energy density slice even at the end of inflation, i.e there will be no curvature perturbation. But if the end of inflation value ϕ_c is modulated by some other perturbed field, $\phi_c(\sigma(\mathbf{x}))$, then the uniform density slice at the end of inflation no longer coincides with the flat slice. According to the section 2.3.4, this produces a perturbation in the amount of expansion between the flat and uniform energy density slices which, from Eq. (2.138), is equal to the curvature perturbation ζ_{end}.

If, on the other hand, $\delta\phi$ is not negligible, then the same argument holds, but ζ_{end} will correspond to the perturbation in the amount of expansion between the uniform energy density just before the end of inflation and the one just after the end of inflation. However, ζ_{end} may be still large enough to dominate the curvature perturbation which is generated during inflation, $\zeta_{\text{end}} \gg \zeta_{\text{inf}}$.

For the following discussion we will assume that after inflation the Universe undergoes prompt reheating, i.e. the inflaton field energy is promptly converted into radiation. Then from Eq. (2.140) up to the second order we can write

$$\zeta_{\text{end}} = N_c \delta\phi_c + N_{cc} (\delta\phi_c)^2 , \tag{2.172}$$

where we have defined

$$N_c \equiv \frac{\partial N}{\partial \phi_c} \quad \text{and} \quad N_{cc} \equiv \frac{\partial^2 N}{\partial \phi_c^2}. \tag{2.173}$$

In the former expression, $\delta\phi_c$ is the perturbation of the end-of-inflation field value due to the coupling to the field σ. Because, as was argued before, ϕ_c depends only on the perturbation $\delta\sigma$, we can expand to the second order

$$\delta\phi_c = \phi_c'\delta\sigma + \frac{1}{2}\phi_c''\left(\delta\sigma\right)^2, \tag{2.174}$$

where $\phi_c' \equiv \partial\phi_c/\partial\sigma$ and $\phi_c'' \equiv \partial^2\phi_c/\partial\sigma^2$. Keeping terms only to the second order in $\delta\sigma$ the curvature perturbation $\zeta_{\rm end}$ becomes

$$\zeta_{\rm end} = N_c\phi_c'\delta\sigma + \frac{1}{2}\left[N_c\phi_c'' + N_{cc}\left(\phi_c'\right)^2\right]\left(\delta\sigma\right)^2. \tag{2.175}$$

To illustrate this model in a concrete example, let us consider the hybrid inflation scenario [59, 60, 61] first. In this scenario the slowly rolling inflaton field is coupled to another scalar field, called the waterfall field. The potential for this type of models is given by

$$V\left(\phi,\chi\right) = V\left(\phi\right) - \frac{1}{2}m_\chi^2\chi^2 + \frac{1}{4}\lambda\chi^4 + \frac{1}{2}\lambda_\phi\phi^2\chi^2, \tag{2.176}$$

where ϕ is the slow rolling inflaton field and χ is the waterfall field. From this expression it can be seen that the effective mass of χ is

$$m_{\rm eff}^2 = \lambda_\phi\phi^2 - m_\chi^2. \tag{2.177}$$

Initial conditions are such that $m_{\rm eff}^2 > 0$ and the waterfall field is located at $\chi = 0$ while the inflaton field ϕ slowly rolls towards zero. With this configuration the dominant term of the potential in Eq. (2.176) is the term $V\left(\phi\right)$.

Inflation ends when the inflaton field reaches a critical value ϕ_c and the effective mass in Eq. (2.177) becomes negative. This destabilizes the waterfall field which very rapidly rolls down to the minimum of the potential and acquires the vacuum expectation value. This also very promptly changes the evolution of the inflaton ϕ: instead of slowly rolling it is quickly driven towards zero.

For the end-of-inflation scenario the critical value ϕ_c is additionally modulated by including one more field σ, which is coupled to the waterfall field. The potential in Eq. (2.176) then becomes

$$V\left(\phi,\chi,\sigma\right) = V\left(\phi\right) - \frac{1}{2}m_\chi^2\chi^2 + \frac{1}{4}\lambda\chi^4 + \frac{1}{2}\lambda_\phi\phi^2\chi^2 + \frac{1}{2}\lambda_\sigma\sigma^2\chi^2 + V\left(\sigma\right). \tag{2.178}$$

It is clear that the effective mass of χ becomes

$$m_{\rm eff}^2 = \lambda_\phi\phi^2 + \lambda_\sigma\sigma^2 - m_\chi^2. \tag{2.179}$$

In this case the waterfall field is destabilized and inflation ends when

$$\lambda_\phi \phi_c^2 = m_\chi^2 - \lambda_\sigma \sigma^2. \tag{2.180}$$

As we can see, the critical value ϕ_c is a function of another field, $\phi_c(\sigma)$. The first and second derivative with respect to this field are

$$\begin{aligned} \phi_c' &= -\frac{\lambda_\sigma}{\lambda_\phi}\frac{\sigma}{\phi_c}, & (2.181)\\ \phi_c'' &= -\frac{\lambda_\sigma}{\lambda_\phi \phi_c}\left[1 + \frac{\lambda_\sigma}{\lambda_\phi}\left(\frac{\sigma}{\phi_c}\right)^2\right]. & (2.182)\end{aligned}$$

In Ref. [58] it was considered that the curvature perturbation generated at the end of inflation dominates over the one generated at the horizon exit. This happens if $N_c \phi_c' \gg N_\phi$, where N_ϕ is given in Eq. (2.155). Using Eq. (2.181) this condition can be rewritten as

$$\left(\frac{\lambda_\sigma}{\lambda_\phi}\frac{\sigma}{\phi_c}\right)^2 \gg \frac{\epsilon_c}{\epsilon_k}, \tag{2.183}$$

where ϵ_c is the slow-roll parameter just before the end of inflation. N_c in this expression was taken from Eq. (2.155) giving $N_c^2 = 1/\left(2m_{\rm Pl}^2\epsilon_c\right)$.

With $\zeta_{\rm end}$ dominating over $\zeta_{\rm inf}$ from Eq. (2.175) we obtain the power spectrum of the produced curvature perturbation in this model as

$$\mathcal{P}_{\zeta_{\rm end}} = \frac{1}{2m_{\rm Pl}^2\epsilon_c}\left(\frac{\lambda_\sigma \sigma_{\rm end}}{\lambda_\phi \phi_c}\right)^2\left(\frac{H_k}{2\pi}\right)^2, \tag{2.184}$$

where $\sigma_{\rm end}$ is the value of σ at the end of inflation. Since the potential of the field σ is flat during inflation, we may apply calculations in section 2.4.1 for its perturbation spectrum. Therefore, using Eq. (2.167) we find the spectral tilt as

$$n - 1 = 2\eta_\sigma, \tag{2.185}$$

where η_σ is the second slow roll parameter for the field σ.

The non-Gaussianity parameter for this model when $\zeta_{\rm end}$ is dominant can be calculated as follows. From the expression of $f_{\rm NL}$ in Eq. (2.145) and the curvature perturbation in Eq. (2.175) one finds

$$\frac{6}{5}f_{\rm NL} = -\left[\frac{\phi''}{N_c\left(\phi_c'\right)^2} + \frac{N_{cc}}{N_c^2}\right]. \tag{2.186}$$

Since we already know that N_{cc}/N_c^2 is of the order of slow roll parameters, the first term dominates. And using Eqs. (2.181) and (2.182) we arrive at

$$\frac{6}{5}f_{\rm NL} = \eta\left[\frac{\lambda_\phi}{\lambda_\sigma}\left(\frac{\phi_c}{\sigma_{\rm end}}\right)^2\right]. \tag{2.187}$$

Expressions of the power spectrum and non-Gaussianity parameter $f_{\rm NL}$ in Eqs. (2.184) and (2.187) involve the homogeneous mode of the field σ. This mode is defined as the average value of the field, where the averaging is done over the same comoving box in which perturbations are defined. The comoving box size L must be larger than the observable Universe, but not too large [5, 62, 35]. As discussed in subsection 2.1.1, depending on the accuracy required the box should be such that $\ln(LH_0) \sim \mathcal{O}(1)$.

In single field inflation the value of the field at horizon exit may be calculated from the number of e-folds of remaining inflation. However, in general it is not possible to calculate the unperturbed value of the field and it must be specified as the free parameter of the model. In some cases, it can be evaluated using the stochastic formalism and assuming that our Universe is the typical realization of the whole ensemble.

2.4.3. The Curvaton Mechanism

In section 2.4.1 we have demonstrated the mechanism for the generation of the curvature perturbation at the horizon exit during single field inflation. In section 2.4.2 the curvature perturbation was generated at the end of inflation. Here we will discuss a mechanism by which ζ is generated some time after inflation when the Universe is radiation dominated. Such model is called the curvaton model and was first introduced in Refs. [63, 64, 65, 66]. It is possible that the total curvature perturbation is generated by the curvaton mechanism, or only a part of it. For simplicity, let us assume first, that the curvature perturbation is generated only by the curvaton mechanism.

In these models the field which is responsible for the curvature perturbation is different from the field which drives inflation. In fact, it is not even necessary to assume any particular model of inflation and validity of Einstein gravity during that era: it might be slow roll inflation due to scalar fields, due to modified gravity or any other mechanism. The only assumptions necessary are that inflationary expansion is almost exponential and that after inflation the Universe undergoes reheating and becomes radiation dominated.

The curvaton mechanism liberates inflation models from the need for the inflaton field to drive inflationary expansion as well as generate the primordial curvature perturbation. Therefore, it substantially increases the available parameter space for viable inflation models [67].

Let us denote the scalar curvaton field by σ. Although it is a different field from the one in the previous section, in this case too σ during inflation is subdominant with a sufficiently flat potential, $|V_{\sigma\sigma}| \ll H^2$, where $V_{\sigma\sigma} \equiv \mathrm{d}^2V/\mathrm{d}\sigma^2$. The unperturbed curvaton field satisfies equation of motion

$$\ddot{\sigma} + 3H\dot{\sigma} + V_\sigma = 0. \tag{2.188}$$

As for all light fields, quantum fluctuations of the curvaton field during inflation are promoted to classical perturbations after horizon exit. Then the curvaton perturbations $\delta\sigma(\mathbf{x})$ satisfy

$$\ddot{\delta\sigma} + 3H\dot{\sigma} + V_{\sigma\sigma}\delta\sigma = 0, \tag{2.189}$$

where we have taken into account that on superhorizon scales all gradients vanish since $k/a \to 0$. With vacuum initial conditions the power spectrum for the curvaton field perturbations is given by Eq. (2.106)

$$\mathcal{P}_\sigma = \left(\frac{H_k}{2\pi}\right)^2. \tag{2.190}$$

At some epoch after inflation when $V_{\sigma\sigma} \sim H^2(t)$ the curvaton field starts to oscillate around its VEV. Let us assume that at this epoch Einstein gravity is already valid and that this happens during the radiation domination (the energy density of the dominant contribution, i.e. radiation, decreases as $\rho_\gamma \propto a^{-4}$). Then $3m_{\mathrm{Pl}}^2H^2 = \rho_\gamma$ and the Hubble parameter decreases as $H \propto a^{-2}$.

The onset of oscillations of the curvaton field is taken to occur much before the cosmological scales enter the horizon. The potential of the curvaton near its VEV can be approximated as

$$V(\sigma) = \frac{1}{2}m_\sigma^2\sigma^2. \tag{2.191}$$

Then oscillations start when $H \sim m_\sigma$. However, even if the potential is not of that form at the start of oscillations, after a few Hubble times, when the amplitude decreases, it can be approximated to high accuracy by this quadratic form. Then, from Eqs. (2.188) and (2.189), it is clear that the unperturbed and perturbed values of the curvaton field satisfy the same equation of motion resulting in $\delta\sigma/\sigma = \mathrm{const}$. But even if this condition is not satisfied, it will result only in a scale independent factor which does not spoil scale invariance [65].

During oscillations, the curvaton field evolves as the underdamped harmonic oscillator with the energy density decreasing as $\rho_\sigma(t,\mathbf{x}) \approx \frac{1}{2}m_\sigma^2\sigma_{\mathrm{A}}^2(t,\mathbf{x}) \propto a^{-3}(t,\mathbf{x})$, where σ_{A} is the amplitude of oscillations. This decrease is slower than that of radiation. Therefore, the relative energy density of the curvaton increases, $\rho_\sigma/\rho_\gamma \propto a$. If the curvaton decay rate is small enough it can dominate the Universe, resulting in the second reheating at its decay. Or the curvaton can decay when it is still subdominant. In both cases during the period when the relative energy density of the curvaton field increases the total pressure of the Universe is not adiabatic. According to Eq. (2.134) this results in a growth of the curvature perturbation ζ which settles at its constant value soon after the curvaton decay.

To calculate the power spectrum and non-Gaussianity of the curvature perturbation we use the δN formalism (see Ref. [57]). This can be applied with the sudden decay approximation. The curvature perturbation without this approximation can be calculated using Eq. (2.134) and knowing the decay rate of the curvaton. Such calculation can only be done numerically. However, in Refs. [68, 69] it was shown that the sudden decay approximation agrees with the numerical results within 10%.

As we have assumed that the curvature perturbation in the radiation dominated Universe is negligible before the curvaton starts oscillating, the number of e-folds from the end of inflation until oscillations is unperturbed. Therefore, the initial epoch for the δN formula can be taken just after oscillations commence. Let us denote the value of the curvaton field at the onset of oscillations by $\sigma_{\mathrm{osc}}(\mathbf{x})$. In general this will depend on the value of the curvaton field σ_* a few Hubble times after horizon exit of a given scale k, i.e. $\sigma_{\mathrm{osc}} = \sigma_{\mathrm{osc}}(\sigma_*)$. Then the number of e-folds from the beginning of oscillations until the curvaton decay is

$$N(\rho_{\mathrm{dec}}, \rho_{\mathrm{osc}}, \sigma_*) = \ln\frac{a_{\mathrm{dec}}}{a_{\mathrm{osc}}} = \frac{1}{3}\ln\frac{\rho_{\sigma,\mathrm{osc}}}{\rho_{\sigma,\mathrm{dec}}}, \tag{2.192}$$

where 'osc' and 'dec' denotes values at the start of curvaton oscillations and at the decay respectively. On the other hand, for the energy density we have $\rho_{\mathrm{osc}}(\mathbf{x}) \approx \frac{1}{2}m_\sigma^2\sigma_{\mathrm{osc}}^2$, which gives

$$N(\rho_{\mathrm{dec}}, \rho_{\mathrm{osc}}, \sigma_*) = \frac{1}{3}\ln\frac{\frac{1}{2}m_\sigma^2\sigma_{\mathrm{osc}}^2}{\rho_{\sigma,\mathrm{dec}}}. \tag{2.193}$$

Let us denote the total energy density at the start of oscillations ρ_{osc}. Because the curvaton energy density at this epoch is negligible ρ_{osc} corresponds primarily to the radiation energy density $\rho_{\mathrm{osc}} \simeq \rho_{\gamma,\mathrm{osc}}$. But the curvaton is not negligible just before its decay, making the total energy density at this epoch $\rho_{\mathrm{dec}} = \rho_{\gamma,\mathrm{dec}} + \rho_{\sigma,\mathrm{dec}}$. From the scaling laws of matter and radiation (see Eq. (1.8)) we find $\rho_{\sigma,\mathrm{dec}}/\rho_{\sigma,\mathrm{osc}} = (\rho_{\gamma,\mathrm{dec}}/\rho_{\gamma,\mathrm{osc}})^{3/4}$. Putting everything together we find

$$\rho_{\sigma,\mathrm{dec}} = \frac{1}{2} m_\sigma^2 \sigma_{\mathrm{osc}}^2 \left(\frac{\rho_{\mathrm{dec}} - \rho_{\sigma,\mathrm{dec}}}{\rho_{\mathrm{osc}}} \right)^{3/4}. \tag{2.194}$$

The derivative of N with respect to the curvaton field may be found using the chain rule $\partial/\partial\sigma_* = \sigma'_{\mathrm{osc}} \cdot \partial/\partial\sigma_{\mathrm{osc}}$ and keeping ρ_{osc} and ρ_{dec} fixed (the prime in this relation denotes differentiation with respect to σ_*). Then from Eq. (2.194) we find

$$\frac{\partial_{\sigma_{\mathrm{osc}}} \rho_{\sigma,\mathrm{dec}}}{\rho_{\sigma,\mathrm{dec}}} = \frac{8}{\sigma_{\mathrm{osc}}} \frac{\rho_{\mathrm{dec}} - \rho_{\sigma,\mathrm{dec}}}{4\rho_{\mathrm{dec}} - \rho_{\sigma,\mathrm{dec}}}. \tag{2.195}$$

Using this relation, from Eq. (2.193) we calculate

$$N_{\sigma_*} = \frac{2}{3} \hat{\Omega}_\sigma \frac{\sigma'_{\mathrm{osc}}}{\sigma_{\mathrm{osc}}}, \tag{2.196}$$

where

$$\hat{\Omega}_\sigma \equiv \frac{3\rho_{\sigma,\mathrm{dec}}}{3\rho_{\sigma,\mathrm{dec}} + 4\rho_{\gamma,\mathrm{dec}}}. \tag{2.197}$$

If the curvaton energy density is subdominant at the decay, then $\rho_{\sigma,\mathrm{dec}} \ll \rho_{\gamma,\mathrm{dec}}$, giving $\hat{\Omega}_\sigma = \frac{3}{4}\Omega_\sigma$, where $\Omega_\sigma \equiv \rho_{\sigma,\mathrm{dec}}/\rho_{\mathrm{dec}}$ is the density parameter of σ at decay. If, on the other hand, the curvaton is dominant at that epoch, then $\hat{\Omega}_\sigma = \Omega_\sigma = 1$. In both cases it is a good approximation to write $\hat{\Omega}_\sigma \approx \Omega_\sigma$. The error introduced by such approximation is not bigger than that of the sudden decay approximation [5].

Inserting Eq. (2.196) into the equation (2.140) we find that the curvature perturbation generated by the curvaton mechanism to first order is

$$\zeta_\sigma = \frac{2}{3} \Omega_\sigma \frac{\sigma'_{\mathrm{osc}}}{\sigma_{\mathrm{osc}}} \delta\sigma_*. \tag{2.198}$$

The power spectrum becomes

$$\mathcal{P}_{\zeta_\sigma} = N_{\sigma_*} \left(\frac{H_k}{2\pi} \right)^2 = \frac{4}{9} \Omega_\sigma^2 \left(\frac{\sigma'_{\mathrm{osc}}}{\sigma_{\mathrm{osc}}} \right)^2 \left(\frac{H_k}{2\pi} \right)^2, \tag{2.199}$$

where for the curvaton field perturbation the power spectrum was given in Eq. (2.190).

To find the non-Gaussianity parameter f_{NL} for this model we have to know ζ_σ at least up to second order. Calculating the second derivative of Eq. (2.196) we find

$$\frac{6}{5}f_{\rm NL}=-\frac{N_{\sigma_*\sigma_*}}{N_{\sigma_*}^2}=-2+\Omega_\sigma+\frac{3}{2\Omega_\sigma}\left(1+\frac{\sigma_{\rm osc}\sigma_{\rm osc}''}{\sigma_{\rm osc}'^2}\right). \tag{2.200}$$

Eqs. (2.199) and (2.200) were calculated using the δN formalism and they agree very well with the calculations performed using the first and second order perturbation theory in Refs. [57, 63].

In the previous calculations we have $\sigma_{\rm osc}(\sigma_*)$ as a general function. In some models it might happen that this is a highly non-trivial function (for example Ref. [70]) but for the future reference let us consider the case when $\sigma_{\rm osc}\simeq\sigma_*$. Then the power spectrum in Eq. (2.199) becomes

$$\mathcal{P}_{\zeta_\sigma}\simeq\frac{4}{9}\Omega_\sigma^2\left(\frac{H_k}{2\pi\sigma_*}\right)^2. \tag{2.201}$$

And the non-Gaussianity from Eq. (2.200) is

$$\frac{6}{5}f_{\rm NL}=\frac{3}{2\Omega_\sigma}, \tag{2.202}$$

where we have considered the case when $f_{\rm NL}\gg 1$.

The result shows that in the curvaton scenario the non-Gaussianity can be very large, $f_{\rm NL}\gg 1$. This is in contrast to single field inflation, where $f_{\rm NL}$ was of order of the slow-roll parameters (see Eq. (2.171)). In the curvaton case the non-Gaussianity can be large because N_σ and $N_{\sigma\sigma}$ have nothing to do with the slow roll parameters.

There is another notable difference from single field inflation - the power spectrum of the curvature perturbation in Eq. (2.201) depends on the homogeneous mode of the field. This situation is analogous to the one discussed in subsection 2.4.2. The value of $\sigma_*(\mathbf{x})$ should be taken as an average within the observable Universe. Then it can be calculated assuming that our Universe is typical.

Until now we have considered only the case when the curvaton is the only source of the curvature perturbation ζ. In other words, we have assumed that the perturbation in the Universe is negligible prior to the domination (or near domination) of the curvaton. But if we drop this assumption then Eq. (2.152) can be used to calculate the resulting curvature perturbation from both components:

$$\zeta=\frac{4\rho_{\gamma,\rm dec}}{3\rho_{\sigma,\rm dec}+4\rho_{\gamma,\rm dec}}\zeta_\gamma+\frac{3\rho_{\sigma,\rm dec}}{3\rho_{\sigma,\rm dec}+4\rho_{\gamma,\rm dec}}\zeta_\sigma, \tag{2.203}$$

where ζ_γ is the curvature perturbation in the radiation dominated background. Using the definition in Eq. (2.197) this equation becomes

$$\zeta = \left(1 - \hat{\Omega}_\sigma\right)\zeta_\gamma + \hat{\Omega}_\sigma\zeta_\sigma. \tag{2.204}$$

Such curvaton models with a two component contribution to the total curvature perturbation were considered in Refs. [71, 62, 72]. While for the negligible curvature perturbation from the inflaton this equation reduces to

$$\zeta \approx \hat{\Omega}_\sigma\zeta_\sigma. \tag{2.205}$$

3. The Primordial Curvature Perturbation from Vector Fields

3.1. Vector Fields in Cosmology

In the previous chapter we have discussed the generation of the primordial curvature perturbation in the Universe. It was shown that during inflation the quantum mechanical fluctuations of light scalar fields are transformed into the classical curvature perturbation. Long after inflation upon horizon entry this perturbation seeds the formation of large scale structure. Until very recently the generation of the curvature perturbation by this mechanism was assigned solely to scalar fields. The main reason for this is that scalar degrees of freedom are the simplest ones. The preference for scalar fields is further supported by the observational fact that the Universe on large scales is predominantly isotropic and statistical properties of the temperature perturbation in the CMB sky are predominantly isotropic too. In addition, particle physics theories beyond the Standard Model are abundant with scalar fields.

In the rest of this thesis we will show that quantum fluctuations of vector fields may influence or even generate the total curvature perturbation in the Universe. But why should we consider something else than the scalar field? The motivation comes from both sides: theoretical as well as observational.

From the theoretical side, for the inflationary model building only scalar fields have been used to generate the curvature perturbation, even if no fundamental scalar field is discovered yet. Although it is widely accepted that all elementary particles possess masses due to the Higgs scalar field, this might be explained by other mechanisms, without invoking fundamental scalar fields (see e.g. the technicolor model in Ref. [73]). It is expected that in the near future the Large Hadron Collider (LHC) in CERN will discover the Higgs boson and prove the existence of the fundamental scalar fields. But if the Higgs boson is not discovered, the alternative models explaining masses of elementary particles will become more favorable. On the other hand, in the case of inflation, the generation of the curvature perturbation from scalar fields will become much less attractive. Furthermore, even if scalar fields are discovered, particle physics theories, such as supersymmetry or supergravity, incorporate many vector bosons and fermions. However, despite this, the possible contribution to the curvature perturbation from other kind of fields usually has been ignored.

Observationally there is some indication that the simplest scalar field scenario may not be sufficient to explain some features of the CMB sky. These features, which challenge the simple homogeneous and isotropic model of the Universe, were first discovered already after the release of the first year WMAP satellite data. For example it was found that the quadrupole moment of the power spectrum of the temperature perturbation was too small compared with predictions of the currently favored ΛCDM cosmology. The lack of power in the quadrupole was still present in the five year data (see Ref. [17]). Another discovery was that 2-4-8-16 spherical harmonics of the CMB temperature map seem to be aligned, suggesting the presence of a preferred direction in the Universe, so called the "Axis of Evil" [74]. In addition this preferred direction is aligned with the large cold spot in the CMB [75], with the large void in the radio galaxy distribution [76] and with the galaxy spin directions [77]. Although currently these anomalies are under intense debate about their statistical significance, they might be an indication that the Universe is mildly anisotropic on large scales.

If these anomalies are confirmed it will prove the existence of the preferred direction in the Universe and this can not be explained solely by scalar fields. On the other hand, for vector fields the existence of the preferred direction is natural. But employing vector fields for the generation of the curvature perturbation we encounter with two complications: conformal invariance and excessive large scale anisotropy of the Universe.

In this Chapter it will be enough to approximate the inflationary expansion of the Universe to be exactly exponential, i.e. with the constant Hubble parameter $\dot{H}=0$.

3.1.1. Conformal Invariance

The evolution of the conformally flat space-time, such as de Sitter or matter/radiation dominated FRW universes, can be modeled as the conformal rescaling of the Minkowski space-time, i.e. by transforming the metric $g_{\mu\nu}(\mathbf{x}) \rightarrow a^2(\tau)\, g_{\mu\nu}(\mathbf{x})$, where τ is the conformal time. As we have seen in section 2.2.2 for a light, minimally coupled scalar field on superhorizon scales this leads to the amplification of vacuum fluctuations. But this is not the case for the conformally trivial theories, for which the field equations are invariant under the rescaling of the metric. For such theories the form of the field equations is time independent in the conformally flat space-times.

This is the case, indeed, for the massless U(1) vector field with the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{3.1}$$

where $F_{\mu\nu}$ is the field strength tensor

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3.2}$$

and A_μ is the four-vector. In contrast to the light scalar field case discussed so far, quantum fluctuations of a vector field with this Lagrangian do not undergo amplification after the horizon exit. But if the vector field is to generate the non-negligible curvature perturbation, its fluctuations have to undergo amplification. Therefore, the conformal invariance of the U(1) massless vector field must be broken.

This problem is well known in the literature on the generation of primordial magnetic fields (PMF) during inflation [78, 79, 80, 81, 82]. In this literature there are numerous suggested ways of breaking the conformal invariance for vector fields: (i) introducing a mass for the vector field, (ii) making the kinetic term time dependent, (iii) introducing an anomaly term (iv) coupling a vector field to another field which is not conformally coupled to gravity, (v) or using non-Abelian vector fields. In this thesis we use only the first two methods. In the context of the curvature perturbation the breaking of the conformal invariance of U(1) field by introducing the mass term was first considered in Ref. [83] and using the time dependent kinetic term in Ref. [84]. The first attempt to calculate the generation of the curvature perturbation by the non-Abelian SU(2) vector fields was reported in Refs. [85, 86]. The other methods of breaking the conformal invariance were investigated only for the generation of PMF.

3.1.2. Large Scale Anisotropy

If the vector field is to influence or generate the curvature perturbation, its energy density must dominate or nearly dominate the Universe for the effect to be non-negligible. But the energy-momentum tensor of the light vector field has an anisotropic stress. For example the energy-momentum tensor of the light Abelian vector field can be written as [83]

$$T_{\mu\nu} = \mathrm{diag}\left(\rho, -p_\perp, -p_\perp, +p_\perp\right), \tag{3.3}$$

where we have chosen the coordinate axis in a way that spatial components of the homogeneous vector field are equal to $\mathbf{A} = (0, 0, A)$ and we use this choice of coordinates in the rest of this thesis. From Eq. (3.3) it is clear that if such a vector field is to dominate or nearly dominate the Universe, the expansion along the vector field direction will be different from the transverse directions. This would induce an excessive large scale anisotropy which is ruled out by observations.

To bypass this problem there are four methods proposed in the literature. The author of the earliest one in Ref. [87] considered three orthogonal identical vector fields. In this model the total energy-momentum tensor, which is the sum of all three vector fields is isotropic. Therefore, these fields can dominate the Universe and even drive the inflationary expansion. Another mechanism, with vector fields responsible for the inflationary expansion, was proposed in Ref. [88]. The authors introduced a large number of identical vector fields which are randomly oriented in space with identical initial conditions. Due to random orientation, the average pressure becomes almost isotropic. The residual anisotropy is proportional to $N^{-1/2}$, where N is the number of vector fields. If this number is sufficiently large, the induced large scale anisotropy can be small enough to agree with observational bounds. In another model in Ref. [89] the vector field is always subdominant and therefore does not generate excessive large scale anisotropy. It cannot be responsible for the inflationary expansion, but the vector field influences the generation of the curvature perturbation by coupling to the scalar field. In particular, the authors of this paper consider that the vector field modulates the end of inflation. Finally in Ref. [83] the excessive large scale anisotropy is avoided by introducing the vector curvaton scenario. In this scenario, a massive vector field is subdominant during inflation and afterwards until it becomes massive. As was demonstrated in that work, when the vector field becomes massive it starts to oscillate with a frequency much larger than the Hubble time. The pressure components in Eq. (3.3) induced by the oscillating vector field oscillates rapidly themselves. Therefore, the time averaged value of the pressure over one Hubble time is zero and the vector field on average acts as the pressureless, isotropic matter. It can dominate the Universe without generating excessive large scale anisotropy.

In this thesis we will use the end-of-inflation scenario of Ref. [89] as an example to calculate the statistical properties of the curvature perturbation (section 3.6). But mostly we will be occupied with the vector curvaton scenario (section 3.3).

3.1.3. The Physical Vector Field

Before going into the description of the vector field quantization and the generation of the curvature perturbation let us make a comment about the distinction between the field which appears in the Lagrangian and the physical vector field. Consider, for example, the Lagrangian of the massive Abelian vector field[1]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A^{\mu}A_{\mu}, \tag{3.4}$$

with the field strength tensor $F_{\mu\nu}$ defined in Eq. (3.2). Using the FRW metric the mass term may be expanded as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_t^2 - \frac{1}{2}m^2 a^{-2}\mathbf{A}^2, \tag{3.5}$$

from which one notices that the spatial term dependents on the scale factor. This might be alarming, because the Lagrangian is the physical quantity and cannot depend on the arbitrary choice of the normalization of the scale factor. This shows that the vector field $\mathbf{A}$ in the Lagrangian is the comoving field defined with respect to the comoving, Cartesian space coordinates x^i. The physical four-vector field is

$$W_{\mu} = \left(A_0,\, A_i/a\right), \tag{3.6}$$

which is defined in in the basis of the physical coordinate system $a\,(t)\,x^i$. For the comoving and physical vector fields the corresponding upper-index quantities are $A^i = -a^{-2}A_i$ and $W^i = -W_i$.

This is the case for the Lagrangian with the canonically normalized field such as in Eq. (3.4). In section 3.5 we will consider a vector field with the time dependent kinetic term. In this case W_{μ} will denote a canonically normalized physical vector field (see Eq. (3.174)).

3.2. Vector Field Quantization and the Curvature Perturbation

3.2.1. δN Formula with the Vector Field

In this section we will generalize the δN formalism introduced in section 2.3.4 to include perturbations of the vector field. For simplicity we will assume that only one perturbed vector field affects the local expansion rate in the Universe. And keeping one scalar field

[1] To be more precise, the Lagrangian of the Abelian vector field whose gauge symmetry is broken by an explicit mass term.

we can write to the second order (c.f. Eq. (2.141))

$$\begin{aligned}\zeta(t,\mathbf{x}) &= \delta N(\phi(\mathbf{x}), W_i(\mathbf{x}), t) \\ &= N_\phi\delta\phi + N_W^i\delta W_i + \frac{1}{2}N_{\phi\phi}(\delta\phi)^2 + N_{\phi W}^i\delta\phi\delta W_i + \frac{1}{2}N_W^{ij}\delta W_i\delta W_j + ...,\end{aligned} \tag{3.7}$$

where

$$N_\phi \equiv \frac{\partial N}{\partial\phi},\ N_W^i \equiv \frac{\partial N}{\partial W_i},\ N_{\phi\phi} \equiv \frac{\partial^2 N}{\partial\phi^2},\ N_W^{ij} \equiv \frac{\partial^2 N}{\partial W_i\partial W_j},\ N_{\phi W}^i \equiv \frac{\partial^2 N}{\partial W_i\partial\phi}, \tag{3.8}$$

with i and j denoting spatial indices running from 1 to 3. As with scalar fields, the unperturbed vector field values are defined as averages within the chosen box (see the discussion below Eq. (2.187)).

For this expression there is no need to define W_i as components of the vector field. Even more, this expression is valid not only for the isotropic background expansion, but for anisotropic as well. Although for the aim of this thesis it will be enough to consider spatially flat isotropic geometry with the line element in the conformal time

$$\mathrm{d}^2 s = a^2(\tau)\left(\mathrm{d}^2\tau - \mathrm{d}^2\mathbf{x}\right). \tag{3.9}$$

3.2.2. The Vector Field Quantization

To quantize the vector field, let us expand perturbations of the field in Fourier modes, similarly to the case of the scalar field

$$\delta W_i(\tau,\mathbf{x}) = \int \delta\mathcal{W}_i(\tau,\mathbf{k})\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\mathrm{d}\mathbf{k}. \tag{3.10}$$

The massive vector field has three degrees of freedom, and the massless vector field has two, in contrast to the scalar field case which has only one degree of freedom. In Eq. (3.10) we have included only spatial components of the vector field, because the temporal component is non-dynamical, i.e. it is not a degree of freedom and is related to the spatial components through the equation of motion (for the massive vector field). The perturbation of each degree of freedom may be parametrized using polarization vectors as

$$\delta\mathcal{W}_i(\tau,\mathbf{k}) = \sum_\lambda e_i^\lambda\left(\hat{\mathbf{k}}\right) w_\lambda(\tau,k), \tag{3.11}$$

where e_i^λ are polarization vectors, $\hat{\mathbf{k}} \equiv \mathbf{k}/k$ is the unit vector in the direction of $\mathbf{k}$ and k is the modulus $k \equiv |\mathbf{k}|$. The most convenient choice is the circular polarization for which two transverse vectors have different handedness. Because both of them transform differently under rotations, the rotational invariance of the Lagrangian prevents any coupling between them. Choosing the coordinate z axis to point into the direction of $\mathbf{k}$, the circular polarization vectors e_i^λ take the form

$$e_i^{\mathrm{L}} = (1, i, 0)/\sqrt{2}, \quad e_i^{\mathrm{R}} = (1, -i, 0)/\sqrt{2} \quad \text{and} \quad e_i^{||} = (0, 0, 1). \tag{3.12}$$

In these expressions superscripts 'L', 'R' and '||' indicate the left-handed, right-handed and longitudinal polarizations respectively. For the massive vector field all three polarizations are present, but for the massless one $w_{||} = 0$ in Eq. (3.11). These expressions define polarization vectors only up to a rotation about the $\mathbf{k}$ direction, but this is enough for the present purpose. Under the transformation $\mathbf{k} \to -\mathbf{k}$ one of the axis x or y change the sign as well. We will choose that x changes the sign and y stays the same. Then $e_i^\lambda\left(-\hat{\mathbf{k}}\right) = -e_i^{\lambda *}\left(\hat{\mathbf{k}}\right)$ and because $\delta W_i(\tau, \mathbf{x})$ is real, imposing the reality condition onto Eqs. (3.10) and (3.11) gives $w_\lambda^*(-\mathbf{k}) = -w_\lambda(\mathbf{k})$.

Later it will be useful to rewrite polarization vectors in Eq. (3.12) when $\mathbf{k}$ points to an arbitrary direction, not only along the z axis. Using the Cartesian coordinate system they become

$$\begin{aligned} e_i^{||}(\hat{\mathbf{k}}) &= \hat{\mathbf{k}} = (k_x, k_y, k_z), \\ e_i^{\mathrm{L}}(\hat{\mathbf{k}}) &= \frac{1}{\sqrt{2\left(k_x^2 + k_y^2\right)}}\left(-k_y + ik_xk_z,\ k_x + ik_yk_z,\ -i\left(k_x^2 + k_y^2\right)\right), \\ e_i^{\mathrm{R}}(\hat{\mathbf{k}}) &= e_i^{\mathrm{L}*}(\hat{\mathbf{k}}). \end{aligned} \tag{3.13}$$

Quantization of the vector field proceeds in the same way as for the scalar field: we expand each degree of freedom in a complete set of orthonormal mode functions and promote the expansion coefficients to operators with appropriate commutation relations. Vector field components satisfy the Klein-Gordon equation and the field lives in the homogeneous and isotropic FRW space-time, therefore, the complete set of orthonormal mode functions was already chosen to be $\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}$ in Eq. (3.10). The quantized vector field then takes the form[2]

$$\delta\hat{W}_i = \sum_\lambda \int \left[e_i^\lambda(\hat{\mathbf{k}}) w_\lambda(\tau, k)\, \hat{a}_\lambda(\mathbf{k})\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} + e_i^{\lambda *}(\hat{\mathbf{k}}) w_\lambda^*(\tau, k)\, \hat{a}_\lambda^\dagger(\mathbf{k})\, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\right] \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}, \tag{3.14}$$

[2]Note that in this chapter we have changed our normalization of Fourier modes from $(2\pi)^{-3/2}$ to $(2\pi)^{-3}$. This resulted in the $(2\pi)^3$ factor in the commutation relations for creation and annihilation operators.

where

$$\left[\hat{a}_\lambda(\mathbf{k}), \hat{a}_{\lambda'}\left(\mathbf{k}'\right)\right] = (2\pi)^3 \delta\left(\mathbf{k}-\mathbf{k}'\right)\delta_{\lambda\lambda'} \tag{3.15}$$

and other commutators being zero. As in the scalar field case, after the horizon exit vector field perturbations become classical in the sense that the commutator $\left[\delta\hat{W}_i(\mathbf{x}), \partial_\tau\delta\hat{W}_j\left(\mathbf{x}'\right)\right]$ approaches zero.

In later sections we will discuss several mechanisms to generate scale invariant perturbations of the vector field. In all of these mechanisms perturbations will be Gaussian, with no correlation between different polarizations λ or between perturbations of scalar and vector fields. With these conditions it is sufficient to consider only the spectra of vector field perturbations $\mathcal{P}_\lambda \equiv \mathcal{P}_{w_\lambda}$. They are defined by the analogue to Eqs. (2.3) and (2.4) as

$$\left\langle w_\lambda(\mathbf{k})\, w^*_{\lambda'}\left(\mathbf{k}'\right)\right\rangle = (2\pi)^3 \delta_{\lambda\lambda'}\delta\left(\mathbf{k}-\mathbf{k}'\right)\frac{2\pi^2}{k^3}\mathcal{P}_\lambda(\mathbf{k}), \tag{3.16}$$

$$\left\langle w_\lambda(\mathbf{k})\, w_{\lambda'}\left(\mathbf{k}'\right)\right\rangle = -(2\pi)^3 \delta_{\lambda\lambda'}\delta\left(\mathbf{k}+\mathbf{k}'\right)\frac{2\pi^2}{k^3}\mathcal{P}_\lambda(\mathbf{k}), \tag{3.17}$$

where $\mathcal{P}_\lambda(\mathbf{k}) \equiv P_\lambda(\mathbf{k})\, k^3/\left(2\pi^2\right)$ as defined in Eq. (2.6), and we have suppressed the notation of time, i.e. $w_\lambda(\tau, \mathbf{k}) \equiv w_\lambda(\mathbf{k})$.

If the Lagrangian is parity conserving then $\mathcal{P}_{\mathrm{L}} = \mathcal{P}_{\mathrm{R}}$, which will be the case in all models considered in this thesis. Parity violation is introduced by terms involving the dual of $F_{\mu\nu}$, i.e. $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor. Examples of such theories may be found in Refs. [90, 91, 92]. The difference between the left-handed and right-handed power spectra would indicate a parity violation. Therefore, it is convenient to define parity conserving $\mathcal{P}_+$ and parity violating $\mathcal{P}_-$ power spectra by

$$\mathcal{P}_\pm \equiv \frac{\mathcal{P}_{\mathrm{R}} \pm \mathcal{P}_{\mathrm{L}}}{2}. \tag{3.18}$$

We also define two parameters which quantify the anisotropy in the particle production of the vector field

$$p(k) \equiv \frac{\mathcal{P}_{||}(k) - \mathcal{P}_+(k)}{\mathcal{P}_+(k)} \quad \text{and} \quad q(k) \equiv \frac{\mathcal{P}_-(k)}{\mathcal{P}_+(k)}, \tag{3.19}$$

where we have also assumed that the expansion during inflation is isotropic making $\mathcal{P}_{||}$ and $\mathcal{P}_\pm$ dependent only on the modulus of $\mathbf{k}$. By the isotropic particle production it is meant that the perturbation spectrum for all three degrees of freedom is the same, i.e. $p = 0$ and $q = 0$. In this case the curvature perturbation generated by the vector field does not differ from the scalar field. But if $p \neq 0$ and/or $q \neq 0$ the particle production of

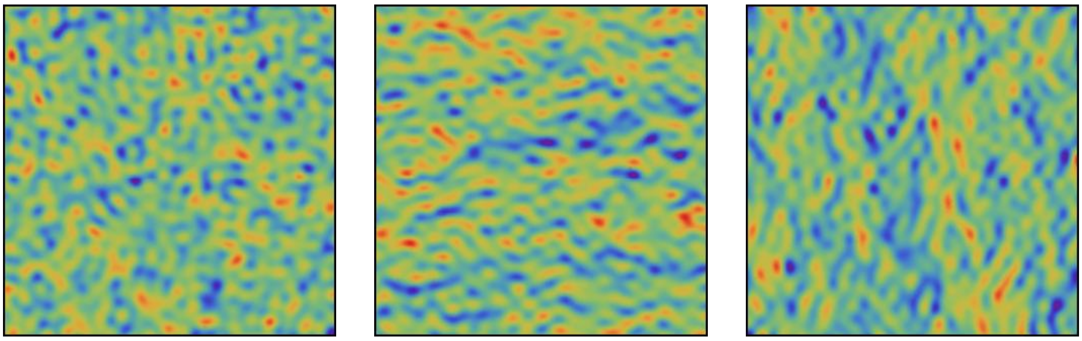

Figure 3.1.: Random field maps. The first is statistically isotropic random field. If the vector field undergoes isotropic particle production with $p = 0$ and $q = 0$, the curvature perturbation generated by such a vector resembles the first map. If, on the other hand, $p \neq 0$ and/or $q \neq 0$ the curvature perturbation generated by the vector field is statistically anisotropic and resembles the second or the third map. The second map is of the statistically anisotropic random field with the preferred direction pointing vertically, while the preferred direction of the third map is horizontal.

the vector field is anisotropic. If such a vector field generates the curvature perturbation, its statistical properties are not invariant under the rotations, i.e. it is statistically anisotropic (see Figure 3.1)

Calculating the two-point correlators of $\delta W_i(\mathbf{k})$ we find

$$\left\langle \delta W_i(\mathbf{k})\, \delta W_j(\mathbf{k}')\right\rangle = (2\pi)^3\, \delta\left(\mathbf{k}+\mathbf{k}'\right) \frac{2\pi^2}{k^3} \left[e_i^{\mathrm{L}} e_j^{\mathrm{L}*}\, \mathcal{P}_{\mathrm{L}} + e_i^{\mathrm{R}} e_j^{\mathrm{R}*}\, \mathcal{P}_{\mathrm{R}} + e_i^{||} e_j^{||}\, \mathcal{P}_{||} \right], \quad (3.20)$$

where we have suppressed the notation of $\hat{\mathbf{k}}$ for $e_i^{\lambda}(\hat{\mathbf{k}})$ and of k for $\mathcal{P}_\lambda(k)$. This expression may be rewritten in terms of $\mathcal{P}_\pm$ in Eq. (3.18) as

$$\begin{aligned} \left\langle \delta W_i(\mathbf{k})\, \delta W_j(\mathbf{k}')\right\rangle &= (2\pi)^3\, \delta\left(\mathbf{k}+\mathbf{k}'\right) \frac{2\pi^2}{k^3} \times \\ &\times \left[T_{ij}^{\mathrm{even}}(\hat{\mathbf{k}}) \mathcal{P}_+(k) + i T_{ij}^{\mathrm{odd}}(\hat{\mathbf{k}}) \mathcal{P}_-(k) + T_{ij}^{\mathrm{long}}(\hat{\mathbf{k}}) \mathcal{P}_{||}(k) \right], \quad (3.21) \end{aligned}$$

where we have introduced tensors

$$\begin{aligned} T_{ij}^{\mathrm{even}}(\hat{\mathbf{k}}) &\equiv e_i^{\mathrm{L}}(\hat{\mathbf{k}}) e_j^{\mathrm{L}*}(\hat{\mathbf{k}}) + e_i^{\mathrm{R}}(\hat{\mathbf{k}}) e_j^{\mathrm{R}*}(\hat{\mathbf{k}}) = e_i^{\mathrm{L}}(\hat{\mathbf{k}}) e_j^{\mathrm{R}}(\hat{\mathbf{k}}) + e_i^{\mathrm{R}}(\hat{\mathbf{k}}) e_j^{\mathrm{L}}(\hat{\mathbf{k}}), \\ T_{ij}^{\mathrm{odd}}(\hat{\mathbf{k}}) &\equiv i \left[e_i^{\mathrm{L}}(\hat{\mathbf{k}}) e_j^{\mathrm{L}*}(\hat{\mathbf{k}}) - e_i^{\mathrm{R}}(\hat{\mathbf{k}}) e_j^{\mathrm{R}*}(\hat{\mathbf{k}}) \right] = i \left[e_i^{\mathrm{L}}(\hat{\mathbf{k}}) e_j^{\mathrm{R}}(\hat{\mathbf{k}}) - e_i^{\mathrm{R}}(\hat{\mathbf{k}}) e_j^{\mathrm{L}}(\hat{\mathbf{k}}) \right], \quad (3.22) \\ T_{ij}^{\mathrm{long}}(\hat{\mathbf{k}}) &\equiv e_i^{||}(\hat{\mathbf{k}}) e_j^{||}(\hat{\mathbf{k}}). \end{aligned}$$

With the circular polarization vectors derived in Eq. (3.13) these tensors take a simple form

$$T_{ij}^{\rm even}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i \hat{k}_j, \tag{3.23}$$
$$T_{ij}^{\rm odd}(\hat{\mathbf{k}}) = \epsilon_{ijk} \hat{k}_k, \tag{3.24}$$
$$T_{ij}^{\rm long}(\hat{\mathbf{k}}) = \hat{k}_i \hat{k}_j. \tag{3.25}$$

3.2.3. The Power Spectrum

Since ζ is Gaussian to high accuracy, it seems reasonable to expect that ζ will be dominated by one or more of the linear terms in Eq. (3.7). Keeping only them (corresponding to what is called the tree-level contribution) we find

$$\begin{aligned} \mathcal{P}_\zeta(\mathbf{k}) &= N_\phi^2 \mathcal{P}_\phi(k) + N_W^i N_W^j \left[T_{ij}^{\rm even}(\hat{\mathbf{k}}) \mathcal{P}_+(k) + T_{ij}^{\rm long} \mathcal{P}_{||}(k) \right] & (3.26) \\ &= N_\phi^2 \mathcal{P}_\phi(k) + N_W^2 \mathcal{P}_+(k) + \left(\mathbf{N}_W \cdot \hat{\mathbf{k}} \right)^2 \left(\mathcal{P}_{||} - \mathcal{P}_+ \right). & (3.27) \end{aligned}$$

Note that the power spectrum of ζ is dependent on the direction of $\mathbf{k}$. In the upcoming discussion we will frequently use the modulus of $\mathbf{N}_W$ and the unit vector along its direction defined by

$$N_W \equiv |\mathbf{N}_W| = \sqrt{N_W^i N_W^i} \quad \text{and} \quad \hat{\mathbf{N}}_W \equiv \frac{\mathbf{N}_W}{N_W}. \tag{3.28}$$

The curvature perturbation power spectrum $\mathcal{P}_\zeta(\mathbf{k})$ may be further separated into isotropic and anisotropic parts

$$\mathcal{P}_\zeta(\mathbf{k}) = \mathcal{P}_\zeta^{\rm iso}(k) \left[1 + g(k) \left(\hat{\mathbf{N}}_W \cdot \hat{\mathbf{k}} \right)^2 \right], \tag{3.29}$$

which has the same form as anisotropic power spectrum in Eq. (2.7) keeping only up to the quadratic term. Comparing this expression with Eq. (3.27) we find that the isotropic part of the spectrum is

$$\mathcal{P}_\zeta^{\rm iso}(k) = N_\phi^2 \mathcal{P}_\phi(k) + N_W^2 \mathcal{P}_+(k) = N_\phi^2 \mathcal{P}_\phi(k) \left[1 + \xi \frac{\mathcal{P}_+(k)}{\mathcal{P}_\phi(k)} \right], \tag{3.30}$$

where we have introduced the parameter ξ

$$\xi \equiv \left(\frac{N_W}{N_\phi}\right)^2 . \tag{3.31}$$

This parameter specifies the relative contribution from the vector field to the statistically isotropic part of the curvature perturbation.

From Eq. (3.27) we can also find that the anisotropy in the curvature perturbation power spectrum $\mathcal{P}_\zeta$ is equal to

$$g(k) = N_W^2 \frac{\mathcal{P}_{||}(k) - \mathcal{P}_+(k)}{\mathcal{P}_\zeta^{\mathrm{iso}}(k)} = \frac{\xi}{[\mathcal{P}_\phi(k)/\mathcal{P}_+(k)] + \xi} p(k) . \tag{3.32}$$

If the vector field perturbation dominates ζ, i.e. $\xi \gg 1$, the anisotropy in the power spectrum of the curvature perturbation is equal to the anisotropy in the particle production of the vector field $g \approx p$. As was mentioned in section 2.1.2.1 the observational bound for the anisotropy in the power spectrum of the curvature perturbation is $g \lesssim 0.3$. Therefore, if the anisotropy in the particle production of the vector field is larger than this bound and there is no other vector field contribution, the produced statistical anisotropy would violate observational constraints. To prevent this, the dominant contribution to ζ must come from one or more statistically isotropic scalar field perturbations.

Equations (3.26) and (3.27) for the power spectrum of the curvature perturbation were calculated only at the tree level. Analogous calculation for the one-loop contribution may be found in Ref. [93].

3.2.4. The Bispectrum

Working to the leading order in the quadratic terms of the δN formula, we calculate the tree-level contribution to the bispectrum defined in Eq. (2.13)

$$B_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = B_\phi(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) + B_{\phi W}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) + B_W(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) , \tag{3.33}$$

where we have separated the bispectrum into three parts: one due to perturbations in the scalar field, another part due to the vector field perturbations and the mixed term.

These terms are given by

$$\begin{aligned} B_{\phi}\left(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\right) &= N_{\phi}^2 N_{\phi\phi}\left[\frac{4\pi^4}{k_1^3 k_2^3}\mathcal{P}_{\phi}^2+\text{c.p.}\right], \\ B_{\phi W}\left(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\right) &= -N_{\phi}N_{\phi W}^i\left[\frac{4\pi^4}{k_1^3 k_2^3}\mathcal{P}_{\phi}\mathcal{M}_i\left(\mathbf{k}_2\right)+\text{c.p.}\right], \\ B_{W}\left(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\right) &= \frac{4\pi^4}{k_1^3 k_2^3}\mathcal{M}_i\left(\mathbf{k}_1\right)N_W^{ij}\mathcal{M}_j\left(\mathbf{k}_2\right)+\text{c.p.} \end{aligned} \tag{3.34}$$

The power spectrum $\mathcal{P}_{\phi}\left(k\right)$ in the above equations depends only on the modulus of $\mathbf{k}$ because we assumed that the expansion during inflation is isotropic, and the vector $\mathcal{M}_i\left(\mathbf{k}\right)$ characterizes perturbations of the vector field:

$$\mathcal{M}_i\left(\mathbf{k}\right)\equiv\mathcal{P}_+N_W\left[\hat{N}_W^i+\hat{k}_i\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{N}}_W\right)p+i\left(\hat{\mathbf{k}}\times\hat{\mathbf{N}}_W\right)_i q\right]. \tag{3.35}$$

Reversal of the three wave-vectors in Eq. (3.34) corresponds to the parity transformation, and using the reality condition $\zeta\left(-\mathbf{k}\right)=\zeta^*\left(\mathbf{k}\right)$ we find that it changes each correlator into its complex conjugate. This does not affect the power spectrum because the reality condition also makes the spectrum real. This may not affect the isotropic bispectrum as well, because the reality condition and statistical isotropy make the bispectrum real. In our case, the bispectrum is anisotropic, and is guaranteed to be real only if the theory is parity conserving, i.e. if $q=0$.

The second order contribution of the quadratic terms in the δN formula gives the one-loop contribution to the bispectrum. It could be significant or even dominant. It has been calculated for the scalar case in Ref. [94], and has been investigated for the case of multifield inflation in Refs. [95, 96] for example. The one-loop contribution from the vector perturbation is calculated in Refs. [97, 98].

3.2.5. The Non-Linearity Parameter f_{NL}

In calculating the non-linearity parameter defined in Eq. (2.20) we will be interested in two configurations: equilateral, with $k_1=k_2=k_3$, and squeezed, with $k_1\simeq k_2\gg k_3$. In the equilateral configuration the bispectra from Eqs. (3.34) become

$$\begin{aligned} \mathcal{B}_{\phi}^{\text{equil}}\left(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\right) &= 3N_{\phi}^2N_{\phi\phi}\mathcal{P}_{\phi}^2, \\ \mathcal{B}_{W\phi}^{\text{equil}}\left(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\right) &= -N_{\phi}N_{\phi W}^i\mathcal{P}_{\phi}\left[\mathcal{M}_i\left(\mathbf{k}_1\right)+\mathcal{M}_i\left(\mathbf{k}_2\right)+\mathcal{M}_i\left(\mathbf{k}_3\right)\right], \\ \mathcal{B}_{W}^{\text{equil}}\left(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\right) &= \mathcal{M}_i\left(\mathbf{k}_1\right)N_W^{ij}\mathcal{M}_j\left(\mathbf{k}_2\right)+\text{c.p.}, \end{aligned} \tag{3.36}$$

where we have defined for the equilateral configuration

$$\mathcal{B}_{\zeta}^{\mathrm{equil}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \equiv \left(\frac{k_1^3}{2\pi^2}\right)^2 B_{\zeta}^{\mathrm{equil}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3), \tag{3.37}$$

and $\mathcal{B}_{\zeta}^{\mathrm{equil}} = \mathcal{B}_{\phi}^{\mathrm{equil}} + \mathcal{B}_{W\phi}^{\mathrm{equil}} + \mathcal{B}_{W}^{\mathrm{equil}}$. In this case the non-linearity parameter $f_{\mathrm{NL}}^{\mathrm{equil}}$ is expressed using the power spectrum and the bispectrum as:

$$\frac{6}{5} f_{\mathrm{NL}}^{\mathrm{equil}} = \frac{\mathcal{B}_{\zeta}^{\mathrm{equil}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)}{3\left(\mathcal{P}_{\zeta}^{\mathrm{iso}}\right)^2}. \tag{3.38}$$

Observations give a limit on the anisotropy $g \lesssim 0.3$ (see the discussion above Eq. (2.19)). Therefore, since the anisotropic contribution to the curvature perturbation is subdominant compared to the isotropic one, we have included only $\mathcal{P}_{\zeta}^{\mathrm{iso}}$ into the expression of $f_{\mathrm{NL}}^{\mathrm{equil}}$.

In the squeezed configuration we have for the two vectors $\mathbf{k}_1 \simeq -\mathbf{k}_2$, but the third vector $\mathbf{k}_3$ is of much smaller modulus than the other two and almost perpendicular to them. For this configuration Eqs. (3.34) take the form

$$\begin{aligned}
\mathcal{B}_{\phi}^{\mathrm{local}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &= 2\, N_{\phi}^2 N_{\phi\phi} \mathcal{P}_{\phi}(k_1)\, \mathcal{P}_{\phi}(k_3), \\
\mathcal{B}_{W\phi}^{\mathrm{local}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &= -N_{\phi} N_{\phi W}^{i} \left\{\mathcal{P}_{\phi}(k_1)\, \mathcal{M}_i(\mathbf{k}_3) + \mathcal{P}_{\phi}(k_3)\, \mathrm{Re}\left[\mathcal{M}_i(\mathbf{k}_1)\right]\right\}, \\
\mathcal{B}_{W}^{\mathrm{local}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &= 2\,\mathrm{Re}\left[\mathcal{M}_i(\mathbf{k}_1)\right] N_{W}^{ij} \mathrm{Re}\left[\mathcal{M}_j(\mathbf{k}_3)\right],
\end{aligned} \tag{3.39}$$

where $\mathrm{Re}\,[\ldots]$ denotes the real part and $\mathcal{B}_{\zeta}^{\mathrm{local}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ is defined similarly to Eq. (3.37)

$$\mathcal{B}_{\zeta}^{\mathrm{local}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \equiv \frac{k_1^3 k_3^3}{4\pi^4} B_{\zeta}^{\mathrm{local}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3). \tag{3.40}$$

Then, the non-linearity parameter $f_{\mathrm{NL}}^{\mathrm{local}}$ in the squeezed configuration becomes

$$\frac{6}{5} f_{\mathrm{NL}}^{\mathrm{local}} = \frac{\mathcal{B}_{\zeta}^{\mathrm{local}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)}{2\mathcal{P}_{\zeta}^{\mathrm{iso}}(k_1)\, \mathcal{P}_{\zeta}^{\mathrm{iso}}(k_3)}. \tag{3.41}$$

In the treatment so far we have calculated the curvature perturbation generated by the vector field as well as the resulting anisotropic power spectrum and the non-linearity parameter f_{NL} in equilateral and squeezed configurations. However, we didn't discuss how the perturbation of the vector field is generated and which mechanism transformed the field perturbation into the curvature perturbation. In the rest of the thesis we consider several examples of the conformal invariance breaking for the vector field, which generates scale invariant perturbation spectrum and determines the values of p and q parameters. We also consider two scenarios in which the vector field perturbation influences or generates the curvature perturbation ζ.

3.3. The Vector Curvaton Scenario

In section 3.1.2 it was discussed that a dominant light vector field would generate large scale anisotropy in the Universe which violates observational bounds. But to generate or influence the curvature perturbation by a vector field it has to dominate or nearly dominate. One of the ways to overcome this difficulty is the curvaton scenario. This scenario was summarized in section 2.4.3 with the curvaton acted by a scalar field. In this section we consider scenarios with the curvaton acted by a vector field, which was first introduced in Ref. [83].

3.3.1. The Vector Curvaton Dynamics

Let us consider a massive Abelian vector field in the Universe dominated by matter (radiation) with the barotropic parameter $w = 0$ ($w = 1/3$). The Lagrangian of the vector field is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu. \tag{3.42}$$

And let us choose a coordinate system in such a way that spatial part of the homogeneous mode of the vector field has components $A_i = (0, 0, A)$. In Ref. [83] it was shown that the energy momentum tensor for this field may be written as

$$T_\mu^\nu = \mathrm{diag}\left(\rho_W, -p_\perp, -p_\perp, +p_\perp\right), \tag{3.43}$$

where

$$\rho_W \equiv \rho_{\mathrm{kin}} + V_W, \qquad p_\perp \equiv \rho_{\mathrm{kin}} - V_W \tag{3.44}$$

with

$$\rho_{\mathrm{kin}} \equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\frac{\dot{A}}{a}\right)^2, \tag{3.45}$$

$$V_W \equiv -\frac{1}{2}m^2 A_\mu A^\mu = \frac{1}{2}m^2\left(\frac{A}{a}\right)^2. \tag{3.46}$$

From which we see that in general the energy momentum tensor will have anisotropic stress due to the opposite sign of the pressure components in the direction parallel to the field and the perpendicular one. Indeed, the equation of motion for the homogeneous mode of the vector field with the Lagrangian in Eq. (3.42) is given by [83]

$$\ddot{\mathbf{A}} + H\dot{\mathbf{A}} + m^2\mathbf{A} = 0. \tag{3.47}$$

In the matter or radiation dominated Universe the solution of this equation is

$$A\left(t\right) = t^v\left[C_1 J_v\left(mt\right) + C_2 J_{-v}\left(mt\right)\right], \tag{3.48}$$

and the time derivative of the vector field is given by

$$\dot{A}\left(t\right) = \frac{t^v}{m}\left[C_1 J_{v-1}\left(mt\right) - C_2 J_{1-v}\left(mt\right)\right], \tag{3.49}$$

where C_1 and C_2 are constants of integration, J_v is the Bessel function of the first kind and

$$v \equiv \frac{1+3w}{6\left(1+w\right)}. \tag{3.50}$$

In the FRW Universe the Hubble parameter is $H \sim t^{-1}$ and the light field corresponds to $mt \ll 1$. In this limit Bessel functions can be approximated by power law functions and the growing mode of the vector field changes with time as

$$A\left(t\right) \propto t^{2v}. \tag{3.51}$$

Inserting this into Eqs. (3.45), (3.46) and using Eq. (3.43) it is clear that the energy-momentum tensor of the light vector field has anisotropic stress. If such a vector field dominated the Universe, the expansion rate in the direction of the field would be different from the transverse directions and the Universe would become predominantly anisotropic. Such excessive *large scale* anisotropy is forbidden by observations, therefore a light vector field cannot dominate the Universe.

In the opposite regime, when the vector field is heavy, $mt \gg 1$, the Bessel functions may be approximated by trigonometric functions giving

$$A\left(t\right) = t^v\sqrt{\frac{2}{\pi mt}}\left[C_1\sin\left(mt + \frac{1-2v}{4}\pi\right) + C_2\cos\left(mt - \frac{1-2v}{4}\pi\right)\right], \tag{3.52}$$

$$\dot{A}\left(t\right) = -mt^v\sqrt{\frac{2}{\pi mt}}\left[C_1\cos\left(mt + \frac{1-2v}{4}\pi\right) - C_2\sin\left(mt - \frac{1-2v}{4}\pi\right)\right]. \tag{3.53}$$

From these equations it is clear that the evolution of the heavy vector field resembles the evolution of the underdamped harmonic oscillator with decreasing amplitude of oscillations. To see this let us rewrite Eqs. (3.52) and (3.53) in the form

$$A(t) = \Lambda t^{v}\sqrt{\frac{2}{\pi mt}}\sin(mt+\varphi), \tag{3.54}$$

$$\dot{A}(t) = -\Lambda m t^{v}\sqrt{\frac{2}{\pi mt}}\cos(mt+\varphi), \tag{3.55}$$

where constants Λ and φ are related to the original constants by

$$\Lambda \equiv \sqrt{C_1^2 + C_2^2 + 2C_1C_2\cos(\pi v)}, \tag{3.56}$$

and

$$\varphi \equiv \arccos\left[\frac{C_1\cos\left(\frac{1-2v}{4}\pi\right) + C_2\sin\left(\frac{1-2v}{4}\pi\right)}{\Lambda}\right]. \tag{3.57}$$

Calculating the energy density and pressure from Eq. (3.44) we find

$$\rho_W = a^{-3}\frac{m}{\pi}\Lambda^2, \tag{3.58}$$

$$p_\perp = a^{-3}\frac{m}{\pi}\Lambda^2\cos(2mt+2\varphi). \tag{3.59}$$

Since the vector field is heavy, i.e. $m \gg H$, the frequency of oscillating functions in Eq. (3.59) is much larger than the Hubble parameter. Therefore, during one Hubble time the average pressure of the heavy vector field is zero and we can write

$$\rho_W \propto a^{-3} \quad \text{and} \quad \overline{p_\perp} \approx 0. \tag{3.60}$$

Thus the heavy vector field acts as the pressureless isotropic matter and it can dominate the Universe without generating excessive large scale anisotropy.

This property of the vector field is utilized in the vector curvaton scenario (see Figure 3.2). During inflation the vector field is light, and although its energy-momentum tensor has non-vanishing stress in accordance with the curvaton scenario, it is subdominant, allowing the expansion of the Universe to be isotropic. During this period the vector field with broken conformal invariance undergoes particle production. (In sections 3.4 and 3.5 we consider two mechanisms of breaking the conformal invariance of the vector field and producing a flat perturbation spectrum.) After inflation, the Hubble parameter decreases as $H \propto t^{-1}$. When it becomes smaller than the mass of the vector field, the latter starts to oscillate and, as was shown in Eq. (3.60), acts as the pressureless

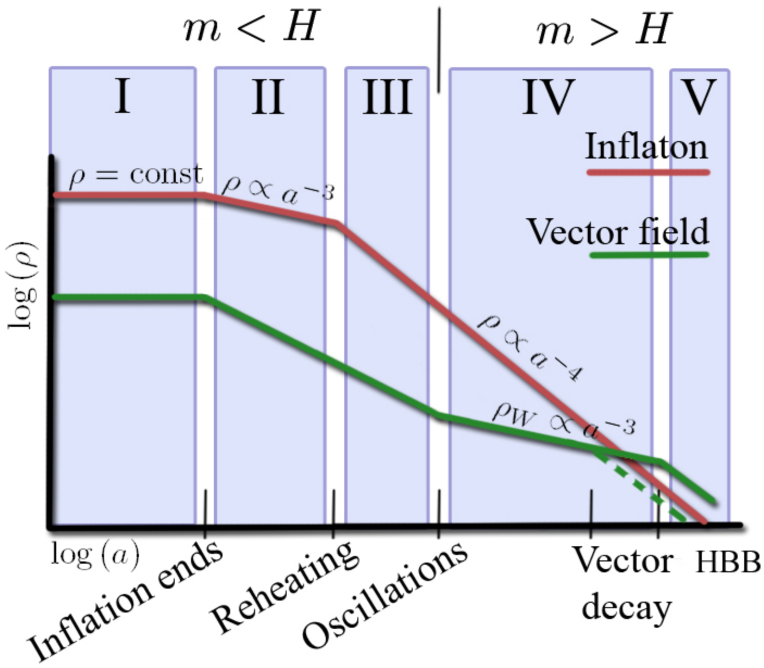

Figure 3.2.: The schematic graph of the vector curvaton scenario: I. during inflation the vector field is subdominant but it acquires a scale invariant perturbation spectrum; II. inflation ends and the inflaton field starts oscillating around its minimum, while the energy density of the Universe scales as $\rho \propto a^{-3}$ (see section 1.5.3); III. after reheating the Universe is radiation dominated and the vector field is still subdominant until it becomes heavy; IV. the heavy vector field oscillates and acts as the pressureless isotropic matter with the energy density decreasing as $\rho_W \propto a^{-3}$; V. the vector curvaton dominates (solid line) or nearly dominates (dashed line) the Universe, imprints its perturbation spectrum and decays, recovering the standard HBB cosmology. Note, that in this graph we depicted the situation when the vector field becomes heavy after reheating. However, this might happen before reheating as well.

isotropic matter. After reheating the relative energy density of the vector field increases as $\rho_W/\rho_\gamma \propto a$, where ρ_γ is the energy density of the radiation dominated Universe. As in the original curvaton scenario (see section 2.4.3) the curvaton field during the radiation dominated period dominates (or nearly dominates) the Universe, imprints its perturbation spectrum and decays, recovering the standard Hot Big Bang cosmology. If the curvaton decays when it is dominant, the Universe undergoes a second reheating.

3.3.2. The Generic Treatment of $f_{\rm NL}$

In this section we obtain analytic expressions for the non-linearity parameter $f_{\rm NL}$ without assuming a specific vector curvaton model. In contrast to the original curvaton idea we include perturbations already present in the radiation dominated Universe when the vector field energy density is still negligible (for a similar study in the scalar curvaton case see Refs. [71, 72]).

Some time after inflation the mass of the vector field becomes bigger than the Hubble parameter. Then the field starts to oscillate and acts as the pressureless isotropic matter. The total contribution to the curvature perturbation by the vector field before its decay can be found using Eqs. (2.151) and (2.204)

$$\zeta_{\rm vec} \equiv \hat{\Omega}_W \zeta_W = \frac{1}{3}\hat{\Omega}_W \frac{\delta\rho_W}{\rho_W}, \tag{3.61}$$

where $\hat{\Omega}_W$ is defined similarly to the Eq. (2.197)

$$\hat{\Omega}_W \equiv \frac{3\rho_W}{3\rho_W + 4\rho_\gamma} = \frac{3\Omega_W}{4-\Omega_W} \tag{3.62}$$

and energy densities ρ_W and ρ_γ are evaluated at the curvaton decay with $\Omega_W \equiv \rho_W/\rho$ and $\rho = \rho_W + \rho_\gamma$. This expression is valid to the first order in $\delta\rho_W$, which is evaluated on a flat slice, where $a\left(t, \mathbf{x}\right)$ is unperturbed.

We assume that the curvaton decays instantly (sudden-decay approximation) and evaluate ζ_W just before the curvaton decays, leaving ζ constant thereafter. Evaluating $\delta\rho_W$ to the second order we have [99]

$$\zeta_{\rm vec} = \frac{2}{3}\hat{\Omega}_W \frac{W_i}{W^2}\delta W_i + \frac{1}{3}\hat{\Omega}_W \frac{1}{W^2}\delta W_i \delta W_i, \tag{3.63}$$

where $W \equiv |\mathbf{W}|$ is evaluated just before the vector field decays. This is valid only for $\hat{\Omega}_W \ll 1$. To calculate the same expression when $\hat{\Omega}_W \simeq 1$ one could evaluate N and hence δN directly. All of this is the same as for a scalar field contribution, where the calculation of N was done in Ref. [99].

Comparing Eq. (3.63) with (3.7) we find N_W^i and N_W^{ij} to be equal to

$$N_W^i = \frac{2}{3}\hat{\Omega}_W \frac{W_i}{W^2}, \tag{3.64}$$
$$N_W^{ij} = \frac{2}{3}\hat{\Omega}_W \frac{\delta_{ij}}{W^2}. \tag{3.65}$$

Using Eq. (3.64) the isotropic part of the total power spectrum in Eq. (3.30) becomes

$$\mathcal{P}_\zeta^{\rm iso}(k) = N_\phi^2 \mathcal{P}_\phi(k)\left(1 + \xi\,\frac{\mathcal{P}_+(k)}{\mathcal{P}_\phi(k)}\right), \tag{3.66}$$

and from Eq. (3.64) the preferred direction in the power spectrum in Eq. (3.29) is

$$\hat{\mathbf{N}}_W = \hat{\mathbf{W}}, \tag{3.67}$$

where $\hat{\mathbf{W}} \equiv \mathbf{W}/W$.

Then the vector part of the bispectrum for equilateral configuration in Eq. (3.36) reduces to

$$\begin{aligned}
\mathcal{B}_W^{\rm equil}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = {} & \left(\frac{2}{3}\frac{\hat{\Omega}_W}{W}\right)^3 \frac{1}{W}\mathcal{P}_+(k_1)\,\mathcal{P}_+(k_2)\left\{1 + p(k_1)\,W_1^2 + p(k_2)\,W_2^2 + \right. \\
& + W_1 W_2\left[q(k_1)\,q(k_2) - \frac{1}{2}p(k_1)\,p(k_2)\right] + \\
& + i\sqrt{\frac{3}{4} - \left(W_1^2 + W_1W_2 + W_2^2\right)}\,\left[W_1 p(k_1)\,q(k_2) - \right. \\
& \left.\left. - W_2 p(k_2)\,q(k_1)\right] + \frac{1}{2}q(k_1)\,q(k_2)\right\} + \text{c.p.}
\end{aligned} \tag{3.68}$$

In the above we used the notation $W_1 \equiv \hat{\mathbf{W}}\cdot\hat{\mathbf{k}}_1$ etc. Because the configuration of wave vectors $\hat{\mathbf{k}}_1$, $\hat{\mathbf{k}}_2$ and $\hat{\mathbf{k}}_3$ is equilateral, with the angle between any two of them being $2\pi/3$, we find $\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2 = \hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_3 = \hat{\mathbf{k}}_2\cdot\hat{\mathbf{k}}_3 = -\frac{1}{2}$. Eq.(3.68) simplifies further if we consider a scale invariant power spectrum, then the expression for $f_{\rm NL}^{\rm equil}$ becomes:

$$\frac{6}{5}f_{\rm NL}^{\rm equil} = \xi^2\mathcal{P}_+^2\,\frac{3}{2\hat{\Omega}_W}\,\frac{\left(1+\frac{1}{2}q^2\right) + \left[p + \frac{1}{8}\left(p^2 - 2q^2\right)\right]W_\perp^2}{\left(\mathcal{P}_\phi + \xi\,\mathcal{P}_+\right)^2}, \tag{3.69}$$

where we have taken into account that the non-Gaussianity generated during the single field inflation is negligible (see Eq. (2.171)). The quantity $W_\perp \leq 1$ is the modulus of the projection of the unit vector $\hat{\mathbf{W}}$ onto the plane containing the three vectors $\hat{\mathbf{k}}_1$, $\hat{\mathbf{k}}_2$ and

$\hat{\mathbf{k}}_3$. The calculation of $W_\perp$ in the equilateral configuration is explained in more detail in the Appendix A.

Similarly to the definition of the anisotropic power spectrum in Eq. (3.29) we may separate the non-linearity parameter $f_{\rm NL}$ in Eq. (3.69) into the isotropic and anisotropic parts

$$\frac{6}{5} f_{\rm NL} = f_{\rm NL}^{\rm iso}\left(1 + \mathcal{G}\cdot W_\perp^2\right), \tag{3.70}$$

where $\mathcal{G}$ parametrizes the anisotropy in $f_{\rm NL}$. Comparing this equation with Eq. (3.69) we find that in the equilateral configuration

$$f_{\rm NL,\,iso}^{\rm equil} = \xi^2 \mathcal{P}_+^2 \frac{3}{2\hat{\Omega}_W} \frac{\left(1+\frac{1}{2}q^2\right)}{\left(\mathcal{P}_\phi + \xi\,\mathcal{P}_+\right)^2}, \tag{3.71}$$

and

$$\mathcal{G}^{\rm equil} = \frac{p + \frac{1}{8}\left(p^2 - 2q^2\right)}{1+\frac{1}{2}q^2}. \tag{3.72}$$

For the squeezed configuration the bispectrum from the vector field perturbation in Eqs. (3.39) becomes

$$\begin{aligned} \mathcal{B}_W^{\rm local}\left(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\right) \;=\; & 2\left(\frac{2}{3}\frac{\hat{\Omega}_W}{W}\right)^3 \frac{1}{W}\mathcal{P}_+\left(k_1\right)\mathcal{P}_+\left(k_3\right) \times \\ & \times\left[1 + p\left(k_1\right)W_1^2 + p\left(k_3\right)W_3^2\right]. \end{aligned} \tag{3.73}$$

Working as in the equilateral case, we find that the non-linearity parameter for the scale invariant power spectra is

$$\frac{6}{5} f_{\rm NL}^{\rm local} = \xi^2 \mathcal{P}_+^2 \frac{3}{2\hat{\Omega}_W} \frac{1 + pW_\perp^2}{\left(\mathcal{P}_\phi + \xi\,\mathcal{P}_+\right)^2}. \tag{3.74}$$

Using the parametrization of Eq. (3.70) we write for the squeezed configuration

$$f_{\rm NL,\,iso}^{\rm local} = \xi^2 \mathcal{P}_+^2 \frac{3}{2\hat{\Omega}_W} \frac{1}{\left(\mathcal{P}_\phi + \xi\,\mathcal{P}_+\right)^2}, \tag{3.75}$$

and

$$\mathcal{G}^{\rm local} = p. \tag{3.76}$$

As one can see from the above equations $f_{\rm NL}$ in general depends on $W_\perp$ in both configurations, i.e. $\mathcal{G}^{\rm equil} \neq 0$ and $\mathcal{G}^{\rm local} \neq 0$. This means that $f_{\rm NL}$ is anisotropic, with the same preferred direction as in the power spectrum (c.f. Eqs. (3.29) and (3.67)). The isotropic parts of $f_{\rm NL}$ in both configurations may be rewritten as

$$f_{\rm NL,\,iso}^{\rm equil} = f_{\rm NL,\,iso}^{\rm local}\left(1+\frac{1}{2}q^2\right) = g^2\frac{3}{2\hat{\Omega}_W}\cdot\frac{1+\frac{1}{2}q^2}{p^2}. \tag{3.77}$$

Given the (quasi) exponential expansion of the Universe during inflation, the value of p depends only on the Lagrangian of the vector field. As Eq. (3.32) suggests it relates to g only indirectly, through parameters determining the generation of anisotropic as well as isotropic parts of ζ. In other words, specifying the value of p does not determine g. In view of this, from Eq. (3.77) we see that the amount of non-Gaussianity is correlated with the statistical anisotropy in the spectrum, $f_{\rm NL} \propto g^2$. If, instead, the particle production is isotropic (i.e. $\mathcal{P}_\parallel = \mathcal{P}_+$ and $\mathcal{P}_- = 0$) Eqs. (3.18) and (3.19) give $p = q = 0$ and therefore $\mathcal{G}^{\rm equil} = \mathcal{G}^{\rm local} = 0$. In this case $f_{\rm NL}^{\rm equil}$ and $f_{\rm NL}^{\rm local}$ become isotropic too and both reduce to $f_{\rm NL} = 5/4\hat{\Omega}_W$ as in the scalar curvaton scenario, where we used Eq. (3.32) with the assumption $\mathcal{P}_\phi \ll \mathcal{P}_+$, i.e. that the dominant contribution to the curvature perturbation is due to the vector curvaton field only.

In addition to the $f_{\rm NL}$ being anisotropic, having the same preferred direction as the spectrum and its magnitude being correlated with the anisotropy in the spectrum g from Eqs. (3.71), (3.72) and (3.75), (3.76) we find more observational signatures. From Eq. (3.77) it is clear that for parity conserving vector fields with $q = 0$, the isotropic parts of $f_{\rm NL}$ are identical, i.e. $f_{\rm NL,\,iso}^{\rm equil} = f_{\rm NL,\,iso}^{\rm local}$. Any departure from this equality would indicate parity violating terms in the Lagrangian of the vector field. But the anisotropy in the non-linearity parameter is configuration dependent with $\mathcal{G}^{\rm equil} \neq \mathcal{G}^{\rm local}$ in both - parity conserving as well as parity violating - theories. In the squeezed configuration, $\mathcal{G}^{\rm local}$ is sensitive only to the anisotropy in the parity conserving perturbations of the vector field. But in the equilateral configuration, $\mathcal{G}^{\rm equil}$ is also correlated with the amount of parity violation of the field. In both cases the anisotropy in $f_{\rm NL}$ is proportional to the anisotropy in the particle production of the vector field, p and q. Therefore, if the anisotropy in particle production is of order one or larger, anisotropic parts of $f_{\rm NL}$ in both configurations *are not subdominant*. At the moment, observations do not provide any information about the values of $\mathcal{G}^{\rm equil}$ and $\mathcal{G}^{\rm local}$. However, as can be seen from Eqs. (3.72) and (3.76) the observational detection of $\mathcal{G}^{\rm equil}$ and $\mathcal{G}^{\rm local}$ would allow to determine uniquely the values of parameters p and q, therefore, allowing to constraint very narrowly the possible range of conformal invariance breaking Lagrangians for the vector field.

3.3.3. Generation of ζ

To calculate the curvature perturbation generated by the vector curvaton field consider an era after reheating when the Universe is radiation dominated, with the energy density decreasing as $\rho_\gamma \propto a^{-4}$. As was discussed at the end of section 3.3.1 the relative energy density of the heavy vector field during this era increases as $\rho_W/\rho_\gamma \propto a$ (see Eq. (3.60)). When the vector field becomes dominant (or nearly dominant) it imprints its perturbation spectrum onto the Universe.

The curvature perturbation ζ_W generated by the vector field is calculated as follows. On the spatially flat slicing of space-time using Eq. (2.151) we can write for the vector field

$$\zeta_W = \left.\frac{\delta\rho_W}{3\rho_W}\right|_{\rm dec}, \tag{3.78}$$

where we considered that the decay of the vector field (labeled by 'dec') occurs after the onset of its oscillations so that it is pressureless, as shown in Eq. (3.60). Note that, since ζ_W is determined by the fractional perturbation of the field's density, which is a scalar quantity, the perturbation ζ_W is scalar and not vector in nature.

Since Eq. (3.47) is a linear differential equation, A and its perturbation δA satisfy the same equation of motion. Therefore, they evolve in the same way, which means that $\delta A/A$ remains constant, before and after the onset of oscillations. As was discussed in section 3.3.1 the massive vector field acts as an underdamped harmonic oscillator. The energy of such oscillator is determined by the amplitude of oscillations. Therefore, we may write $\rho_W = \frac{1}{2}m^2\,||W||^2$, where we used the physical vector field $W = A/a$ defined in Eq. (3.6) and $||W||$ is the amplitude of the oscillating physical vector field. From the above we obtain

$$\zeta_W = \left.\frac{\delta\rho_W}{3\rho_W}\right|_{\rm dec} \approx \left.\frac{2}{3}\,\frac{||\delta W||}{||W||}\right|_{\rm dec} \simeq \left.\frac{2}{3}\,\frac{\delta W}{W}\right|_{\rm osc} \simeq \left.\frac{2}{3}\,\frac{\delta W}{W}\right|_{\rm end}, \tag{3.79}$$

where 'osc' denotes the onset of oscillations and 'end' denotes the time at the end of inflation. Therefore, from this equation we may write

$$\zeta_W \sim \frac{\delta W_{\rm end}}{W_{\rm end}}. \tag{3.80}$$

In the usual scalar curvaton scenario the curvaton field generates the total curvature perturbation in the Universe. This may be realized in the vector curvaton scenario as well if the curvature perturbation ζ_W, generated by the vector field, is statistically isotropic, or if its statistical anisotropy is within the observationally allowed region (see Eq. (2.19)).

However, if ζ_W is predominantly anisotropic it can only be a subdominant contribution to the total curvature perturbation ζ, while the dominant part must be generated by some statistically isotropic source. In analogy to the Eq. (2.204) in this case the total curvature perturbation in the curvaton scenario may be written as

$$\zeta = \left(1 - \hat{\Omega}_W\right) \zeta_\gamma + \hat{\Omega}_W \zeta_W, \tag{3.81}$$

where ζ_γ is the dominant and statistically isotropic curvature perturbation which is present in the radiation dominated Universe before ζ_W is generated. In this equation $\hat{\Omega}_W \approx \frac{3}{4}\Omega_W < 1$ because the vector field must decay before it starts dominating (the dashed line in Figure 3.2). Assuming that ζ_γ is generated by the scalar field ϕ, the anisotropy in the power spectrum g from Eq. (3.32) becomes

$$g \approx \xi \frac{\mathcal{P}_+}{\mathcal{P}_\phi} p, \tag{3.82}$$

were we also assumed scale invariant power spectra for the scalar and vector field perturbations.

Because the isotropic part of the curvature perturbation is dominant, from Eq. (3.29) we may write (see (2.10))

$$\zeta \approx \sqrt{\mathcal{P}_\zeta^{\mathrm{iso}}} = g^{-1/2} N_W \sqrt{\mathcal{P}_+ \left(g + p\right)}, \tag{3.83}$$

were we used Eqs. (3.30) and (3.82).

Using the definition of p in Eq. (3.19) we find that the typical amplitude of the vector field perturbation is

$$\delta W \approx \sqrt{\mathcal{P}_{||} + 2\mathcal{P}_+} = \sqrt{\mathcal{P}_+ \left(p + 3\right)}. \tag{3.84}$$

Combining the last two equations we obtain

$$\zeta \approx g^{-1/2} N_W \delta W \sqrt{\frac{g + p}{3 + p}}, \tag{3.85}$$

where δW is evaluated at the vector field decay. For the vector curvaton scenario the parameter N_W can be found from Eq. (3.64) as $N_W \approx \Omega_W / \left(2W\right)$, where W is evaluated at the field decay. Therefore, using Eq. (3.79) we find that the total curvature perturbation given in Eq. (3.81) is of order

$$\zeta \sim g^{-1/2} \Omega_W \zeta_W, \tag{3.86}$$

were we have taken $\sqrt{\left(g + p\right) / \left(3 + p\right)} \sim \mathcal{O}\left(1\right)$.

After the vector field decays, the curvature perturbation ζ stays constant. This happens at the time Γ_W^{-1}, where $\Gamma_W \sim h^2 m$ is the field decay rate and h is the vector field coupling to its decay products. Due to gravitational decay the lower bound for h is

$$h \gtrsim \frac{m}{m_{\rm Pl}}. \tag{3.87}$$

However, during its oscillations the vector field is subject to thermal evaporation. Were this to occur, all memory of the superhorizon perturbation spectrum would be erased; therefore, no ζ_W would be generated. Considering that the scattering rate of the massive vector boson with the thermal bath is $\Gamma_{\rm sc} \sim h^4 T$ we can obtain a bound such that the condensate does not evaporate before the vector field decays, i.e.

$$\Gamma_{\rm sc} < \Gamma_W, \tag{3.88}$$

which is evaluated at $H \sim \Gamma_W$. The temperature of the Universe at the vector field decay is $T \sim \sqrt{m_{\rm Pl}\Gamma_W}$, giving $\Gamma_{\rm sc} \sim h^5 \sqrt{m_{\rm Pl} m}$. Substituting this into Eq. (3.88) and combining with Eq. (3.87) the range for h becomes

$$\frac{m}{m_{\rm Pl}} \lesssim h \lesssim \left(\frac{m}{m_{\rm Pl}}\right)^{1/6}. \tag{3.89}$$

The lower bound in the above is due to decay through gravitational interactions, while the upper bound is relaxed if the vector field dominates the Universe before it decays. This happens if $\Gamma_W < H_{\rm dom}$, where $H_{\rm dom}$ is the Hubble parameter when the vector field starts to dominate. Then the energy density of the thermal bath is exponentially smaller than ρ_W and the vector field condensate does not evaporate. Thus it is enough to ensure that the vector field condensate does not evaporate before it dominates, i.e. $\Gamma_{\rm sc} < H_{\rm dom}$. For the dominant vector curvaton this bound can be satisfied even if the one in Eq. (3.88) is violated.

Having discussed the general predictions of the vector curvaton scenario for the power spectrum $\mathcal{P}_\zeta$ and non-linearity parameter $f_{\rm NL}$ let us turn now to the realization of this scenario in two concrete examples. In sections 3.4 and 3.5 we present two mechanisms for breaking the conformal invariance of the vector field and find under which conditions the field perturbation power spectrum is scale invariant. This allows us to calculate parameters p and q as well. Then we implement these models into the vector curvaton scenario and compute the parameter space for these models.

3.4. Non-minimally Coupled Vector Curvaton

In Ref. [83] it was shown that a massive vector field may acquire a scale invariant perturbation spectrum if its effective mass during inflation is $-2H^2$. In this section we consider the realization of this scenario. The negative mass squared can be achieved by non-minimally coupling the vector field to gravity through the Ricci scalar term. The vector field Lagrangian for this model is written as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \left(m^2 + \alpha R \right) A_\mu A^\mu , \tag{3.90}$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.91}$$

and R is the Ricci scalar, with α being a real coupling constant. For the further discussion let us define the effective mass of the vector field as

$$M^2 \equiv m^2 + \alpha R. \tag{3.92}$$

Starting from Ref. [79] this action with $m = 0$ was invoked by many authors for the generation of primordial magnetic fields, where A_μ is identified with the electromagnetic field. Note, that due to the non-minimal coupling, this field is no longer gauge invariant. This is in contrast to the electromagnetic field in the Standard Model. But because in the present Universe R is very small, it is thought that at present the electromagnetic field is approximately gauge invariant.

In our case we don't have to worry about the gauge invariance because we don't associate A_μ with the electromagnetic field. Even more so, we don't assume that the vector field A_μ couples to any scalar field through the covariant derivative of the form $\mathcal{D}_\mu \phi \left(\mathcal{D}^\mu \phi \right)^*$.

3.4.1. Equations of Motion

During inflationary stage the spatial curvature of the Universe is inflated away. And in accordance with the curvaton scenario, the vector field during inflation is subdominant and does not influence the expansion of the Universe. Therefore, we can assume to a good approximation that inflationary expansion is homogeneous and isotropic with the flat space-time metric in Cartesian coordinates given by

$$\mathrm{d}^2 s = \mathrm{d}^2 t - a^2 \left(t \right) \mathrm{d}^2 \mathbf{x}. \tag{3.93}$$

In this case the Ricci scalar takes the form

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = -6\left(\dot{H} + 2H^2\right) = 3\left(3w - 1\right)H^2, \tag{3.94}$$

where $w \approx -1$, $w = 1/3$ and $w = 0$ during (quasi) de Sitter inflation, radiation and matter dominated epochs respectively. We will further assume that inflationary expansion is of the (quasi) de Sitter type with $H \simeq$ constant, making the Ricci scalar $R \simeq -12H^2$. With this condition, the effective mass of the vector field during inflation becomes $M^2 \simeq m^2 - 12\alpha H^2 \simeq$ constant.

Calculating equations of motion for the vector field components we will mainly follow Ref. [83]. Using Eq. (3.90) and the variation principle

$$\frac{\partial\left(\sqrt{-\mathcal{D}_g}\mathcal{L}\right)}{\partial A_\nu} - \partial_\mu \frac{\partial\left(\sqrt{-\mathcal{D}_g}\mathcal{L}\right)}{\partial\left(\partial_\mu A_\nu\right)} = 0 \tag{3.95}$$

we find the field equation for the vector field as

$$\left[\partial_\mu + \left(\partial_\mu \ln\sqrt{-\mathcal{D}_g}\right)\right]F^{\mu\nu} + M^2 A^\nu = 0, \tag{3.96}$$

where $\mathcal{D}_g \equiv \det\left(g_{\mu\nu}\right)$. With the FRW metric in Eq. (3.93) and the field equation in Eq. (3.96) the equation of motion for the temporal component ($\nu = 0$) is found to be

$$\nabla\cdot\dot{\mathbf{A}} - \nabla^2 A_t + \left(aM\right)^2 A_t = 0, \tag{3.97}$$

where ∇ is the divergence and $\nabla^2 \equiv \partial_i\partial_i$ is the Laplacian. In the same way we may find the equation of motion for the temporal component ($\nu = i$):

$$\ddot{\mathbf{A}} + H\dot{\mathbf{A}} - a^{-2}\left[\nabla^2\mathbf{A} - \nabla\left(\nabla\cdot\mathbf{A}\right)\right] + M^2\mathbf{A} = \nabla\left(\dot{A}_t + HA_t\right). \tag{3.98}$$

A third useful relation is the integrability condition, which is obtained by contracting Eq. (3.96) with ∂_ν:

$$\left(aM\right)^2\dot{A}_t + 2\left(aM\right)^2\frac{\dot{M}}{M}A_t - M^2\nabla\cdot\mathbf{A} + 3H\left(\nabla^2 A_t - \nabla\cdot\dot{\mathbf{A}}\right) = 0. \tag{3.99}$$

Combining the integrability condition with Eq. (3.97) we find

$$\dot{A}_t + \left(3H + 2\frac{\dot{M}}{M}\right)A_t - a^{-2}\nabla\cdot\mathbf{A} = 0. \tag{3.100}$$

From this equation we can see that the temporal component of the vector field is non-dynamical. Taking the gradient of Eq. (3.100) and plugging it into Eq. (3.98) we arrive at

$$\ddot{\mathbf{A}} + H\dot{\mathbf{A}} + M^2\mathbf{A} - a^{-2}\nabla^2\mathbf{A} = -2\left(H + \frac{\dot{M}}{M}\right)\nabla A_t. \tag{3.101}$$

Classical inhomogeneities of the vector field are diluted by inflation. Therefore, we can neglect all gradient terms

$$\partial_i A_\mu = 0 \quad \forall\, \mu. \tag{3.102}$$

Using this condition in Eqs. (3.97) and (3.98) we find that for the homogeneous mode the temporal and spatial components of the vector field obey

$$\begin{aligned} A_t &= 0, & (3.103)\\ \ddot{A} + H\dot{A} + M^2 A &= 0, & (3.104)\end{aligned}$$

where we have used the choice of the coordinates such that the homogeneous vector field has components $A_\mu = (A_t, 0, 0, A)$. We see that a temporal component of the homogeneous massive vector field in the FRW Universe is zero. The equation of motion for the spatial component in Eq. (3.104) may be rewritten in terms of the physical vector field. For this model it is $W = A/a$. In the (quasi)de Sitter space-time ($\dot{H} \approx 0$) Eq. (3.104) becomes

$$\ddot{W} + 3H\dot{W} + \left(M^2 + 2H^2\right) W = 0. \tag{3.105}$$

However, to quantize the vector field we need to perturb the field

$$A_\mu\left(t,\mathbf{x}\right) = A_\mu\left(t\right) + \delta A_\mu\left(t,\mathbf{x}\right) \Rightarrow \begin{cases} \mathbf{A}\left(t,\mathbf{x}\right) = \mathbf{A}\left(t\right) + \delta\mathbf{A}\left(t,\mathbf{x}\right), \\ A_t\left(t,\mathbf{x}\right) = \delta A_t\left(t,\mathbf{x}\right), \end{cases} \tag{3.106}$$

where we have used Eq. (3.103). From Eqs. (3.97) and (3.101) we find that the evolution of perturbations of the vector field in (quasi)de Sitter space-time ($\dot{H} \approx 0$ and $\dot{M} \approx 0$) follow equations

$$\begin{aligned} \nabla\cdot\delta\dot{\mathbf{A}} - \nabla^2\delta A_t + (aM)^2\,\delta A_t &= 0, & (3.107)\\ \delta\ddot{\mathbf{A}} + H\,\delta\dot{\mathbf{A}} + M^2\delta\mathbf{A} - a^{-2}\nabla^2\delta\mathbf{A} &= -2H\nabla\delta A_t. & (3.108)\end{aligned}$$

Going to the Fourier space (see Eq. (3.10)) the first equation for the temporal component becomes

$$\delta A_{kt} + \frac{i\partial_t \left(\mathbf{k} \cdot \delta \mathbf{A}_k\right)}{k^2 + (aM)^2} = 0, \tag{3.109}$$

where $k^2 \equiv \mathbf{k} \cdot \mathbf{k}$ and the subscript 'k' in $\delta A_{k\mu}$ denotes the Fourier mode of the vector field perturbation. Using the Fourier transform of Eq. (3.108) and plugging it in Eq. (3.109) we find

$$\delta \ddot{\mathbf{A}}_k + H\,\delta \dot{\mathbf{A}}_k + M^2 \delta \mathbf{A}_k + \left(\frac{k}{a}\right)^2 \delta \mathbf{A}_k + 2H \frac{\mathbf{k}\partial_t \left(\mathbf{k} \cdot \delta \mathbf{A}_k\right)}{k^2 + (aM)^2} = 0. \tag{3.110}$$

The massive vector field has three degrees of freedom and all three of them have to be quantized. Similarly to Eq. (3.11) we decompose $\delta \mathbf{A}_k$ into three polarizations

$$\delta A_{ki} = \sum_\lambda e_i^\lambda \left(\hat{\mathbf{k}}\right) \delta A_\lambda, \tag{3.111}$$

and choose e_i^λ to denote three vectors of the circular polarization in Eq. (3.12). Two transverse ones are perpendicular to the wave-vector $\mathbf{k}$ giving $e_i^+ k_i = 0$, where '+' stands for the left-handed 'L' or right-handed 'R' polarizations. Substituting this into Eq. (3.110) we find

$$\left[\partial_t^2 + H\partial_t + M^2 + \left(\frac{k}{a}\right)^2\right] \delta A_+ = 0. \tag{3.112}$$

For the longitudinal polarization $e_i^{||} k_i = k$, and taking into account that $e^{||} = \hat{\mathbf{k}} = \mathbf{k}/k$ from Eq. (3.110) we find

$$\left[\partial_t^2 + \left(1 + \frac{2k^2}{k^2 + (aM)^2}\right) H\partial_t + M^2 + \left(\frac{k}{a}\right)^2\right] \delta A_{||} = 0. \tag{3.113}$$

In the following sections we quantize the transverse and longitudinal degrees of freedom separately.

3.4.2. Transverse Modes

Let us rewrite the equation of motion of the transverse polarizations in Eq. (3.112) in terms of the physical vector field W_μ and conformal time $\tau \equiv \int \mathrm{d}t/a$:

$$\left[\partial_\tau^2 + 2\frac{a'}{a}\partial_\tau + (aM)^2 + k^2 + \frac{a''}{a}\right] w_+ = 0, \tag{3.114}$$

where prime denotes the derivative with respect to the conformal time τ and $w_+(\tau,k) = \delta A_+(\tau,k)/a(\tau)$ is defined in Eq. (3.11).

To find initial conditions for this field let us make a transformation in Eq. (2.81)

$$\varphi_+ \equiv w_+ \mathrm{e}^{\frac{1}{2}\int^\tau \frac{a'}{a}\mathrm{d}\tau} = a w_+ \tag{3.115}$$

and bring the equation of motion into the form of the harmonic oscillator

$$\left[\partial_\tau^2 + (aM)^2 + k^2\right]\varphi_+ = 0. \tag{3.116}$$

In the subhorizon limit, for the modes with $k^2 \gg |aM|^2$, Eq. (3.116) reduces to the flat space-time harmonic oscillator. In other words, if we write the action for functions $\varphi_+(k)$ in the limit $k/aH \to \infty$ it would correspond to the collection of harmonic oscillators. Choosing the initial state for φ_+ to correspond to a vacuum (no particles or minimum energy), from Eq. (2.97) it becomes $\varphi_+(k) = \exp(ik/aH)/\sqrt{2k}$, or going back to the physical field

$$\lim_{\frac{k}{aH}\to+\infty} w_+ = \frac{a^{-1}}{\sqrt{2k}}\mathrm{e}^{ik/aH}. \tag{3.117}$$

We are interested in the power spectrum of classical perturbations of the vector field on the superhorizon scales when $k \ll aH$, which from Eq. (3.16) can be calculated as

$$\mathcal{P}_\lambda = \frac{k^3}{2\pi^2}\lim_{\frac{k}{aH}\to 0}|w_\lambda|^2. \tag{3.118}$$

To find the power spectrum we need to solve the equation of motion in Eq. (3.114) in the limit $k/aH \to 0$ with the vacuum initial conditions in Eq. (3.117). For this purpose it is convenient to rewrite this equation in the form

$$\left[\partial_x^2 - \frac{2}{x}\partial_x + 1 + \frac{2+(M/H)^2}{x^2}\right] w_+ = 0, \tag{3.119}$$

where

$$x \equiv \frac{k}{aH}. \tag{3.120}$$

This is a Bessel equation with a general solution of the form

$$w_+ = x^{3/2}\left[C_1\mathcal{H}_\nu^1(x) + C_2\mathcal{H}_\nu^2(x)\right], \tag{3.121}$$

where C_1, C_2 are constants of integration and $\mathcal{H}_\nu^1$, $\mathcal{H}_\nu^2$ are the Hankel functions of the first and second kind respectively with

$$\nu \equiv \sqrt{\frac{1}{4} - \left(\frac{M}{H}\right)^2} = \sqrt{\frac{1}{4} + 12\alpha - \left(\frac{m}{H}\right)^2} \tag{3.122}$$

Taking the limit $x \to \infty$ and matching to the vacuum value in Eq. (3.117) we get

$$w_+ = a^{-1}\sqrt{\frac{\pi}{2k}}\,\mathrm{e}^{i(2\nu+1)\pi/4}\left(\frac{x}{2}\right)^{\frac{1}{2}}\mathcal{H}^1_\nu\left(x\right). \tag{3.123}$$

For the superhorizon perturbations ($x \to 0$) this solution becomes

$$w_+ = a^{-1}\frac{\Gamma\left(\nu\right)}{\sqrt{2\pi k}}\,\mathrm{e}^{i(2\nu-1)\pi/4}\left(\frac{x}{2}\right)^{\frac{1}{2}-\nu}, \tag{3.124}$$

where $\Gamma\left(\nu\right)$ is the Gamma function. Plugging this result into Eq. (3.118) we find the power spectrum for the transverse polarization to be equal to

$$\mathcal{P}_+ = \frac{4}{\pi}\Gamma^2\left(\nu\right)\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{2aH}\right)^{3-2\nu}. \tag{3.125}$$

Note that for the light vector field, when $M \to 0$, the power spectrum becomes

$$\mathcal{P}_+^{\mathrm{vac}} = \left(\frac{k}{2\pi a}\right)^2. \tag{3.126}$$

Comparing with Eq. (3.117) we see that it is simply the vacuum value. This is in accord with the expectation that the massless vector field is conformally invariant and does not undergo particle production.

The scale dependence of the power spectrum can be parametrized in the usual way as $\mathcal{P}_+ \propto k^{n_\mathrm{v}-1}$, so that $n_\mathrm{v} = 1$ corresponds to a flat spectrum. Comparing this with Eq. (3.125) we find that the spectral index is $n_\mathrm{v} - 1 = 3 - 2\nu$ and the scale invariant spectrum of the vector field perturbation is achieved if

$$n_\mathrm{v} = 1 \quad \Rightarrow \quad \nu = \frac{3}{2} \quad \Rightarrow \quad M^2 = -2H^2, \tag{3.127}$$

which agrees with the findings of Ref. [83]. With this condition the power spectrum becomes

$$\mathcal{P}_+ = \left(\frac{H}{2\pi}\right)^2, \tag{3.128}$$

the same as for the massless scalar field.

The condition in Eq. (3.127) is satisfied if the coupling constant of the vector field to gravity is

$$\alpha \approx \frac{1}{6}\left[1 + \frac{1}{2}\left(\frac{m}{H}\right)^2\right], \tag{3.129}$$

from which it is clear that for scale invariance we need $\alpha \gtrsim 1/6$. If $m \gtrsim H$ then scale invariance is attained only when α is tuned according to Eq. (3.129). However, if $m \ll H$ then scale-invariance simply requires $\alpha \approx 1/6$. In the latter case m and H do not have to balance each other through the condition in Eq. (3.129) and can be treated as free parameters. We feel that this is a more natural setup.

With $\alpha = 1/6$ the ν parameter in Eq. (3.122) becomes

$$\nu = \sqrt{\frac{9}{4} - \left(\frac{m}{H}\right)^2}, \tag{3.130}$$

which is reminiscent of the scalar field case, where perturbations become classical in the superhorizon limit if ν is real, corresponding to $m^2 < 9H^2/4$ (see the discussion below Eq. (2.113)).

3.4.3. The Longitudinal Mode

Let us first rewrite the equation for longitudinal perturbations of the vector field in Eq. (3.113) in terms of the conformal time τ

$$\left[\partial_\tau^2 + \frac{2k^2 aH}{k^2 + a^2M^2}\partial_\tau + \left(k^2 + a^2M^2\right)\right]\delta A_{||} = 0. \tag{3.131}$$

In the previous discussion on the perturbations of the transverse components, we found that the scale invariant spectrum is achieved if the effective mass squared of the field is negative and equal to $M^2 = -2H^2$. But in this case the second term in the above equation becomes singular when $(k/a)^2 = 2H^2$. This might indicate that the longitudinal vector field perturbation becomes unstable when approaching the horizon exit. But the two independent solutions of this equation

$$\delta A_{||}^{\pm} \propto \left(-k\tau + \frac{2}{k\tau} \pm 2i\right)\mathrm{e}^{\mp ik\tau} \tag{3.132}$$

show that this is not the case.[3]

$M^2 = -2H^2 =$ constant corresponds to the flat perturbation power spectrum of transverse modes and, as will be seen later, of the longitudinal mode too. However, the

[3] To find this solution we have used the relation $\tau = -\left(aH\right)^{-1}$, which is valid in de Sitter space-time.

exactly flat spectrum is excluded by observations (Eq. (2.16)). Therefore, one would expect that in the realistic theory the condition $M^2 = -2H^2$ is violated by a small amount to give the correct spectral tilt and the Hubble parameter is not exactly constant during inflation. In this case the solution in Eq. (3.132) is not valid and one may be worried that for general effective negative mass $M^2 < 0$ and $\dot{H} \neq 0$ the solution of Eq. (3.131) is still singular at $(k/a)^2 \to |M^2|$. We can prove that this is not the case using the Frobenius method for differential equations with regular singular points (see for example Ref. [100]).

Using Eqs. (3.100) and (3.101) with $\dot{M} \neq 0$ the equation of motion for the longitudinal mode $\delta A_{||}$ becomes

$$\left[\partial_\tau^2 - (3w+1)\frac{k^2 aH}{k^2 + a^2M^2}\partial_\tau + \left(k^2 + a^2M^2\right)\right]\delta A_{||} = 0, \tag{3.133}$$

where w is the barotropic parameter of the dominant component of the Universe which drives inflation. For de Sitter expansion $w = -1$ and we recover Eq. (3.131). However, for this calculation we do not assume de Sitter inflation and consider a constant w in the range $-1 < w < -\frac{1}{3}$, which is necessary for the accelerated expansion of the Universe (see the discussion in section 1.5.2). This equation is valid for a general non-minimal coupling constant α defined in Eq. (3.92) and we used that $\alpha R \gg m^2$ during inflation.

Let us first we make a change of variables

$$y \equiv \left(\frac{k}{a\,|M|}\right)^2 - 1, \tag{3.134}$$

with y varying in the region $-1 < y < \infty$. Eq. (3.131) with this transformation translates into the form

$$\left[\partial_y^2 - \frac{1}{2}\frac{(y+2)}{y\,(y+1)}\partial_y + \frac{|M^2|}{H^2}\frac{y}{(3w+1)^2\,(y+1)^2}\right]\delta A_{||} = 0, \tag{3.135}$$

with $M^2 < 0$ and the regular singular point at $y \to 0$, corresponding to $(k/a)^2 \to |M^2|$. The general solution of this equation can be found using the ansatz

$$\delta A_{||} = \sum_{n=0}^{\infty} D_n y^{s+n}, \tag{3.136}$$

where $D_0 \neq 0$. In this case the series in Eq. (3.136) is convergent at least in the region $-1 < y < 1$ (corresponding to $|M^2| < (k/a)^2 < 2\,|M^2|$) without a singular point at

$y = 0$. Our aim is to prove that the solution in Eq. (3.136) is not singular even at $(k/a)^2 = |M^2|$, i.e. $y = 0$. This will be the case if the power series ansatz in Eq. (3.136) has two independent solutions and if $s + n > 0$ for all n, i.e. there are no negative powers of y in the series. To show this let us substitute Eq. (3.136) into Eq. (3.135) giving

$$\sum_{n=0}^{\infty} D_n \left[4(s+n)(s+n-2)\, y^{s+n-2} + 8(s+n)\left(s+n-\frac{7}{4}\right) y^{s+n-1} + \right.$$
$$\left. +4(s+n)\left(s+n-\frac{3}{2}\right) y^{s+n} + 12\alpha \frac{|3w-1|}{(3w+1)^2} y^{s+n+1} \right] = 0, \quad (3.137)$$

where we also used Eq. (3.94). In order for this equality to be valid, coefficients in front of each y with the same power must vanish. The coefficient in front of the term with the smallest power, i.e. y^{s-2}, is $4D_0 s(s-2)$. Because $D_0 \neq 0$, from the indicial equation $s(s-2) = 0$ we find

$$s = 0 \quad \text{or} \quad s = 2. \quad (3.138)$$

Because these two solutions differ by an integer, it might be alarming that both series in Eq. (3.136) with $s = 0$ and $s = 2$ do not provide two independent solutions. In this case the second independent solution would involve the term $\ln y$, which indeed diverges at $y \to 0$. However, by closer inspection of Eq. (3.137) we find that the coefficient D_2 of the series with $s = 0$ is arbitrary, thus the power series in Eq. (3.136) with $s = 0$ and $s = 2$ do give two independent solutions. In addition they do not involve negative powers of y, i.e. $s \geq 0$, therefore, the solution with the ansatz in Eq. (3.136) converges at the singular point $y = 0$. This proves that during inflation, when $M^2 < 0$, the solution of Eq. (3.133) is stable when the wavelength of the perturbation approaches $(k/a)^2 \to |M^2|$.

Let us turn now to the quantization of the longitudinal mode in (quasi) de Sitter space-time. From Eq. (3.113) the equation of motion for the longitudinal physical field in the conformal time is

$$\left[\partial_\tau^2 + 2\frac{a'}{a}\left(1 + \frac{k^2}{k^2 + (aM)^2}\right)\partial_\tau + k^2 + (aM)^2 + 2\left(\frac{a'}{a}\right)^2 \frac{k^2}{k^2 + (aM)^2} + \frac{a''}{a} \right] w_{||} = 0. \quad (3.139)$$

To quantize the longitudinal mode, let us write the Lagrangian corresponding to the equation of motion in Eq. (3.131)

$$\mathcal{L} = M^2 \left[\frac{\left|\delta A'_{||}(\tau, \mathbf{k})\right|^2}{(k/a)^2 + M^2} - a^2 \left|\delta A_{||}(\tau, \mathbf{k})\right|^2 \right]. \quad (3.140)$$

This Lagrangian can also be achieved by perturbing the full Lagrangian in Eq. (3.90) and it is unique up to the total derivative. To set the initial conditions for the subhorizon modes we use the transformation

$$\varphi_{\|} \equiv \gamma^{-1}\delta A_{\|}, \tag{3.141}$$

where γ is the Lorentz boost factor

$$\gamma = \frac{E}{|M|} = \frac{\sqrt{(k/a)^2 + |M^2|}}{|M|} \approx \frac{k/a}{|M|}, \tag{3.142}$$

and the last equality is taken in the limit $k/a \gg |M|$. With this transformation the Lagrangian for subhorizon modes reduces to that of the simple harmonic oscillator

$$\mathcal{L} = \pm\left(\left|\varphi_{\|}'\right|^2 - k^2\left|\varphi_{\|}\right|^2\right), \tag{3.143}$$

where the sign $\pm$ is that of M^2, hence negative for the case of interest $M^2 \simeq -2H^2$.

The Lagrangian in Eq. (3.143) is the same as of the harmonic oscillator. Choosing initial conditions to correspond to the vacuum state, we have

$$\varphi_{\|} = \frac{1}{\sqrt{2k}}\mathrm{e}^{-ik\tau}. \tag{3.144}$$

This is similar as for the scalar field case, except that for $M^2 < 0$ the Lagrangian has a negative sign. Because of the wrong sign, initial conditions in Eq. (3.144) are not identical to the scalar field case, since for the longitudinal mode occupied initial states would have negative energy density and pressure. As the pressure is negative it is not dangerous for inflation. Instead, it is the negative energy density that is dangerous. As the total energy density is required to be positive, the negative contribution of occupied states has to be less than the total at the beginning of inflation. This is satisfied by assuming that initially the occupation number is much less than 1 (as in the scalar field case), justifying both the choice of initial mode function and the assumption of the vacuum state.

Matching the solution in Eq. (3.132) to the vacuum initial conditions from Eq. (3.144), $\delta A_{\|,\mathrm{vac}} = aw_{\|} = \frac{\gamma}{\sqrt{2k}}\exp\left(-ik\tau\right)$, and using Eq. (3.118) we find that the power spectrum for the longitudinal mode with $\alpha = 1/6$ is

$$\mathcal{P}_{\|} = 2\left(\frac{H}{2\pi}\right)^2 = 2\mathcal{P}_{+}. \tag{3.145}$$

This corresponds to $p = 1$ in Eq. (3.19), meaning that the particle production of the vector field is anisotropic.

3.4.4. The Stability of the Longitudinal Mode

As shown above, the possible instability of the longitudinal mode when $(k/a)^2 \to |M^2|$ is absent. However, in Ref. [101] it has been noted that other instabilities might be present for non-minimally coupled vector field. The first concern is that for $m^2 \ll |R|$ the kinetic term of the longitudinal mode is negative on the subhorizon scales (see Eqs. (3.92) and (3.140)). As a result one might suspect that corresponding particles carry a negative energy and they can be created from the vacuum making it unstable. This is indeed the case for minimally coupled scalar field with negative kinetic term. The latter field is called a ghost and is cosmologically unacceptable as it would create too many photons from the present day vacuum (Ref. [102]). However, the flat space-time calculation of Ref. [102] can not be directly applied to a vector field with non-minimal coupling to gravity and currently no such calculation exits. Moreover, the bound on the photon creation from the vacuum at the present Universe is irrelevant for the vector curvaton scenario as its bare mass squared m^2 in Eq. (3.92) dominates, making M^2 positive. But even for models with negligible m^2 one wouldn't expect a large particle creation, as in the present day Universe $|R| \sim 10^{-66}\,\mathrm{eV}^2$, which is extremely small compared to other energy scales.

Another concern is about the singularity when $M^2 \to 0$. After inflation $R \propto t^{-2}$, and when both terms in Eq. (3.92) cancel each other out M^2 vanishes. As shown in Ref. [101] this results in a singularity which invalidates linear calculations around this point. To evaluate the effects of this, one needs to perform a full non-linear calculation which has not been done to the present moment. However, as the period of non-linear evolution is very brief, one might expect that linear calculations before and after $M^2 = 0$ will match. Or one can assume that $m^2 = 0$ and consider more complicated models to generate mass term for the vector field, in which case the aforementioned singularity can be avoided.[4]

3.4.5. Statistical Anisotropy and Non-Gaussianity

Let us calculate the statistical anisotropy and non-Gaussianity for this model. Because $p = 1$, the dominant contribution to the curvature perturbation is assumed to be generated by the scalar field. The parity conserving transverse power spectrum of the vector field perturbation in Eq. (3.128) and the power spectrum generated during the single

[4] A more thorough discussion of these issues can be found in Ref. [103].

scalar field inflation are equal, i.e. $\mathcal{P}_+ = \mathcal{P}_\phi$. Thus the isotropic part of the curvature perturbation spectrum can be written as

$$\mathcal{P}_\zeta^{\rm iso} = \mathcal{P}_\phi N_\phi^2 \left(1+\xi\right). \tag{3.146}$$

While the anisotropy parameter from Eq. (3.32) becomes

$$g = \frac{\xi}{1+\xi}. \tag{3.147}$$

This model does not have parity violating terms, and from Eqs. (3.19) and (3.145) we find

$$p = 1 \quad \text{and} \quad q = 0. \tag{3.148}$$

Thus, the anisotropy in the vector field is rather strong, which means that it will have to remain subdominant, i.e. $\Omega_W \ll 1$. Using this and Eq. (3.69), the $f_{\rm NL}^{\rm equil}$ for the non-minimally coupled vector curvaton is found to be

$$\frac{6}{5} f_{\rm NL}^{\rm equil} = 2\frac{\xi^2}{\Omega_W}\left(1+\frac{9}{8}W_\perp^2\right), \tag{3.149}$$

Similarly, $f_{\rm NL}^{\rm local}$ for the squeezed configuration in Eq. (3.74) is

$$\frac{6}{5} f_{\rm NL}^{\rm local} = 2\frac{\xi^2}{\Omega_W}\left(1+W_\perp^2\right) \tag{3.150}$$

Since $\mathcal{P}_+ = \frac{1}{2}\mathcal{P}_{||} = \mathcal{P}_\phi = (H/2\pi)^2$, for the typical values of the perturbation we have $\delta\phi \sim \delta W_i \sim H$. This means that, in order for the vector field contribution to be subdominant, we require $N_W \ll N_\phi$ (c.f. Eq. (3.7)), which from Eq. (3.31) gives $\xi \ll 1$. Using these results and $p = 1$ (see Eq. (3.148)) from Eq. (3.32) we find $g \simeq \xi$. Thus, in view of Eqs. (3.149) and (3.150), we see that $f_{\rm NL} \sim g^2/\Omega_W$. Therefore, we find that the non-Gaussianity is determined by the magnitude of the anisotropy in the power spectrum.

This prediction is valid in the regime $|\delta W/W| \ll 1$ which corresponds to $\Omega_W^2 \gtrsim \mathcal{P}_\zeta \xi$, which implies $f_{\rm NL} \lesssim g^{3/2}/\sqrt{\mathcal{P}_\zeta}$. For smaller Ω_W, the contribution of the vector field perturbation to ζ is of order $\Omega_W[\delta W/(\overline{\delta W^2})^{1/2}]$. In other words, it is of order Ω_W and is the square of a Gaussian quantity. The resulting prediction for its contribution to $f_{\rm NL}$ would be given by the one-loop formula which is calculated in Ref. [97].

3.4.6. The Energy-Momentum Tensor

Let us now study the evolution of the vector field. For the scale invariant perturbation spectrum with $\alpha = 1/6$ from Eq. (3.104) we find the equation of motion for the homogeneous mode of the vector field

$$\ddot{W} + 3H\dot{W} + m^2 W = 0, \tag{3.151}$$

which is identical to the one of a massive scalar field. When $m \ll H$ it has the solution

$$W = W_0 + C a^{\frac{3}{2}(w-1)}, \tag{3.152}$$

where W_0 and C are constants of integration. Because the second term in Eq. (3.152) is decaying, as long as $m \ll H$ the physical vector field develops a condensate which remains constant $W \simeq W_0$.

We can follow the evolution of the vector field condensate by considering the energy momentum tensor, which can be written in the form

$$T_\mu^\nu = \mathrm{diag}\left(\rho_W, -p_\perp, -p_\perp, -p_{||}\right), \tag{3.153}$$

where [88]

$$\rho_W = \frac{1}{2}\dot{W}^2 + \frac{1}{2}m^2 W^2 \tag{3.154}$$

and the transverse and longitudinal pressures are

$$\begin{aligned} p_\perp &= \frac{5}{6}\left(\dot{W}^2 - m^2 W^2\right) + \frac{1}{3}\left(2H\dot{W} + \dot{H}W + 3H^2 W\right) W, \\ p_{||} &= -\frac{1}{6}\left(\dot{W}^2 - m^2 W^2\right) - \frac{2}{3}\left(2H\dot{W} + \dot{H}W + 3H^2 W\right) W. \end{aligned} \tag{3.155}$$

Thus, the energy-momentum tensor for the homogeneous vector field is, in general, anisotropic because $p_{||} \neq p_\perp$. This is why the vector field cannot be taken to drive inflation, for if it did it would generate a substantial large-scale anisotropy, which would be in conflict with the predominant isotropy in the CMB. Therefore, we have to investigate whether, *after* inflation, there is a period in which the vector field becomes isotropic (i.e. $p_\perp \approx p_{||}$) and can imprint its perturbation spectrum onto the Universe without such problems.

Considering the growing mode in Eq. (3.152) and Eqs. (3.154), (3.155) we see that, during and after inflation, when $m \ll H$, we have

$$\rho_W \simeq \frac{1}{2} m^2 W_0^2 \quad \text{and} \quad p_\perp \simeq -\frac{1}{2} p_\| \simeq \frac{1}{2} (1-w) H^2 W_0^2. \tag{3.156}$$

Hence, the density of the vector field remains roughly constant, while the vector field condensate remains anisotropic after inflation.

The above are valid under the condition $m \ll H$. However, after the end of inflation $H(t) \propto t^{-1}$, so there will be a moment when $m \sim H$. After this moment, due to Eq. (3.94), the curvature coupling becomes negligible and the vector field behaves as a massive minimally-coupled Abelian vector field. As shown in Eq. (3.52), when $m \gtrsim H$ a massive vector field undergoes (quasi)harmonic oscillations of frequency $\sim m$, because the friction term in Eq. (3.151) becomes negligible. In this case, on average over many oscillations $\overline{\dot{W}^2} \approx m^2 \overline{W^2}$. Hence, Eqs. (3.154) and Eq. (3.155) become

$$\begin{aligned} \rho_W &\simeq m^2 \overline{W^2}, \\ p_\perp &\simeq -\frac{1}{2} p_\| \simeq \frac{2}{3} m H \left[1 + \frac{3}{4}(1-w)\left(\frac{H}{m}\right)\right] \overline{W^2}. \end{aligned} \tag{3.157}$$

The effective barotropic parameters of the vector field are

$$0 < w_\perp \simeq -\frac{1}{2} w_\| = \frac{2}{3}\left[1 + \frac{3}{4}(1-w)\left(\frac{H}{m}\right)\right]\left(\frac{H}{m}\right) \ll 1, \tag{3.158}$$

where $w_\perp = p_\perp/\rho_W$ and $w_\| = p_\|/\rho_W$. By virtue of the condition $m \gg H$, we see that, after the onset of the oscillations, $w_\perp, w_\| \to 0$. This means that the oscillating massive vector field behaves as a pressureless isotropic matter, which can dominate the Universe without generating an excessive large-scale anisotropy. Moreover, as was shown in Eq. (3.60) the energy density decreases as $\rho_W \propto a^{-3}$, i.e. like dust. Thus, if the Universe is radiation dominated, $\rho_W/\rho \propto a$ while oscillations occur, so the field has a chance to dominate the Universe and imprint its curvature perturbation according to the curvaton scenario.

3.4.7. Curvaton Physics

As we have seen in section 3.4.5 for the non-minimally coupled vector field with the Lagrangian in Eq. (3.90) the particle production is anisotropic, and the curvature perturbation generated by such a field is statistically anisotropic. Therefore, the non-minimally coupled vector curvaton may generate only the subdominant contribution to the total curvature perturbation ζ, while the dominant part must be produced by a statistically isotropic source. In such scenario the total curvature perturbation with statistically anisotropic contribution ζ_W was calculated in Eq. (3.86).

If $m \ll H$ during inflation the physical vector field (being non-conformally invariant) undergoes particle production and obtains an approximately flat superhorizon spectrum of perturbations, as shown. Indeed, if in Eq. (3.130) $\nu \approx \frac{3}{2}$ from Eq. (3.145) we find that the typical value of the vector field perturbation is (see Eq. (2.10))

$$\delta W_{\rm end} \approx \sqrt{\mathcal{P}_{||} + 2\mathcal{P}_{+}} \approx \frac{H_*}{\pi}, \tag{3.159}$$

where 'end' denotes the typical value of the vector field perturbation at the end of inflation and H_* is the Hubble parameter during inflation. The curvature perturbation generated by the vector curvaton field was calculated in Eq. (3.80). Using this equation and considering that $W \approx$ constant during inflation (see Eq. (3.152)), i.e. $W_{\rm end} = W_0$, we may write

$$\zeta_W \sim \frac{H_*}{W_0}. \tag{3.160}$$

Thus, from this result and Eq. (3.86) we obtain

$$\zeta \sim g^{-1/2}\Omega_W \frac{H_*}{W_0}. \tag{3.161}$$

At the onset of vector field oscillations the density parameter of the vector field is

$$\Omega_W \equiv \frac{\rho_W}{\rho} \sim \left(\frac{W_0}{m_{\rm Pl}}\right)^2, \tag{3.162}$$

where we have used the flat Friedman equation (1.16) $\rho = 3m_{\rm Pl}^2 H^2$. To avoid excessive large scale anisotropy the density of the vector field must be subdominant before the onset of oscillations, which means that $W_0 < m_{\rm Pl}$.

Let us assume that inflation is driven by some inflaton field, which after inflation ends, oscillates around its VEV until its decay into a thermal bath of relativistic particles at reheating. In this scenario the Universe is matter dominated (by inflaton particles) until reheating. Using the above findings we can estimate the density ratio of the vector field at decay

$$\Omega_{\rm dec} \sim \left(\frac{\min\{m, \Gamma\}}{\Gamma_W}\right)^{1/2} \left(\frac{W_0}{m_{\rm Pl}}\right)^2, \tag{3.163}$$

where Γ_W is the vector field decay rate and Γ is the decay rate of the inflaton field. If inflation gives away directly to a thermal bath of particles then we have prompt reheating and $\Gamma \to H_*$, where H_* is the Hubble scale of inflation.

Using Eqs. (3.161) and (3.163) and considering that at the vector field decay $\Omega_{\rm dec} = \Omega_W$ we get

$$\frac{H_*}{m_{\rm Pl}} \sim \zeta \left(\frac{g}{\Omega_{\rm dec}}\right)^{1/2} \left(\frac{\Gamma_W}{\min\{m,\Gamma\}}\right)^{1/4}, \tag{3.164}$$

The Hot Big Bang has to begin before nucleosynthesis (which occurs at the temperature $T_{\rm BBN} \sim 1\,{\rm MeV}$). Hence, $\Gamma_W \gtrsim T_{\rm BBN}^2/m_{\rm Pl}$. Using this and also $\min\{m,\Gamma\} \lesssim H_*$, we obtain the bound

$$H_* \gtrsim \zeta^{4/5} \left(\frac{g}{\Omega_{\rm dec}}\right)^{2/5} \left(T_{\rm BBN}^2 m_{\rm Pl}^3\right)^{1/5} \Longleftrightarrow V_*^{1/4} \gtrsim g^{1/5}\, 10^{12}\,{\rm GeV}, \tag{3.165}$$

where we used that $\Omega_{\rm dec} \lesssim 1$ and $\zeta = 4.8 \times 10^{-5}$ from COBE observations. For $g \lesssim 0.3$ this is similar to the case of a scalar field curvaton [104].[5]

Another bound on the inflation scale is obtained by considering that $\Gamma_W \sim h^2 m$, where h is the vector field coupling to its decay products, for which $h \gtrsim m/m_{\rm Pl}$ due to gravitational decay. Thus, $\Gamma_W \gtrsim m^3/m_{\rm Pl}^2$. Combining with Eq. (3.164) we obtain the bound

$$H_* \gtrsim \zeta \left(\frac{g}{\Omega_{\rm dec}}\right)^{1/2} (m_{\rm Pl} m)^{1/2} \Longleftrightarrow V_*^{1/4} \gtrsim g^{1/4}\, 10^{11}\,{\rm GeV}, \tag{3.166}$$

where we took $1\,{\rm TeV} \lesssim m < \Gamma$.

Finally, an upper bound on inflation scale can be obtained by combining Eq. (3.161) with the requirement $W_0 < m_{\rm Pl}$, thereby finding

$$H_* < g^{1/2} \zeta \Omega_{\rm dec}^{-1} m_{\rm Pl} \Longleftrightarrow V_*^{1/4} < g^{1/4}\, 10^{16}\,{\rm GeV}, \tag{3.167}$$

where we considered that $\Omega_{\rm dec} \gtrsim 10^{-3}$, in order to avoid excessive non-Gaussianity in the CMB. This bound on $\Omega_{\rm dec}$ may be found considering that $f_{\rm NL} \sim g^2/\Omega_{\rm dec}$ (see Eq. (3.77)) and the observational constraints on $|f_{\rm NL}| \lesssim 100$ (Eq. (2.22)).

As was discussed in section 3.3.3 we also need to consider the hazardous possibility of the thermal evaporation of the vector field condensate. If it evaporates all memory of the superhorizon perturbation spectrum is erased and no ζ_W is generated. This puts a bound on the allowed values of the vector field coupling constant h to its decay products which was calculated in Eq. (3.89).

The above lower bounds on H_* can be substantially relaxed by employing the so-called mass increment mechanism according to which, the vector field obtains its bare mass at a phase transition (denoted by 'pt') with $m/H_{\rm pt} \gg 1$. The mechanism was

[5]The cosmological scales re-enter the horizon at temperatures $T \lesssim 1\,{\rm keV}$, i.e. much later than nucleosynthesis and well after our vector field condensate decays restoring local Lorentz invariance.

firstly introduced for the scalar curvaton in Ref. [105] and has been already implemented in the vector curvaton case in Ref. [84].

Let us consider now the case when $\alpha \neq \frac{1}{6}$. If $\alpha = \mathcal{O}(1)$ then, according to Eq. (3.129), a scale invariant spectrum is possible only if $m \sim H_*$. Hence, the oscillations begin immediately after the end of inflation. With this in mind the previous analysis remains valid. In particular, the bound in Eq. (3.165) remains the same. However, the bound in Eq. (3.166) becomes much more stringent:

$$H_* \gtrsim g\, 10^{10}\,\mathrm{GeV} \Longleftrightarrow V_*^{1/4} \gtrsim g^{1/2}\, 10^{14}\,\mathrm{GeV}. \tag{3.168}$$

3.4.8. A Concrete Example

To illustrate our findings let us consider a specific example. Let us choose $\alpha \approx \frac{1}{6}$, $m \sim 10\,\mathrm{TeV}$ and also $\Gamma_W \sim 10^{-10}\,\mathrm{GeV}$ such that the temperature at the vector field decay is $T_{\mathrm{dec}} \sim 10\,\mathrm{TeV}$. Such a particle may be potentially observable in the LHC. These values suggest $h \sim 10^{-7}$, which lies comfortably within the range in Eq. (3.89). For the decay rate of the inflaton let us chose $\Gamma \sim 10^{-2}\,\mathrm{GeV}$ so that the reheating temperature satisfies the gravitino overproduction constraint $T_{\mathrm{reh}} \sim \sqrt{m_{\mathrm{Pl}}\Gamma} \sim 10^{8}\,\mathrm{GeV}$. Then Eq. (3.164) reduces to $H_*/m_{\mathrm{Pl}} \sim 10^{-4}\zeta\,(g/\Omega_{\mathrm{dec}})^{1/2}$. Using this and Eq. (3.161) we get $W_0/m_{\mathrm{Pl}} \sim 10^{-4}\sqrt{\Omega_{\mathrm{dec}}}$. Hence, with the maximum observationally allowed statistical anisotropy $g \sim 0.1$ the lowest value for the inflationary Hubble scale is $H_* > 10^{9}\,\mathrm{GeV}$.

3.4.9. Summary of the RA^2 Model

In Ref. [83] it was demonstrated for the first time that the vector field may influence or generate the curvature perturbation in the Universe. It was shown that a massive vector field may act as a curvaton field without producing excessive large scale anisotropy. In this reference it was also calculated that the perturbation spectrum of a massive Abelian vector field is scale invariant if the mass of the field is equal to $M^2 = -2H^2$. Section 3.4 of this thesis explored the possibility of realizing the negative mass squared by non-minimal coupling of the vector field to gravity through the term $\alpha R A_\mu A^\mu$, where R is the Ricci scalar and ε is the non-minimal coupling constant. We have calculated the vector field perturbation spectrum for the transverse and longitudinal degrees of freedom and found that they are scale invariant if $\alpha = 1/6$. However, the magnitude of the longitudinal power spectrum is twice the transverse ones, indicating that the particle production of the vector field is anisotropic. If such a vector field generated the total curvature perturbation in the Universe, the resulting magnitude of statistical anisotropy in ζ would violate observational bounds obtained from CMB measurements. Therefore, the vector curvaton considered in this section may generate only a subdominant contribution to ζ.

We have also explored the parameter space of the proposed scenario. In this thesis calculations of the constraints for the non-minimally coupled vector curvaton model, with the statistical anisotropy taken into account, were performed for the first time. We have shown that there is an ample parameter space for the model to work by considering all relevant constraints in the cosmology.

Some of recently raised concerns [101, 106] about the stability of the model were also addressed. It was shown that although the longitudinal mode is a ghost when it is subhorizon, but it may not be dangerous during inflation if we assume no-particle (vacuum) initial conditions (as in the scalar field case) and negligible coupling to other fields. It was also emphasized that the equation of motion of the longitudinal mode has a singular point at $(k/a)^2 = |M^2|$, which might indicate that the longitudinal mode becomes singular at horizon exit ($|M^2| \approx H^2$). We have obtained an exact solution for non-zero bare mass of the vector field, i.e. $m \neq 0$, and demonstrated that it is well behaved at all time during inflation. However, we have not addressed the instability of the longitudinal mode when the effective mass of the vector field becomes zero after inflation, i.e. when $M \to 0$.

3.5. Vector Curvaton with a Time Varying Kinetic Function

In this section we consider a vector curvaton scenario with the vector field Lagrangian during inflation

$$\mathcal{L} = -\frac{1}{4} f F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu, \tag{3.169}$$

where $f = f(t)$ is the kinetic function and $m = m(t)$ is the mass and both are functions of the cosmic time t. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor. If f is time-independent it can be set equal to 1 because any constant value can be absorbed into A_μ. Otherwise, f represents a time-dependent coupling.

The above Lagrangian density can be of a massive Abelian gauge field, in which case f is the gauge kinetic function. However, we need not restrict ourselves to gauge fields only. If no gauge symmetry is considered the argument in support of the above Maxwell type kinetic term is that it is one of the few (three) choices [107] which avoids introducing instabilities, such as ghosts [106].

3.5.1. Equations of Motion

We focus, at first, on a period of cosmic inflation, during which we assume that the contribution of the vector field to the energy budget of the Universe is negligible. Thus, we take the inflationary expansion to be isotropic. As in the previous model in section 3.4 we also assume that inflation is of (quasi)de Sitter type, i.e. the Hubble parameter is $H \approx$ constant.

Inflation is expected to homogenize the vector field. Following the analogous calculations as in section 3.4 and Ref. [84], we find that the temporal component of the homogeneous vector field has to be zero, while the spatial components satisfy the equation of motion

$$\ddot{\mathbf{A}} + \left(H + \frac{\dot{f}}{f} \right) \dot{\mathbf{A}} + \frac{m^2}{f} \mathbf{A} = 0, \tag{3.170}$$

where the dot denotes derivative with respect to t. From the above it is evident that the effective mass of the vector field is

$$M \equiv \frac{m}{\sqrt{f}}\,, \tag{3.171}$$

where we assumed that $m,\ f > 0$.

We perturb the vector field according to Eq. (3.106) and going to the Fourier space we calculate equations of motions for the transverse and longitudinal polarizations as

$$\left\{ \partial_t^2 + \left(H + \frac{\dot{f}}{f} \right) \partial_t + \frac{m^2}{f} + \left(\frac{k}{a} \right)^2 \right\} \delta A_+ \quad = \quad 0, \tag{3.172}$$

$$\left\{ \partial_t^2 + \left[H + \frac{\dot{f}}{f} + \left(2H + 2\frac{\dot{m}}{m} - \frac{\dot{f}}{f} \right) \frac{\left(\frac{k}{a}\right)^2}{\left(\frac{k}{a}\right)^2 + \frac{m^2}{f}} \right] \partial_t + \frac{m^2}{f} + \left(\frac{k}{a} \right)^2 \right\} \delta A_\parallel \quad = \quad 0, \tag{3.173}$$

where δA_+ and $\delta A_\parallel$ are defined in Eq. (3.111).

To continue we need to employ the physical (in contrast to comoving), canonically normalized vector field

$$\mathbf{W} = \sqrt{f} \frac{\mathbf{A}}{a}. \tag{3.174}$$

Note that the definition of $\mathbf{W}$ differs from the one in Eq. (3.6) because in this section $\mathbf{W}$ gets an additional factor of $\sqrt{f}$ due to canonical normalization.

Expressing Eqs. (3.172) and (3.173) in terms of the physical vector field we obtain

$$\left\{\partial_t^2 + 3H\partial_t + \frac{1}{2}\left[\frac{1}{2}\left(\frac{\dot{f}}{f}\right)^2 - \frac{\ddot{f}}{f} - \frac{\dot{f}}{f}H + 4H^2\right] + M^2 + \left(\frac{k}{a}\right)^2\right\} w_+ = 0 \tag{3.175}$$

and

$$\left\{\partial_t^2 + \left[3H + \left(2H + 2\frac{\dot{M}}{M}\right)\frac{(k/a)^2}{(k/a)^2 + M^2}\right]\partial_t + \frac{1}{2}\left[\frac{1}{2}\left(\frac{\dot{f}}{f}\right)^2 - \frac{\ddot{f}}{f} - \frac{\dot{f}}{f}H + 4H^2\right] + \right.$$
$$\left. + \left(H - \frac{1}{2}\frac{\dot{f}}{f}\right)\left(2H + 2\frac{\dot{M}}{M}\right)\frac{\left(\frac{k}{a}\right)^2}{\left(\frac{k}{a}\right)^2 + M^2} + M^2 + \left(\frac{k}{a}\right)^2\right\} w_\parallel = 0. \tag{3.176}$$

Because the theory is parity conserving the Fourier mode w_+ of $\delta\mathbf{W}(t,\mathbf{x})$ perturbations denotes both polarizations: the left-handed and right-handed, i.e. $w_+ = \sqrt{f}\delta A_+/a$.

Let us use the following ansatz for the time dependence of the kinetic function and the mass

$$f \propto a^\alpha \quad \text{and} \quad m \propto a^\beta, \tag{3.177}$$

where α and β are real constants. We will also assume that $f \to 1$ at the end of inflation so that, after inflation, the vector field is canonically normalized. Then Eqs. (3.175) and (3.176) become

$$\ddot{w}_+ + 3H\dot{w}_+ + \left[-\frac{1}{4}(\alpha+4)(\alpha-2)H^2 + M^2 + \left(\frac{k}{a}\right)^2\right] w_+ = 0 \tag{3.178}$$

and

$$\begin{aligned} \ddot{w}_\parallel + \left(3 + \frac{2-\alpha+2\beta}{1+r^2}\right)H\dot{w}_\parallel + & \\ + \left[\frac{1}{2}(2-\alpha)\left(\alpha + 4 + \frac{2-\alpha+2\beta}{1+r^2}\right)H^2 + \left(\frac{k}{a}\right)^2(1+r^2)\right] w_\parallel &= 0, \end{aligned} \tag{3.179}$$

where r is defined as

$$r \equiv \frac{M}{k/a}. \tag{3.180}$$

3.5.2. The Power Spectrum

To calculate the power spectrum one can proceed as in section 3.4: calculate general solutions of Eqs. (3.178) and (3.179), determine integration constants by matching the solution to the vacuum at the subhorizon limit, $k/a \gg H$, and calculating the field amplitude at the superhorizon regime when $k/a \ll H$. However, it is difficult to find general solutions for these equations, therefore one needs to use approximate methods.

In Appendix B it is shown that, in analogy to the equation of motion of a scalar field during quasi de Sitter inflation and with initial conditions in Eq. (3.117), the scale invariant perturbation spectrum for transverse polarizations in Eq. (3.178) is achieved if

$$\alpha = -1 \pm 3 \tag{3.181}$$

(i.e. either $f \propto a^2$ or $f \propto a^{-4}$) and

$$M_* \ll H, \tag{3.182}$$

where the star denotes the time when cosmological scales exit the horizon. The latter condition simply requires that the physical vector field W_μ is effectively massless at that time.[6]

For the longitudinal polarization the initial condition reads

$$\lim_{\frac{k}{aH}\to+\infty} w_\parallel = \gamma \frac{a^{-1}}{\sqrt{2k}} \mathrm{e}^{ik/aH}, \tag{3.183}$$

where the Lorentz boost factor is

$$\gamma = \frac{E}{M} = \frac{\sqrt{\left(\frac{k}{a}\right)^2 + M^2}}{M} = \sqrt{1 + \frac{1}{r^2}}. \tag{3.184}$$

In the subhorizon limit $r \ll 1$. After finding the solution of Eq. (3.179) with this condition and matching it to the vacuum solution in Eq. (3.183), one can calculate the power spectrum of $w_\parallel$ in the superhorizon limit. In Ref. [108] it was shown that the spectrum is scale invariant if

$$\beta = -\frac{1}{2}\left(3 \pm 5\right). \tag{3.185}$$

As explained in the Appendix B the value $\beta = -4$ must be disregarded because it implies

[6]Note that this is not the same as having A_μ being effectively massless. In the latter case the vector field is approximately conformally invariant and does not undergo particle production. However, the conformal invariance of the massless physical vector field W_μ is broken.

the massive physical vector field in the subhorizon limit. This contradicts the requirement for the scale invariant perturbation spectrum of the transverse modes.

Having evaluated α and β in Eqs. (3.181) and (3.185) to give the scale invariant perturbation spectrum of the transverse and longitudinal modes one can analyze equations of motion in Eqs. (3.178) and (3.179) in more detail. In Ref. [108] these equations were solved in different approximation regimes as well as solved numerically. Below we provide the summary of results.

Case: $f \propto a^{-4}$ & $m \propto a$

For $\alpha = -4$ and $\beta = 1$, the equation of motion for the transverse mode functions in Eq. (3.178) become

$$\ddot{w}_+ + 3H\dot{w}_+ + \left(\frac{k}{a}\right)^2 (1+r^2)w_+ = 0. \tag{3.186}$$

When the kinetic function of the vector field scales as $f \propto a^{-4}$, from Eqs. (3.180) and (3.171) we find that $r \propto a^4$. In addition we assume that cosmological scales exit the horizon when the vector field is light. Therefore, for subhorizon perturbations when $x \gtrsim 1$, where x was defined in Eq. (3.120) as $x \equiv k/(aH)$, the r parameter is very small, i.e. $r \ll 1$. When the mode leaves the horizon $x \lesssim 1$ and for cosmological scales $r < 1$. However, because r is a growing function, at some later time it may become large, i.e. $r \gtrsim 1$. Assuming the Bunch-Davies vacuum initial conditions, when $x \gg 1$, the solution of Eq. (3.186) in these three different regimes are given by [108]

$$w_+ = a^{-3/2}\sqrt{\frac{\pi}{4H}}\left[J_{3/2}(x) - iJ_{-3/2}(x)\right] \quad \text{for} \quad x \gtrsim 1\,, \tag{3.187}$$

$$w_+ = \frac{i}{\sqrt{2k}}\left(\frac{H}{k}\right)\left[1 + \frac{i}{3}x^3\right] \simeq \frac{i}{\sqrt{2k}}\left(\frac{H}{k}\right) \quad \text{for} \quad x \ll 1 \ll \frac{1}{z}\,, \tag{3.188}$$

$$w_+ = \frac{1}{\sqrt{2k}}\left(\frac{H}{k}\right)\sqrt{\frac{z\pi}{2}}\left[\frac{x^3}{3}J_{-1/2}(z) + iz^{-1}J_{1/2}(z)\right] \quad \text{for} \quad \frac{1}{z} \lesssim 1\,, \tag{3.189}$$

where z is defined as

$$z \equiv \frac{M}{3H}\,, \tag{3.190}$$

and $r = z/(3x)$. The solution of Eq. (3.186) was calculated using numerical methods as well and it was found that they agree with Eqs. (3.187)-(3.189) remarkably well.

The equation of motion for the longitudinal component with the same scaling of f and

m is

$$\ddot{w}_{\|} + \left(3 + \frac{8}{1+r^2}\right) H\dot{w}_{\|} + \left[\frac{24}{1+r^2}H^2 + \left(\frac{k}{a}\right)^2 (1+r^2)\right] w_{\|} = 0\,. \tag{3.191}$$

And the solution of this equation in the same three regimes was found to be

$$w_{\|} = -\frac{i}{6}a^{-9/2}\sqrt{\frac{\pi}{H}}x^{-2}z^{-1}\left[J_{5/2}(x) - iJ_{-5/2}(x)\right] \quad \text{for} \quad x \gtrsim 1\,, \tag{3.192}$$

$$w_{\|} \simeq -a^{-1/2}\frac{3a_k^4}{\sqrt{2H}}x^{5/2} = -\frac{1}{\sqrt{2k}}\left(\frac{H}{k}\right)z^{-1} \quad \text{for} \quad x \ll 1 \ll \frac{1}{z}\,, \tag{3.193}$$

$$w_{\|} = -\frac{1}{\sqrt{2k}}\left(\frac{H}{k}\right)\sqrt{\frac{z\pi}{2}}\left[z^{-1}J_{-1/2}(z) - i\frac{x^3}{3}J_{1/2}(z)\right] \quad \text{for} \quad \frac{1}{z} \lesssim 1\,, \tag{3.194}$$

which agrees with the numerical solution of Eq. (3.191) very well too.

As is seen from Eqs. (3.188), (3.189) and (3.193), (3.194) on the superhorizon scales modes w_+ and $w_{\|}$ evolves differently if the vector field is light, $M \lesssim H$, or heavy, $M \gtrsim H$. When the field is light, w_+ is constant and $w_{\|} \propto a^{-1}$. Therefore using Eq. (3.118) we find

$$\mathcal{P}_+ = \left(\frac{H}{2\pi}\right)^2 \quad \text{and} \quad \mathcal{P}_{\|} = \frac{1}{z^2}\left(\frac{H}{2\pi}\right)^2 \propto a^{-6} \quad \text{for} \quad M < H. \tag{3.195}$$

Thus the typical value of the vector field perturbation is (see Eq. (2.10))

$$\delta W \approx \sqrt{\mathcal{P}_{\|}} = \frac{3H}{M}\frac{H}{2\pi} \propto a^{-3}, \tag{3.196}$$

where $\delta W \equiv |\delta \mathbf{W}|$ and we used $\mathcal{P}_{\|} \gg \mathcal{P}_+$.

On the other hand, when the mass of the vector field becomes comparable with the inflationary Hubble parameter, i.e. $M \sim H$, the transverse and longitudinal mode functions on the superhorizon scales, with $x \ll 1$, become

$$\left.\begin{aligned} w_+ &= \tfrac{i}{\sqrt{2k}}\left(\tfrac{H}{k}\right)\sqrt{\tfrac{\pi}{2}}\tfrac{J_{1/2}(z)}{\sqrt{z}} \\ w_{\|} &= \tfrac{-1}{\sqrt{2k}}\left(\tfrac{H}{k}\right)\sqrt{\tfrac{\pi}{2}}\tfrac{J_{-1/2}(z)}{\sqrt{z}} \end{aligned}\right\} \;\Rightarrow\; ||w_+|| \approx \left|\left|w_{\|}\right|\right|, \quad z \gtrsim 1. \tag{3.197}$$

When the vector field becomes heavy $M \gg H$ from Eqs. (3.189) and (3.194) we find

$$w_+ = \frac{i}{\sqrt{2k}}\left(\frac{H}{k}\right)\frac{\sin(z)}{z}, \tag{3.198}$$

$$w_{\|} = -\frac{1}{\sqrt{2k}}\left(\frac{H}{k}\right)\frac{\cos(z)}{z}, \tag{3.199}$$

i.e. they oscillate with the same amplitude, $||w_+|| = ||w_{||}||$, but with the phase difference of $\pi/2$. The frequency of oscillations is much larger than the Hubble parameter because $z \gg 1$, therefore it makes sense to use the average values of the power spectra over many oscillations. Using Eq. (3.118) we find

$$\overline{\mathcal{P}_+} = \overline{\mathcal{P}_{||}} = \frac{1}{2z^2}\left(\frac{H}{2\pi}\right)^2 \quad \text{for} \quad M \gtrsim H. \tag{3.200}$$

Thus, the typical value for the vector field perturbations in this regime is

$$\delta W \approx \frac{1}{\sqrt{2}}\frac{3H}{M}\frac{H}{2\pi} \propto a^{-3}, \tag{3.201}$$

where for the scale invariant perturbations $|\delta \mathbf{W}| \approx \sqrt{\overline{\mathcal{P}_{||}}}$ (see Eq. (2.10)). Thus from Eq. (3.196) we see that the typical value of the vector field perturbation is roughly the same if the field is light or heavy.

Case: $f \propto a^2$ & $m \propto a$

When the vector field kinetic function f is increasing with time, i.e. $\alpha = 2$, and $\beta = 1$ the effective mass of the field is constant, $M =$ constant. The requirement that the field is effectively massless when cosmological scales exit the horizon in Eq. (3.182) suggests that $M/H \ll 1$ at all times when the scaling above holds. Using this condition and scaling we can calculate the power spectra for all components of the superhorizon vector field perturbations generated by the particle production process.

The equation of motion for the transverse mode functions

$$\ddot{w}_+ + 3H\dot{w}_+ + \left(\frac{k}{a}\right)^2 (1+r^2)w_+ = 0. \tag{3.202}$$

is the same as for the $\alpha = -4$ case except that now $M =$ constant. This condition simplifies Eq. (3.202), making it possible to obtain the exact solution. After matching this solution to the initial Bunch-Davies vacuum state, the power spectrum on superhorizon scales becomes [108]

$$\mathcal{P}_+ = \left(\frac{H}{2\pi}\right)^2. \tag{3.203}$$

The equation of motion for the longitudinal component from Eq. (3.179) and $\alpha = -4$ becomes

$$\ddot{w}_{||} + \left(3 + \frac{2}{1+r^2}\right)H\dot{w}_{||} + \left[\left(\frac{k}{a}\right)^2 (1+r^2)\right]w_{||} = 0. \tag{3.204}$$

Again, it is impossible to find an exact solution of this equation. But using the vacuum initial conditions in Eq. (3.183) and solving it in two regimes, $r \ll 1$ and $r \gg 1$, and matching those solutions at $r = 1$ we find

$$w_{\|} = -\frac{x}{2}\sqrt{\frac{\pi}{aH}}\left[J_{-5/2}\left(x\right) + iJ_{5/2}\left(x\right)\right] \quad \text{for} \quad x \gtrsim 1, \tag{3.205}$$

$$w_{\|} = -\frac{z^{-1}}{\sqrt{2k}}\left(\frac{H}{k}\right) \quad \text{for} \quad x \ll 1 \ll \frac{1}{z}. \tag{3.206}$$

From Eq. (3.206) we calculate the power spectrum

$$\mathcal{P}_{||} = 9\left(\frac{H}{M}\right)^2\left(\frac{H}{2\pi}\right)^2, \tag{3.207}$$

same as in Eq. (3.195). Since in this case, we have $M/H = \text{constant} \ll 1$, the longitudinal power spectrum is constant, in contrast to the $\alpha = -4$ case. Also, we see that $\mathcal{P}_{\|} \gg \mathcal{P}_{+}$.

3.5.3. Statistical Anisotropy and Non-Gaussianity

The theory studied in this section has two clear advantages. First we can obtain a completely isotropic perturbation spectrum for the vector field, which has previously never been achieved. As we discuss below, this means that we may consider vector fields as dominating the total energy density of the Universe when the curvature perturbation is formed. The second advantage is that we can also account for a small amount of statistical anisotropy in the curvature perturbation spectrum depending on when inflation ends, again by considering the vector field alone. We also demonstrate this in what follows. Finally, statistical anisotropy can also be present in a correlated manner in the bispectrum as well, which characterizes the non-Gaussian features of the CMB temperature perturbations. In view of the forthcoming observations of the recently launched Planck satellite mission this is a particularly promising and timely result.

Let us first consider the case with $\alpha = -4$. As we have seen in the previous section in this case the effective mass of the vector field is time dependent during inflation, $M \propto a^3$. When $M \lesssim H$ the evolution of the vector field perturbations follows the power law on superhorizon scales (see Eqs. (3.188) and (3.193)) and the power spectra for the transverse and longitudinal modes are given in Eq. (3.195). Using the definition of p in Eq. (3.19) we find that in this regime the anisotropy in the particle production is equal to

$$p = z^{-2} - 1, \tag{3.208}$$

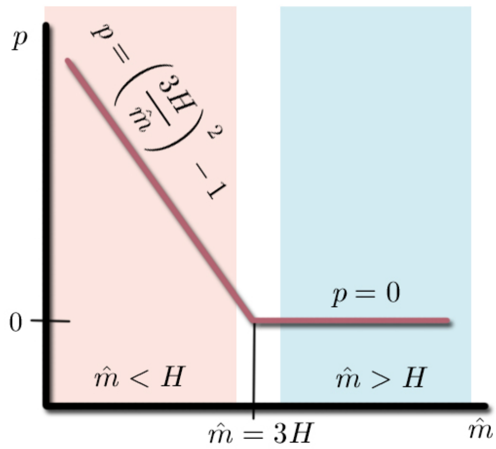

Figure 3.3.: If the vector field is light at the end of inflation, the particle production will be highly anisotropic. If it is heavy, the transverse and longitudinal power spectra are equal and $p = 0$, i.e particle production is isotropic. In the regime where $\hat{m} \sim H$ the anisotropy in the particle production might be non-zero but very small, $p \lesssim 1$.

where $z \lesssim 1$.

In the opposite regime, when $M \gtrsim H$, the vector field perturbations are oscillating (see Eqs. (3.198) and (3.199)) and the average power spectra in Eq. (3.200) are equal giving

$$p = 0. \tag{3.209}$$

At the end of inflation the kinetic function and the mass of the vector field are stabilized: $f_{\rm end} = 1$ and $M_{\rm end} = \text{constant} \equiv \hat{m}$. At this epoch the vector field perturbation power spectra become constant. Therefore, although in the curvaton scenario ζ becomes constant only after the curvaton decay, it is enough to evaluate amplitudes of the vector field perturbations at the end of inflation. Thus, the value of p is frozen at the end of inflation and it depends on the ratio $\hat{m}/H$ (see Figure 3.3). If the vector field is light at the end of inflation (or if its mass is of the order of inflationary Hubble parameter), the particle production is anisotropic and given in Eq. (3.208) with $z = z_{\rm end} \equiv \hat{m}/3H$. If the field is heavy, the transverse and longitudinal power spectra are equal, i.e. the particle production is isotropic giving Eq. (3.209).

If $\hat{m} > H$, the particle production of the vector field is isotropic, $p = 0$, and the vector field generated curvature perturbation is statistically isotropic

$$g = 0. \tag{3.210}$$

Therefore, if the vector field is heavy at the end of inflation the generated curvature perturbation is indistinguishable from the scalar field case. Indeed, if we plug Eq. (3.209) into the expression of the $f_{\rm NL}$ in Eqs. (3.69) and (3.74) we find

$$\frac{6}{5}f_{\rm NL}^{\rm equil} = \frac{6}{5}f_{\rm NL}^{\rm local} = \frac{3}{2\hat{\Omega}_W}, \tag{3.211}$$

exactly the same as in the scalar curvaton case. In this expression we considered that the only contribution to ζ comes from the vector field, i.e. $\mathcal{P}_\phi = 0$.

If, on the other hand, the vector field mass at the end of inflation is $\hat{m} \sim H$, from Eq. (3.208) we find $0 \lesssim p < 1$. Using this and Eqs. (3.72) and (3.76) the anisotropy in $f_{\rm NL}$ becomes

$$\mathcal{G}^{\rm equil} \approx \mathcal{G}^{\rm local} \approx p < 1, \tag{3.212}$$

where again we have considered that only the vector field generates the curvature perturbation. This regime is possible because the observational bound on anisotropy in the spectrum (defined in Eq. (3.32)) is not violated

$$g \approx p < 0.3, \tag{3.213}$$

where $g < 0.3$ is the observational constraint from CMB on the statistical anisotropy (see the discussion above Eq. (2.19)). Using Eq. (3.213) and requiring that $z_{\rm end} \lesssim 1$ from Eq. (3.208) we find the allowed range of the mass values $\hat{m}$ for this case

$$0.3 < \frac{H}{\hat{m}} < 0.4. \tag{3.214}$$

Unfortunately this range is very narrow and initial conditions must be tuned accurately to achieve this possibility.

In the vector curvaton model with the light vector field at the end of inflation the dominant part of ζ must be generated by the scalar field, while the vector field can generate only a subdominant contribution. This is because if $\hat{m} \ll H$, the anisotropy in the vector field particle production is large, i.e. $p \gg 1$, and this would violate observational constraints on g. Assuming a light scalar field with perturbation power spectrum

$\mathcal{P}_\phi = (H/2\pi)^2 = \mathcal{P}_+$ from Eq. (3.32) we find

$$g = \frac{\xi}{1+\xi}\,p \approx \frac{\xi}{1+\xi}\left(\frac{3H}{\hat{m}}\right)^2 , \tag{3.215}$$

where in the last expression we have used $z \ll 1$. Similarly from Eq. (3.77) we find that the isotropic part of $f_{\rm NL}$ is

$$f^{\rm local}_{\rm NL,\,iso} = f^{\rm equil}_{\rm NL,\,iso} = g^2 \frac{2}{\Omega_W}\left(\frac{3H}{\hat{m}}\right)^4 . \tag{3.216}$$

While amplitudes of the angular modulation of $f_{\rm NL}$ in the equilateral and squeezed configurations with $z \ll 1$ are

$$\mathcal{G}^{\rm local} = p \approx \left(\frac{3H}{\hat{m}}\right)^2 \tag{3.217}$$

and

$$\mathcal{G}^{\rm equil} \approx \frac{1}{8}p^2 \approx \frac{1}{8}\left(\frac{3H}{\hat{m}}\right)^4 , \tag{3.218}$$

which are much larger than one in both cases.

The values of the non-linearity parameter $f_{\rm NL}$ calculated in this section correspond to the scaling of the kinetic function with $\alpha = -4$. But in the limit $z \ll 1$ Eqs. (3.215)-(3.217) are also applicable for $\alpha = 2$. In the latter case $z = \text{constant} \ll 1$ therefore, if $f \propto a^2$ the vector field may only generate a subdominant contribution to the curvature perturbation without violating observational bounds on statistical anisotropy, where the dominant part is produced by a scalar field. But for $f \propto a^{-4}$, as we have seen in Eq. (3.210), the vector field can also produce the total curvature perturbation in the Universe. If it is heavy at the end of inflation, i.e. $\hat{m} \gg H$, the generated ζ will be statistically isotropic and indistinguishable from the scalar curvaton case. If, on the other hand, $\hat{m} \sim H$, it may still generate the total ζ which is approximately statistically anisotropic within the observational bounds.

3.5.4. Evolution of the Zero Mode

In order to calculate the curvature perturbation associated with the vector field one needs to study also the evolution of the homogeneous zero mode W. Combining Eqs. (3.170) and (3.174) and using Eq. (3.177), we obtain

$$\ddot{\mathbf{W}} + 3H\dot{\mathbf{W}} + \left[\left(1 - \frac{1}{2}\alpha\right)\dot{H} - \frac{1}{4}(\alpha+4)(\alpha-2)H^2 + M^2\right]\mathbf{W} = 0\,, \tag{3.219}$$

where we also used the definition of the effective mass in Eq. (3.171).

3.5.4.1. During Inflation

As shown in Appendix B, to obtain a scale invariant spectrum for the transverse components of the vector field perturbation we require $f(a)$ to scale according to Eq. (3.181), i.e. $\alpha = -1 \pm 3$. Using this and considering the (quasi)de Sitter inflation (with $\dot{H} \approx 0$) the above becomes

$$\ddot{\mathbf{W}} + 3H\dot{\mathbf{W}} + M^2\mathbf{W} = 0\,. \tag{3.220}$$

We show below that, when $M \ll H$ (true at early times when $\alpha = -4$; always true when $\alpha = 2$), the solution of the above is well approximated by

$$W \simeq \hat{C}_1 + \hat{C}_2 a^{-3}, \tag{3.221}$$

where $\hat{C}_i$ are constants. The dominant term to the solution of Eq. (3.221) is determined by the initial conditions. We choose initial conditions for the vector field zero-mode based on energy equipartition grounds. As is demonstrated in what follows, if the energy equipartition is assumed at the onset of inflation, the dominant term turns out to be the decaying mode $W \propto a^{-3}$ when $\alpha = -4$, and the "growing" mode $W = \text{constant}$ when $\alpha = 2$.

To apply energy equipartition in the initial conditions we need to consider the energy-momentum tensor for this theory, which, from Eq. (3.169) is given by [84]

$$T_{\mu\nu} \;=\; f\left(\frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - F_{\mu\rho}F_{\nu}^{\ \rho}\right) + m^2\left(A_\mu A_\nu - \frac{1}{2}g_{\mu\nu}A_\rho A^\rho\right). \tag{3.222}$$

If we assume that the homogenized vector field lies along the z-direction, we can write the above as [84]

$$T_\mu^\nu = \mathrm{diag}(\rho_W, -p_\perp, -p_\perp, +p_\perp)\,, \tag{3.223}$$

where

$$\rho_W \equiv \rho_{\rm kin} + V_W\,, \qquad p_\perp \equiv \rho_{\rm kin} - V_W\,, \tag{3.224}$$

with

$$\rho_{\rm kin} \equiv -\frac{1}{4} f F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} a^{-2} f \dot{A}^2 = \frac{1}{2}\left[\dot{W} - \frac{1}{2}(\alpha-2) HW\right]^2, \tag{3.225}$$

$$V_W \equiv -\frac{1}{2} m^2 A_\mu A^\mu = \frac{1}{2} a^{-2} m^2 A^2 = \frac{1}{2} M^2 W^2, \tag{3.226}$$

where $A \equiv |\mathbf{A}|$, we used Eqs. (3.174) and (3.177), and we assumed a negative signature for the metric.

Energy equipartition corresponds to

$$(\rho_{\rm kin})_0 \simeq (V_W)_0 \,, \tag{3.227}$$

where the subscript '0' indicates the values at some initial time, e.g. near the onset of inflation.

Case: $f \propto a^{-4}$

In this case $M \propto a^3$ and the solution to Eq. (3.220) is

$$W = a^{-3}\left[\hat{C}_3 \sin\left(\frac{M}{3H}\right) + \hat{C}_2 \cos\left(\frac{M}{3H}\right)\right]. \tag{3.228}$$

When $M \gtrsim H$ the above shows that the amplitude of the oscillating zero mode is decreasing as $||W|| \propto a^{-3}$. In the opposite regime, when $M \ll H$ the solution above is well approximated by Eq. (3.221) with $\hat{C}_1 = \hat{C}_3 a_0^{-3} M_0/3H$, where we considered that $a^{-3}M = a_0^{-3}M_0 =$ constant. Using this, the constants $\hat{C}_2$ and $\hat{C}_3$ in Eq. (3.228) can be expressed in terms of initial values of the field amplitude W_0 and it's velocity $\dot{W}_0$:

$$\hat{C}_2 = -\frac{\dot{W}_0}{3H}\, a_0^3 \quad \text{and} \quad \hat{C}_3 = \frac{\left(\dot{W}_0 + 3HW_0\right)}{M_0}\, a_0^3 \,. \tag{3.229}$$

Assuming initial equipartition of energy we can relate W_0 with $\dot{W}_0$. From Eqs. (3.225) and (3.226), setting $\alpha = -4$, we readily obtain

$$\rho_{\rm kin} = \frac{1}{2}(\dot{W} + 3HW)^2 \quad \text{and} \quad V_W = \frac{1}{2} M^2 W^2. \tag{3.230}$$

Then, using Eq. (3.227), we get

$$\dot{W}_0 \simeq W_0 \left(-3H \pm M_0\right). \tag{3.231}$$

Substituting this relation into Eq. (3.229) we find that the evolution of the vector field W in Eq. (3.228) takes the simple form:

$$W = W_0 \left(\frac{a}{a_0}\right)^{-3} \sqrt{2} \cos\left(\frac{M}{3H} \pm \frac{\pi}{4}\right). \tag{3.232}$$

Note that this equation is valid for any value of M. However, we can see that when $M \ll H$ the zero mode of the vector field is decreasing as $W \propto a^{-3}$, but when $M \gg H$ it oscillates rapidly with a decreasing amplitude proportional to a^{-3}. On this basis we can assume that the typical value of the zero mode during inflation always scales as

$$W \propto a^{-3}. \tag{3.233}$$

With the assumption of initial equipartition of energy for the vector field at the onset of inflation we can calculate the kinetic and potential energy densities.[7] Inserting Eq. (3.232) and its derivative into Eqs. (3.225) and (3.226) we find

$$\rho_{\rm kin} = \left[W_0 M_0 \sin\left(\frac{M}{3H} \pm \frac{\pi}{4}\right)\right]^2 \quad \text{and} \quad V_{\rm W} = \left[W_0 M_0 \cos\left(\frac{M}{3H} \pm \frac{\pi}{4}\right)\right]^2. \tag{3.234}$$

Hence, the total energy density is constant

$$\rho_W = M_0^2 W_0^2. \tag{3.235}$$

Because this relation is independent of the vector field mass M it is valid in both regimes: when $M \ll H$ and W follows a power law evolution, and when $M \gg H$ and W oscillates. This is valid as long as $f(a)$ and $m(a)$ are varying with time.

In the vector curvaton scenario the vector field must be subdominant during inflation. From Eq. (3.235) we see that assuming this to be the case at the onset of inflation, it will stay so until the end of inflation irrespective if the field is light or heavy.

Case: $f \propto a^2$

In this case, $M =$ constant, which means that the solution of Eq. (3.220) is

$$W = a^{-3/2} \left[\hat{C}_1 a^{\sqrt{\frac{9}{4} - (\frac{M}{H})^2}} + \hat{C}_2 a^{-\sqrt{\frac{9}{4} - (\frac{M}{H})^2}}\right]. \tag{3.236}$$

Since in this case $M \ll H$, the above solution is always well approximated by Eq. (3.221) and there is no oscillating regime.

[7] By "potential" we refer to the energy density stored in the mass-term $V_W = -\frac{1}{2} m^2 A_\mu A^\mu$.

Now, Eqs. (3.225) and (3.226) take the form

$$\rho_{\rm kin}=\frac{1}{2}\dot{W}^2 \quad \text{and} \quad V_W=\frac{1}{2}M^2W^2. \tag{3.237}$$

Combining Eqs. (3.221) and (3.237), we find

$$\rho_{\rm kin}=\frac{9}{2}H^2\hat{C}_2^2a^{-6}. \tag{3.238}$$

Thus, at the onset of inflation assuming energy equipartition in Eq, (3.227) gives

$$\left(1+\frac{\hat{C}_1}{\hat{C}_2}a_0^3\right)^2=\left(\frac{3H}{M_0}\right)^2\gg 1 \;\Rightarrow\; \hat{C}_1\simeq\pm\frac{3H}{M_0}a_0^{-3}\hat{C}_2\,, \tag{3.239}$$

where we used that $M_0=M\ll H$. Inserting the above into Eq. (3.221) we find

$$W=a_0^{-3}\hat{C}_2\left[\left(\frac{a_0}{a}\right)^3\pm\frac{3H}{M_0}\right]\simeq \text{constant}\simeq W_0\,, \tag{3.240}$$

because, after the onset of inflation, $(a_0/a)^3\ll 1\ll 3H/M_0$.

Therefore, we have found that W remains constant. Since $M=$ constant, this means that V_W also remains constant. On the other hand, Eq. (3.238) suggests that $\rho_{\rm kin}\propto a^{-6}$. Thus, since we assumed energy equipartition at the onset of inflation, we find that, during inflation, $\rho_{\rm kin}\ll V_W$. Hence,

$$\rho_W\approx V_W\simeq M_0^2W_0^2, \tag{3.241}$$

where $M=\text{constant}=M_0$. This result is the same as in the case $f\propto a^{-4}$ in Eq. (3.235) and the vector curvaton field is ensured to be subdominant during inflation (as required by the curvaton mechanism) if it is subdominant at the onset of inflation.

3.5.4.2. After Inflation

At the end of inflation we assume that the scaling of f and m has ended and we have

$$f=1 \quad \text{and} \quad m=\hat{m}\,. \tag{3.242}$$

Hence, Eqs. (3.235) and (3.241) no longer apply. The evolution of ρ_W is determined as follows.

As mentioned already, after the end of scaling, $\alpha=0$ and $M=\hat{m}$. Then, Eqs. (3.225) and (3.226) become

$$\rho_{\rm kin} = \frac{1}{2}(\dot{W} + HW)^2 \quad \text{and} \quad V_W = \frac{1}{2}\hat{m}^2 W^2. \tag{3.243}$$

The behavior of $\rho_{\rm kin}$ and V_W depends on whether the vector field is light or not. To see this let us calculate the evolution of the field after inflation. With the conditions in Eq. (3.242) the physical vector field of Eq. (3.174) is $\mathbf{W} = \mathbf{A}/a$, while Eq. (3.219) becomes

$$\ddot{\mathbf{W}} + 3H\dot{\mathbf{W}} + \left(\dot{H} + 2H^2 + \hat{m}^2\right)\mathbf{W} = 0\,, \tag{3.244}$$

where the Hubble parameter after inflation decreases as $H(t) = \frac{2}{3(1+w)t}$, with $w \equiv p/\rho$ being the barotropic parameter of the Universe. Solving Eq. (3.244) we find

$$W = t^{\frac{1}{2}\frac{w-1}{w+1}}\left[\tilde{C}_1 J_v\left(\hat{m}t\right) + \tilde{C}_2 J_{-v}\left(\hat{m}t\right)\right], \tag{3.245}$$

$$\dot{W} + HW = \hat{m}\,t^{\frac{1}{2}\frac{w-1}{w+1}}\left[\tilde{C}_1 J_{v-1}\left(\hat{m}t\right) - \tilde{C}_2 J_{1-v}\left(\hat{m}t\right)\right], \tag{3.246}$$

where $v = \frac{1+3w}{6(1+w)}$. One can easily see that the vector field behaves differently if it is light, $\hat{m}t \ll 1$, or heavy, $\hat{m}t \gg 1$.

Let us first see what happens if the vector field is light. Then, Eqs. (3.245) and (3.246) can be approximated as

$$W = t^{\frac{1}{2}\frac{w-1}{w+1}}\left[\frac{\tilde{C}_1}{\Gamma\left(1+v\right)}\left(\frac{\hat{m}t}{2}\right)^{v} + \frac{\tilde{C}_2}{\Gamma\left(1-v\right)}\left(\frac{\hat{m}t}{2}\right)^{-v}\right], \tag{3.247}$$

$$\dot{W} + HW = \hat{m}\,t^{\frac{1}{2}\frac{w-1}{w+1}}\left[v\,\frac{\tilde{C}_1}{\Gamma\left(1+v\right)}\left(\frac{\hat{m}t}{2}\right)^{v-1} - \frac{1}{1-v}\frac{\tilde{C}_2}{\Gamma\left(1-v\right)}\left(\frac{\hat{m}t}{2}\right)^{1-v}\right]. \tag{3.248}$$

Although the solution has one decaying and one growing mode, it might happen that the decaying mode stays larger than the growing mode. To check this we calculate constants $\tilde{C}_1$ and $\tilde{C}_2$ by matching the above equations to the values $W_{\rm end}$ and $\dot{W}_{\rm end}$ at the end of inflation (denoted by 'end'). Thus, we find that

$$W = \frac{2}{3w+1}\left(\frac{a}{a_{\rm end}}\right)^{\frac{1}{2}(3w-1)}\left(W_{\rm end} + \frac{\dot{W}_{\rm end}}{H_*}\right), \tag{3.249}$$

$$\dot{W} + HW = H_*\left(\frac{a}{a_{\rm end}}\right)^{-2}\left(W_{\rm end} + \frac{\dot{W}_{\rm end}}{H_*}\right), \tag{3.250}$$

where H_* is the inflationary Hubble scale. Plugging these solutions into Eq. (3.243) (and using that $a^{3(1+w)} \propto t^2$) we obtain

$$\frac{V_W}{\rho_{\rm kin}} = \frac{4}{(3w+1)^2}\left(\frac{\hat{m}}{H_*}\right)^2\left(\frac{t}{t_{\rm end}}\right)^2 \simeq (\hat{m}t)^2 \ll 1\,, \tag{3.251}$$

which implies that the total energy density of the light vector field is

$$\rho_W \simeq \rho_{\rm kin} = \frac{1}{2}\left(\dot{W}_{\rm end} + W_{\rm end}H_*\right)^2\left(\frac{a}{a_{\rm end}}\right)^{-4} \;\Rightarrow\; \rho_W \propto a^{-4}. \tag{3.252}$$

Therefore, we see that the energy density of the light vector field scales as that of relativistic particles. This is in striking difference to the scalar field case, in which when the field is light its density remains constant even after inflation.

On the other hand, if the vector field is heavy, $\hat{m}t \gg 1$, the Bessel functions in Eqs. (3.245) and (3.246) are oscillating. Hence, as was discussed in section 3.3.1, the heavy vector field oscillates with a frequency much larger than the Hubble parameter and with the amplitude decreasing as $t^{-1/(1+w)} \propto a^{-3/2}$. In Eq. (3.60) it was shown that the energy density of such field decreases as $\rho_W \propto a^{-3}$ and the average pressure is zero, i.e. $\overline{p_\perp} \approx 0$. Therefore, on average, the oscillating vector field behaves as pressureless isotropic matter and can dominate the Universe without generating excessive large scale anisotropy. This is crucial for the vector curvaton mechanism because, to produce the curvature perturbation, the field must dominate (or nearly dominate) the Universe without inducing excessive anisotropic expansion.

3.5.5. Curvaton Physics

In this section we calculate constraints for our vector curvaton model assuming that the scaling behavior of $f(t)$ and $m(t)$ ends when inflation is terminated. This implies that the scaling is controlled by some degree of freedom which varies during inflation, e.g. the inflaton field.

In the curvaton scenario the total curvature perturbation can be calculated as the sum of individual curvature perturbations from the constituent components of the Universe multiplied by the appropriate weighting factor. In the current scenario this is written as follows

$$\zeta = (1-\hat{\Omega}_W)\zeta_\gamma + \hat{\Omega}_W\zeta_W, \tag{3.253}$$

where $\hat{\Omega}_W$ is defined in Eq. (2.197). As in the scalar curvaton paradigm, the above is to be evaluated at the time of decay of the curvaton field.

As was discussed in section 3.5.3, if $\hat{m} \gg H_*$ at the end of inflation, then the vector field perturbation spectrum is isotropic and may generate the total curvature perturbation in the Universe without violating observational bounds on the statistical anisotropy of the curvature perturbation. If this is the case, we can assume that $\zeta_\gamma = 0$. On the other hand, when $\hat{m} \ll H_*$, the amplitude of the spectrum of the longitudinal component of the vector field perturbations is substantially larger than the one of the transverse perturbations. Hence, the curvature perturbation due to the vector field is excessively anisotropic. To avoid conflict with observational bounds, the contribution of the vector field to the curvature perturbation has to remain subdominant. Therefore, for this scenario, we have to consider $\zeta_\gamma \neq 0$ and the curvature perturbation already present in the radiation dominated Universe must dominate the one produced by the vector curvaton field.

In Eqs. (3.196) and (3.201) it was shown that the typical value of the field perturbation is $\delta W \sim (3H_*/M)(H_*/2\pi)$. If $M \ll H_*$ this is because the longitudinal component is dominant over the transverse ones (see Eq. (3.195)). If $M \gg H_*$, then the transverse and longitudinal components are oscillating with same amplitudes (see Eq. (3.200)).

For this reason, at the end of inflation, we can write

$$\delta W_{\rm end} \sim \frac{3H_*}{\hat{m}}\frac{H_*}{2\pi} \simeq \frac{H_*^2}{\hat{m}}\,, \tag{3.254}$$

where we have taken $M = \hat{m}$ and $f = 1$ at the end of inflation. $W_{\rm end}$ can be found from Eq. (3.235) by using $(\rho_W)_{\rm end} \simeq W_0 M_0 \simeq W_{\rm end}\hat{m}$ (see Eqs. (3.235) and (3.241)). Thus,

$$W_{\rm end} \sim \frac{\sqrt{(\rho_W)_{\rm end}}}{\hat{m}}\,. \tag{3.255}$$

Hence, from Eq. (3.80) we calculate the curvature perturbation of the vector field

$$\zeta_W \sim \Omega_{\rm end}^{-1/2}\frac{H_*}{m_{\rm Pl}}\,, \tag{3.256}$$

where $\Omega_{\rm end} \equiv (\rho_W/\rho)_{\rm end}$ is the density parameter of the vector field at the end of inflation, $\rho_{\rm end}$ is the total energy density dominated by the inflaton field, and we have used the Friedman equation: $3m_P^2 H_*^2 = \rho_{\rm end}$. Since the vector field must be subdominant during inflation we have $\Omega_{\rm end} \ll 1$.

Eq. (3.256) is valid in both $\alpha = -1 \pm 3$ cases. The only difference is that, in the $f \propto a^2$ case, statistically isotropic curvature perturbations cannot be generated. Hence, only considerations for statistically anisotropic perturbations in Sec. 3.5.3 are relevant.

To calculate the parameter space for this model we note that at the end of inflation the inflaton field starts oscillating and $w \neq -1$. Therefore the Hubble parameter decreases as $H(t) \sim t^{-1}$. In general, the inflaton potential is approximately quadratic around its VEV. Thus, the coherently oscillating inflaton field corresponds to a collection of massive particles (inflatons) whose energy density decreases as a^{-3}. When the Hubble parameter falls bellow the inflaton decay rate Γ, the inflaton particles decay into much lighter relativistic particles reheating the Universe. After reheating, the Universe becomes radiation dominated with the energy density scaling as $\rho_\gamma \propto a^{-4}$.

On the other hand, the evolution of the energy density of the vector field, depends on its mass $\hat{m}$. As discussed in Sec. 3.5.4, if $\hat{m} \ll H_*$ the energy density scales as $\rho_W \propto a^{-4}$ until the vector field becomes heavy and starts oscillating. If $\hat{m} \gg H_*$, however, the vector field has already started oscillating during inflation and $\rho_W \propto a^{-3}$.

To avoid causing an excessive anisotropic expansion period the vector field must be oscillating before it dominates the Universe and decays. This requirement implies that

$$\Gamma,\ \hat{m} > \Gamma_W,\ H_{\rm dom}\ , \tag{3.257}$$

where Γ_W is the decay rate of the vector field and $H_{\rm dom}$ is the value of the Hubble parameter when the vector field dominates the Universe if it has not decayed already. Working as in Ref. [84], we can estimate $H_{\rm dom}$ as

$$H_{\rm dom} \sim \Omega_{\rm end}\Gamma^{1/2}{\rm min}\left\{1;\frac{\hat{m}}{H_*}\right\}^{2/3}{\rm min}\left\{1;\frac{\hat{m}}{\Gamma}\right\}^{-1/6}. \tag{3.258}$$

Similarly, if the vector field decays before it dominates, the density parameter just before the decay is given by

$$\Omega_{\rm dec} \sim \Omega_{\rm end}\left(\frac{\Gamma}{\Gamma_W}\right)^{1/2}{\rm min}\left\{1;\frac{\hat{m}}{H_*}\right\}^{2/3}{\rm min}\left\{1;\frac{\hat{m}}{\Gamma}\right\}^{-1/6}. \tag{3.259}$$

where $\Omega_{\rm dec} \equiv (\Omega_W)_{\rm dec}$. Combining the last two equations and using Eq. (3.256) we can express the inflationary Hubble scale as

$$\frac{H_*}{m_{\rm Pl}} \sim \Omega_{\rm dec}^{1/2}\,\zeta_W\,{\rm min}\left\{1;\frac{\hat{m}}{H_*}\right\}^{-1/3}{\rm min}\left\{1;\frac{\hat{m}}{\Gamma}\right\}^{1/12}\left(\frac{{\rm max}\left\{\Gamma_W;H_{\rm dom}\right\}}{\Gamma}\right)^{1/4}. \tag{3.260}$$

The bound on the inflationary scale can be obtained by considering that the decay rate of the vector field is $\Gamma_W \sim h^2\hat{m}$, where h is the coupling to the decay products. Then we can write ${\rm max}\{\Gamma_W;H_{\rm dom}\} \gtrsim h^2\hat{m}$. Furthermore, we must consider the possibility of thermal evaporation of the vector field condensate during the radiation dominated phase. If this were to occur, all the memory of the superhorizon perturbation spectrum would be erased. The bound on h, such that the condensate does not evaporate before its decay, is given in Eq. (3.89).

From Eq. (3.260) one can see that the parameter space is maximized if the Universe undergoes prompt reheating after inflation, i.e. if $\Gamma \to H_*$. To find the parameter space we investigate two separate cases: when $\hat{m} \gg H_*$ and when $\hat{m} \ll H_*$.

3.5.5.1. The Statistically Isotropic Perturbation

The statistically isotropic perturbation can be realized only in the case when $\alpha = -4$. As mentioned before, if the mass of the vector field at the end of inflation is larger than the Hubble parameter, $\hat{m} > H_*$, then the field has started oscillating already during inflation. In this case amplitudes of the longitudinal and transverse perturbations are equal and therefore the curvature perturbation induced by the vector field is statistically isotropic. We can assume, in this case, that the vector field alone is responsible for the total curvature perturbation in the Universe without the need to invoke additional perturbations from other fields. Thus, we can set $\zeta_\gamma = 0$ in Eq. (3.253) and write

$$\zeta \sim \Omega_{\rm dec}\zeta_W \,. \tag{3.261}$$

Using this and the lower bound on h we find from Eq. (3.89) the lower bound for the inflationary Hubble parameter

$$\frac{H_*}{m_{\rm Pl}} \gtrsim \left(\frac{\zeta}{\sqrt{\Omega_{\rm dec}}}\right)^{4/5}\left(\frac{\hat{m}}{m_{\rm Pl}}\right)^{3/5}, \tag{3.262}$$

where we have taken into account that the parameter space is maximised when the Universe undergoes prompt reheating, i.e. $\Gamma \to H_*$. From this expression it is clear that the lowest bound is attained when the vector field dominates the Universe before its decay, $\Omega_{\rm dec} \to 1$, and when the oscillations of the vector field commence at the very end of inflation, i.e. $\hat{m} \to H_*$. With these values we find the bounds

$$H_* \gtrsim 10^9\,\text{GeV} \quad \Leftrightarrow \quad V_*^{1/4} \gtrsim 10^{14}\,\text{GeV}\,, \tag{3.263}$$

where $V_*^{1/4}$ denotes the inflationary energy scale and we used that $\zeta \approx 5\times10^{-5}$ from the observations of the Cosmic Background Explorer.

In view of the above, we can obtain a lower bound for the decay rate of the vector field. Indeed, using Eqs. (3.89) and (3.263) we find

$$\Gamma_W \gtrsim \frac{\hat{m}^3}{m_{\rm Pl}^2} \gtrsim \frac{H_*^3}{m_{\rm Pl}^2} \gtrsim 10^{-9}\,\mathrm{GeV}\,. \tag{3.264}$$

From the above we find that the temperature of the Universe after the decay of the vector field is $T_{\rm dec} \sim \sqrt{m_{\rm Pl}\Gamma_W} \gtrsim 10^4\,\mathrm{GeV}$, which is comfortably higher than the temperature at BBN $T_{\rm BBN} \sim 1\,\mathrm{MeV}$ (i.e. the decay occurs much earlier than BBN), and also higher than the electroweak phase transition, i.e. the decay precedes possible electroweak baryogenesis processes.

Since $\hat{m} > H_*$, Eq. (3.263) corresponds to a lower bound on $\hat{m}$. An upper bound on $\hat{m}$ can be obtained as follows. Because, $\hat{m} > H_* \gtrsim \Gamma$, Eq. (3.259) becomes

$$\Omega_{\rm dec} \sim \Omega_{\rm end}\sqrt{\frac{\Gamma}{\Gamma_W}}\,. \tag{3.265}$$

From Eq. (3.89) we have $\Gamma_W \gtrsim \hat{m}^3/m_{\rm Pl}^2$. Combining this with the above we obtain

$$\hat{m}^3 \lesssim \left(\frac{\Omega_{\rm end}}{\Omega_{\rm dec}}\right)\Gamma m_{\rm Pl}^2. \tag{3.266}$$

Now, when $\alpha = -4$ we have $M \propto a^3$ during inflation. Since the end of scaling occurs when inflation is terminated, for $a < a_{\rm end}$ we can write

$$\hat{m} = \left(\frac{a_{\rm end}}{a}\right)^3 M \simeq \mathrm{e}^{3N_{\rm osc}} H_*\,, \tag{3.267}$$

where we considered that the field begins oscillating when $M \simeq H_*$ and $N_{\rm osc}$ is the number of remaining e-folds of inflation when the oscillations begin. Inserting the above into Eq. (3.266) we find

$$N_{\rm osc} \lesssim N_{\rm osc}^{\max} \equiv \frac{2}{9}\left[\ln\left(\frac{\Omega_{\rm end}}{\Omega_{\rm dec}}\right) + \ln\sqrt{\frac{\Gamma}{H_*}} + \ln\left(\frac{m_{\rm Pl}}{H_*}\right)\right] < \frac{2}{9}\ln\left(\frac{m_{\rm Pl}}{\Omega_{\rm dec}H_*}\right)\,, \tag{3.268}$$

where in the last inequality we used that $\Omega_{\rm end} < 1$ and $\Gamma \lesssim H_*$. Now, considering that $\hat{m} \gtrsim H_*$, Eq. (3.262) gives

$$\frac{\Omega_{\rm dec}H_*}{m_{\rm Pl}} \gtrsim \zeta^2. \tag{3.269}$$

Hence, combining Eqs. (3.268) and (3.269) we obtain

$$N_{\rm osc}^{\max} < -\frac{4}{9}\ln\zeta = 4.4\,. \tag{3.270}$$

Thus, in view of Eq. (3.267), we obtain the bound $\hat{m} \lesssim \mathrm{e}^{3N_{\rm osc}^{\rm max}} H_*$, which results in the following parameter space for $\hat{m}$:

$$1 \lesssim \hat{m}/H_* < 10^6, \tag{3.271}$$

where we used Eq. (3.270). The above range is reduced if the decay of the curvaton occurs more efficiently than through gravitational couplings, i.e. if $h > \hat{m}/m_{\rm Pl}$. Nevertheless, we see that the parameter space in which the vector field undergoes isotropic particle production and can alone account for the curvature perturbation, is not small but may well be exponentially large. Indeed, repeating the above calculation with $\Gamma_W \sim \hat{m}$ (i.e. $h \sim 1$) it is easy to find that

$$N_{\rm osc} = \frac{2}{3}\left[\ln\left(\frac{\Omega_{\rm end}}{\Omega_{\rm dec}}\right) + \ln\sqrt{\frac{\Gamma}{H_*}}\right]. \tag{3.272}$$

Hence, using that $\Omega_{\rm end} < 1$ and $\Gamma \lesssim H_*$ we obtain

$$N_{\rm osc}^{\rm max} = -\frac{2}{3}\ln\Omega_{\rm dec} \lesssim 3.1 \quad \Longleftrightarrow \quad 1 \lesssim \hat{m}/H_* < 10^4, \tag{3.273}$$

where we used $\Omega_{\rm dec} \gtrsim 10^{-2}$. This is because, in the case considered, $f_{\rm NL}$ is given by Eq. (3.211), so a smaller $\Omega_{\rm dec}$ would violate the current observational bounds on the non-Gaussianity in the CMB temperature perturbations (see the discussion in section 2.1.2.2).

Still, it seems that, to obtain an exponentially large parameter space for $\hat{m}$, we need ρ_W not to be too much smaller that V_* during inflation and also inflationary reheating to be efficient. In the case of gravitational decay ($\Gamma_W \sim \hat{m}^3/m_{\rm Pl}^2$) Eq. (3.268) has a weak dependence on both $\Omega_{\rm end}$ and Γ: $\hat{m} \propto (\Omega_{\rm end}^2\Gamma)^{1/3}$, which means that the allowed range of values for $\hat{m}$ remains large even when $\Omega_{\rm end}$ and Γ are substantially reduced. This is not necessarily so when $\Gamma_W \sim h^2\hat{m}$, with $h \gg \hat{m}/m_{\rm Pl}$. Indeed, in this case it can be easily shown that $\hat{m} \propto h^{-2}\Omega_{\rm end}^2\Gamma$. Therefore, if Γ is very small it may eliminate the available range for $\hat{m}$. Fortunately, the decay coupling h can counteract this effect without being too small.

3.5.5.2. Statistically Anisotropic Perturbations

If the vector field is not responsible for the total curvature perturbation in the Universe, the parameter space is more relaxed. In this case, the vector field may start oscillating after inflation and hence its mass is $\hat{m} \ll H_*$. However, this means that the curvature perturbation due to the vector field is strongly statistically anisotropic. For this reason we can no longer set ζ_γ to zero in Eq. (3.253) because the curvature perturbation present in the radiation dominated Universe must be dominant. In other words, the parameter ξ defined in Eq. (3.31) needs to be very small, $\xi \ll 1$.

In this case the total curvature perturbation is given in Eq. (3.86). Inserting this into Eq. (3.260) and considering again that the lowest decay rate of the vector field is through the gravitational decay, $\max\{\Gamma_W; H_{\rm dom}\} \geq \hat{m}^3/m_{\rm Pl}^2$ we find

$$\frac{H_*}{m_{\rm Pl}} > \left(\frac{g\,\zeta^2}{\Omega_{\rm dec}}\right)^{3/4} \left(\frac{\hat{m}}{m_{\rm Pl}}\right)^{5/8} \left(\frac{\Gamma}{m_{\rm Pl}}\right)^{-3/8} \min\left\{1; \frac{\hat{m}}{\Gamma}\right\}^{1/8}. \tag{3.274}$$

The above suggests that the lower bound on H_* is minimised for prompt reheating with $\Gamma \to H_*$. Also, from observations we know that the statistically anisotropic contribution to the curvature perturbation must be subdominant. Thus, the vector field should not dominate the Universe before its decay. Hence, using $\Gamma \to H_*$ and $\Omega_W < 1$ we obtain

$$\frac{H_*}{m_{\rm Pl}} > \sqrt{g}\,\zeta\sqrt{\frac{\hat{m}}{m_{\rm Pl}}}\,. \tag{3.275}$$

From this expression it is clear that the parameter space for H_* is maximised for the lowest mass value. The minimum mass of the vector field can be estimated from the requirement that the field decays before BBN. Because the lowest decay rate is the gravitational decay, this condition reads $\hat{m}^3/m_{\rm Pl}^2 \gtrsim T_{\rm BBN}^2/m_{\rm Pl}$, with $T_{\rm BBN} \sim 1\,\text{MeV}$, which corresponds to $\hat{m} \gtrsim 10^4$ GeV. Using this, we find that the parameter space for the vector curvaton model with the statistically anisotropic curvature perturbations is

$$H_* > g^{1/2}\,10^7\,\text{GeV} \quad \Leftrightarrow \quad V_*^{1/4} > g^{1/4}\,10^{13}\,\text{GeV}\,, \tag{3.276}$$

i.e. it is somewhat relaxed compared to the statistically isotropic case (c.f. Eq. (3.263)) depending on the magnitude of the statistical anisotropy in the spectrum, for which $g \lesssim 0.3$ (see the discussion above Eq. (2.19)). This result is valid for both $\alpha = -1 \pm 3$ cases. From the above it is evident that there is ample parameter space for the mass of the vector field

$$10\,\text{TeV} \lesssim \hat{m} \ll H_*. \tag{3.277}$$

3.5.6. Summary for the Massive fF^2 Model

In section 3.5 we studied a particularly promising vector curvaton model consisting of a massive Abelian vector field, with a Maxwell type kinetic term and with varying kinetic function f and mass m during inflation. The model is rather generic, it does not suffer from instabilities such as ghosts and may be realized in the context of theories beyond the standard model such as supergravity and superstrings (see two tentative examples in Ref. [108]).

We have parametrised the time dependence of the kinetic function as $f \propto a^{\alpha}$, where $a = a(t)$ is the scale factor. Our model offers two distinct possibilities. If $\hat{m} < H_*$ (possible for $\alpha = -1 \pm 3$) the vector field can only produce a subdominant contribution to the curvature perturbation ζ, but it can be the source of statistical anisotropy in the spectrum and bispectrum. In fact, non-Gaussianity in this case is predominantly anisotropic, which means that, if a non-zero $f_{\rm NL}$ is observed without angular modulation, then our model is falsified in the $\hat{m} < H_*$ case. The second possibility (possible for $\alpha = -4$ only) corresponds to $\hat{m} \gtrsim H_*$. In this case the vector field can alone generate the curvature perturbation ζ without any contribution from other sources such as scalar fields. If $\hat{m} \gg H_*$, particle production is isotropic and the model does not generate any statistical anisotropy. The vector field begins oscillating a few e-folds before the end of inflation but its density remains constant until inflation ends. The parameter space for this case can be exponentially large, i.e. $1 \ll \hat{m}/H_* < 10^6$. Significant non-Gaussianity can be generated, provided the vector field decays before it dominates the Universe, in which case $f_{\rm NL}$ is found to be identical to the scalar curvaton scenario. In other words, if $\hat{m} \gg H_*$, our vector curvaton can reproduce the results of the scalar curvaton paradigm. Finally, if $\hat{m} \sim H_*$ the vector field can alone generate the curvature perturbation ζ but it can also generate statistical anisotropy in the spectrum and bispectrum. In this case, the anisotropy in $f_{\rm NL}$ is subdominant and equal to the statistical anisotropy in the spectrum, which is a characteristic signature of this possibility. However, the allowed range for $\hat{m}$ values in this case is very narrow, as shown in Eq. (3.214), requiring accurate tuning of the initial conditions. We have also found that inflation has to occur at energies of $V_*^{1/4} \gtrsim 10^{14}\,$GeV in the (almost) isotropic and $V_*^{1/4} > g^{1/4}\,10^{13}\,$GeV in the anisotropic case.

3.6. The End-of-Inflation Scenario

3.6.1. Vector Field Perturbations and ζ

In this section we consider another model in which a vector field influences the generation of the curvature perturbation. The model avoids excessive large scale anisotropy in the Universe by a different mechanism than the vector curvaton scenario described in section 3.3. The idea is based on Ref. [58] which was summarized in section 2.4.2, where it was shown that in hybrid inflation models the generation of the curvature perturbation can be realized due to the inhomogeneous end of inflation. Yokoyama and Soda [89] used this idea to generate the anisotropic contribution to the total curvature perturbation. In their model the anisotropy is generated at the end of inflation due to the vector field coupling with the waterfall field. In other words the scalar field σ of section 2.4.2 is changed by the vector field A_μ. In this section we calculate the non-Gaussianity of the model in Ref. [89] using the formalism developed in section 3.2.

This scenario uses the conformal invariance breaking of the $U(1)$ vector field through the non-canonical kinetic function of the form $f(t) F_{\mu\nu}F^{\mu\nu}$:

$$S = \int \sqrt{-\mathcal{D}_g}\left(-\frac{1}{4}f(t)\, F_{\mu\nu}F^{\mu\nu} - \dots\right) \mathrm{d}^4x, \tag{3.278}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A$.

This action is only written for the conformal invariance breaking term, the dots represent other terms which give inflation with practically constant H, and generate $f(t)$ without having any other effect on the evolution of the gauge field during inflation. For the vector field to be gauge invariant any scalar field coupled to A_μ must have zero expectation value (no spontaneous symmetry breaking) with negligible quantum fluctuation around that value.

This form of the conformal invariance breaking was considered in many papers. Starting from Ref. [109] such action was often considered for the generation of the primordial magnetic fields (see for example Refs. [110, 111, 112, 113]) and recently in Refs. [84, 108, 114] it was considered for the generation of ζ by the vector field (see section 3.5). In these papers it was discovered that a scale invariant perturbation spectrum of the physical, canonically normalized vector field W_μ is obtained if $f \propto a^2$ (we showed in section 3.5 that this is the case for $f \propto a^{-4}$ as well):

$$\mathcal{P}_+ = \left(\frac{H}{2\pi}\right)^2. \tag{3.279}$$

In the end-of-inflation scenario of Soda and Yokoyama there are two components of the curvature perturbation: one generated during inflation and an anisotropic one, generated by a vector field at the end of inflation:

$$\zeta = \zeta_{\mathrm{inf}} + \zeta_{\mathrm{end}}. \tag{3.280}$$

The first component is due to the perturbation of the light scalar field, while the second one is due to the perturbation of the vector field with the kinetic term in Eq. (3.278). Without parity violating terms the power spectra for left handed and right handed polarizations are equal, while the longitudinal polarization is absent for a massless vector field. In this situation we find that parameters $p\,(k)$ and $q\,(k)$ defined in Eq. (3.19) become

$$p = -1 \quad \text{and} \quad q = 0. \tag{3.281}$$

ζ_{inf} in this scenario is the statistically isotropic contribution to the total curvature perturbation. In the slow roll inflation the spectrum of the scalar field perturbation is $\mathcal{P}_\phi = (H/2\pi)^2$, so that we get

$$\mathcal{P}_+ = \mathcal{P}_\phi, \tag{3.282}$$

And the total isotropic part of the curvature perturbation from Eq. (3.30) becomes

$$\mathcal{P}_\zeta^{\text{iso}} = \mathcal{P}_\phi N_\phi^2 \left(1 + \xi\right), \tag{3.283}$$

with ξ given by

$$\xi = \left(\frac{N_W}{N_\phi}\right)^2. \tag{3.284}$$

Using the expression for the anisotropy parameter g in Eq. (3.32) we find that in this scenario

$$g = -\frac{\xi}{1+\xi}. \tag{3.285}$$

Taking into account Eq. (3.281), the vector $\mathcal{M}_i\,(\mathbf{k})$ defined in Eq.(3.35) reduces to the simple form

$$\boldsymbol{\mathcal{M}}\,(\mathbf{k}) = N_W \mathcal{P}_\phi \left[\hat{\mathbf{N}}_W - \hat{\mathbf{k}}\left(\hat{\mathbf{N}}_W \cdot \hat{\mathbf{k}}\right)\right]. \tag{3.286}$$

3.6.2. Hybrid Inflation Model

To calculate f_{NL} we consider a specific example of the hybrid inflation with the potential

$$V\left(\phi, \chi, A^\mu\right) = V_0 + \frac{1}{2}m_\phi^2\phi^2 - \frac{1}{2}m_\chi^2\chi^2 + \frac{1}{4}\lambda\chi^4 + \frac{1}{2}\lambda_\phi\phi^2\chi^2 + \frac{1}{2}\lambda_A\chi^2A^\mu A_\mu, \tag{3.287}$$

which contributes to terms in Eq. (3.278) denoted by dots. Here ϕ is the inflaton and χ is the waterfall field (compare this with the scalar field case in Eq. (2.178)). The effective mass of the waterfall field for this potential is

$$m_{\text{eff}}^2 \quad = \quad -m_\chi^2 + \lambda_\phi\phi^2 - \frac{\lambda_A}{f}W_iW_i, \tag{3.288}$$

where we chose the Coulomb gauge with $W_t = 0$ and $\partial_i W^i = 0$ was chosen. Inflation ends when the inflaton reaches a critical value ϕ_c where the effective mass of the waterfall field becomes tachyonic. But one can see from Eq.(3.288) that the critical value is a function of the vector field $\phi_c = \phi_c(W)$. With this in mind N_W^i and N_W^{ij} can be readily calculated:

$$N_W^i = \frac{\partial N}{\partial \phi_c}\frac{\partial \phi_c}{\partial W_i} = N_c \frac{\lambda_A}{f\,\lambda_\phi}\frac{W_i}{\phi_c}, \tag{3.289}$$

and

$$N_W^{ij} = \frac{\partial N}{\partial \phi_c}\frac{\partial^2 \phi_c}{\partial W_i \partial W_j} + \frac{\partial^2 N}{\partial \phi_c^2}\frac{\partial \phi_c}{\partial W_i}\frac{\partial \phi_c}{\partial W_j} = \frac{N_W^2}{\phi_c N_c}\left(C^2 \delta_{ij} - \hat{W}_i \hat{W}_j\right), \tag{3.290}$$

where we have defined

$$N_c \equiv \frac{\partial N}{\partial \phi_c} \quad \text{and} \quad C \equiv \sqrt{\frac{f\,\lambda_\phi}{\lambda_A}\frac{\phi_c}{W}}, \tag{3.291}$$

where $W \equiv |W_i|$ and f are evaluated at the end of inflation and we used the fact that $N_{cc}/N_c^2 \sim N_{\phi\phi}/N_\phi^2 \sim \mathcal{O}(\epsilon)$ under the slow roll approximation, where ϵ is the slow roll parameter defined as $\epsilon \equiv \frac{1}{2} m_{\rm Pl}^2 (V_\phi / V)^2$ (see Eq. (1.55)), with the prime denoting derivatives with respect to the inflaton. As mentioned earlier the total of perturbations consists of two components: perturbations of the scalar and vector fields. This gives the following bispectrum in the equilateral configuration

$$\begin{aligned} \mathcal{B}_\zeta^{\rm equil}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= \mathcal{B}_\phi^{\rm equil}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) + \mathcal{B}_W^{\rm equil}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \\ &= 3\mathcal{P}_\phi^2 N_\phi^2 N_{\phi\phi} + \left[\mathcal{M}_i(\mathbf{k}_1) N_W^{ij} \mathcal{M}_j(\mathbf{k}_2) + \text{c.p.}\right] = \\ &= \mathcal{P}_\phi^2 N_\phi^4 \frac{\xi^2}{N_c \phi_c} 3\left[\left(C^2 - 1\right) - \left(\frac{7}{8}C^2 - 1\right) W_\perp^2 - \frac{3}{16} W_\perp^4\right]. \end{aligned} \tag{3.292}$$

The mixed term $\mathcal{B}_{\phi W}^{\rm equil}$ is absent from Eq.(3.292) because in this model $N_{\phi W}^i = 0$. By using the expression for the isotropic power spectrum in Eq. (3.283) and the bispectrum in Eq. (3.292) from the definition of $f_{\rm NL}^{\rm equil}$ in Eq. (3.38) we obtain

$$\frac{6}{5} f_{\rm NL}^{\rm equil} = \eta g^2 \left[\left(C^2 - 1\right) - \left(\frac{7}{8}C^2 - 1\right) W_\perp^2 - \frac{3}{16} W_\perp^4\right], \tag{3.293}$$

where the slow parameter η is equal to $\eta \equiv m_{\rm Pl}^2 V_{\phi\phi}/V = m_\phi^2 m_{\rm Pl}^2 / V_0$ and $m_{\rm Pl} N_c = 1/\sqrt{2\epsilon_c}$, with ϵ_c being the ϵ parameter evaluated at the end of inflation. Similarly, for the squeezed configuration we find

$$\frac{6}{5}f_{\rm NL}^{\rm local} = \eta g^2 \left[\left(C^2-1\right) - \left(C^2-1\right) W_\perp^2 - \frac{1}{4}(\sin\varphi)^2 W_\perp^4 \right]. \tag{3.294}$$

In this equation φ is the angle between the vectors $\mathbf{k}_1$ and $\mathbf{W}_\perp$ (see Figure A.1).

We find that $f_{\rm NL}^{\rm equil}$ and $f_{\rm NL}^{\rm local}$ are functions of $W_\perp$, i.e. they are anisotropic and correlated with the statistical anisotropy. Also the level of non-Gaussianity is proportional to the anisotropy parameter squared, $f_{\rm NL} \propto g^2$, as in the vector curvaton model. However, the angular modulation of $f_{\rm NL}$ in this scenario is different from the curvaton scenario. From Eqs. (3.293) and (3.294) we see the additional modulation term proportional to $W_\perp^4$. This term is absent in the vector curvaton scenario.

As was mentioned earlier, in this model the vector field during inflation is massless and, therefore, gauge invariant. The homogeneous value of such vector field can be set to zero by an appropriate gauge choice. However, as seen from Eqs. (3.291) and (3.293), (3.294) calculated predictions do depend on the homogeneous value of the vector field W. Therefore, for this model as it stands, the interpretation of the results are not clear. Although terms proportional to $W_\perp^4$ in $f_{\rm NL}$ expressions do not depend on C and consequently on the gauge choice.

3.6.3. Summary of the End-of-Inflation Scenario

In section 3.6 we have considered a model proposed in Ref. [89]. In this model the energy density of the vector field is subdominant throughout the history of the Universe. However, it influences the generation of ζ by modulating the end of inflation through the coupling to the waterfall field. The conformal invariance of the massless vector field is broken by the time dependent kinetic function as in section 3.5. We consider a scale invariant perturbation spectrum of the vector field with the kinetic function scaling as $f \propto a^2$. In this model the vector field is gauge invariant, therefore its particle production is anisotropic and the curvature perturbation generated due to this field is statistically anisotropic.

We have calculated the non-linearity parameter $f_{\rm NL}$ in this model and found that it is correlated with anisotropy in the power spectrum, as in the vector curvaton scenario. In this model too $f_{\rm NL}$ has an angular modulation with the amplitude of the same order as the isotropic part. In addition it has the modulation term, proportional to $W_\perp^4$, which is absent in the vector curvaton model.

4. Summary and Conclusions

The successes of the standard Hot Big Bang theory in explaining the structure and evolution of the Universe since the very first second until today is very impressive. The predictions for abundances of the light elements are in a very good agreement with observations. The origin and the process of formation of galaxies and galaxy clusters are now well understood. However, to reproduce the observable Universe, the initial conditions of the HBB model must be finely tuned. The spatial curvature of the Universe must have been incredibly close to zero near its birth for the Universe to have time to evolve to the present state, and it must have started being extraordinary smooth even in regions which were never in causal contact with each other. In addition, in the framework of the standard HBB model, there are no mechanisms to explain the origin of tiny primordial density perturbations which are almost Gaussian, adiabatic and correlated on superhorizon scales. Such perturbations are observed as temperature fluctuations in the CMB sky and they seed the growth of large scale structure.

The fine tuning problems may be substantially alleviated by postulating a period of accelerated expansion at the earliest stages of the evolution of the Universe. This period is called inflation. In addition to solving the flatness and horizon problems, the greatest achievement of the inflationary paradigm is the explanation of the origin of the primordial density perturbation which has the properties observed in the CMB sky. According to this paradigm, the primordial density perturbation originated as quantum fluctuations during the inflationary period. In Chapter 2 we have demonstrated how the application of quantum field theory on a curved space-time background may lead to the amplification of quantum fluctuations and their conversion into the classical field perturbation. This perturbation, consequently, causes the perturbation in the curvature of space-time. Much later, after inflation, when the wavelengths of the perturbation become smaller than the horizon size, it seeds the formation of structure in the Universe due to the process of gravitational instability.

To describe the formation and evolution of the cosmological perturbation we have used a very important quantity: the curvature perturbation ζ. This quantity is constant throughout the history of the Universe, except during those periods when the total pressure of the Universe is not a unique function of the energy density. In other words, when pressure is not adiabatic. To show how the classical field perturbation, originating from quantum fluctuations, is related to the curvature perturbation ζ, we used the separate universes approach. In this approach the evolution of the Universe on superhorizon scales at each space point is treated as that of the separate, unperturbed Universe with the locally defined expansion rate. The latter is determined by the average energy density on the flat hypersurface at that point, where the averaging is performed on a superhorizon scale of interest.

The statistical properties of ζ provide one of the main tools in cosmology for observational tests of models of the very early Universe. We have shown how these properties may be calculated using the δN formalism. It was applied to calculate the power spectrum and the bispectrum at tree level for three models, namely: the single field inflation, the end-of-inflation and the curvaton scenarios. In the treatment of these three models, we have assumed that ζ is generated solely by quantum fluctuations of scalar fields. In Chapter 3 we showed that quantum fluctuations of vector fields may contribute or even generate the total curvature perturbation in the Universe as well.

However, a massless, canonically normalized $U(1)$ vector field cannot produce ζ because, being conformally invariant, its quantum fluctuations are not amplified during inflation. And even if they were amplified, such a field cannot dominate the Universe without producing excessive large scale anisotropy, i.e. excessive anisotropic expansion of the Universe, although in most scenarios the vector field must dominate or nearly dominate the Universe to generate ζ. In section 3.1 we discuss possibilities of breaking the conformal invariance of vector fields and introduce four mechanisms for the generation of ζ by vector fields without producing an excessive large scale anisotropy.

In section 3.2 we have extended the δN formalism to include perturbations of vector fields. In contrast to the scalar field, which has one degree of freedom (DoF), the massive vector field has three DoF. Therefore, in a theory with a massive vector field we must consider quantum fluctuations for all three of them. To calculate the evolution of each DoF they were decomposed into the longitudinal and two circular polarization vectors. This choice is advantageous because each polarization vector transforms differently under the Lorentz group. Therefore, we can be sure that they do not mix in the course of evolution. We found that in general the amplification of quantum fluctuations is not the same for all three DoF. In other words, the particle production of a vector field is not in general isotropic. This results in different values of n-point correlation functions for each polarization.

To quantify the anisotropy in the particle production we introduced two parameters

$p(k)$ and $q(k)$ in Eq. (3.19), where k is the wavevector. The q parameter quantifies the difference in the power spectra of two transverse polarization modes. It is non-zero only in parity violating theories. The parameter p quantifies the difference in the longitudinal power spectrum and the average of the transverse ones. If both parameters are equal to zero, the particle production of the vector field is isotropic. However, if any of these are non-zero, the particle production is anisotropic. The values of p and q parameters are determined by the mechanism which brakes the conformal invariance.

If the vector field with anisotropic particle production generates or affects the curvature perturbation, the latter is statistically anisotropic, i.e. statistical properties of ζ are not invariant under rotations. The power spectrum of such perturbation will have an angular modulation. To the lowest order we can express it as [36]

$$\mathcal{P}_\zeta = \mathcal{P}_\zeta^{\mathrm{iso}} \left[1 + g \left(\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} \right)^2 \right], \tag{4.1}$$

where $\mathcal{P}_\zeta^{\mathrm{iso}}$ is the isotropic part of the spectrum, $\hat{\mathbf{n}}$ is the unit vector along the preferred direction and g parametrizes the amount of modulation. In the vector field models $\hat{\mathbf{n}}$ is in the direction of the homogeneous vector field. $\mathcal{P}_\zeta^{\mathrm{iso}}$ in these models may be solely due to the vector field, if g satisfies the observational bounds, or it may be dominated by some other, statistically isotropic source of ζ. The present observational bound on the anisotropy in the spectrum of ζ is $g \lesssim 0.3$ (see the discussion above Eq. (2.19)). The value of g is determined by the mechanism which generates the curvature perturbation and by the value of p. In this thesis we consider two such mechanisms: the vector curvaton and the end-of-inflation scenarios.

The vector curvaton scenario, first proposed in Ref. [83], uses the fact that a heavy vector field oscillates rapidly with the frequency much larger than the Hubble parameter. The time averaged pressure of such field is zero and the energy density decreases with the scale factor as a^{-3}. Thus, the heavy vector field acts as pressureless, isotropic matter and can dominate the Universe without producing excessive large scale anisotropy. In accord with the curvaton scenario, the vector field dominates (or nearly dominates) the Universe after reheating, when the latter is radiation dominated. The curvaton imprints its perturbation spectrum and decays before the BBN. The perturbation spectrum of the vector curvaton field is acquired during inflation, when the field is light and its energy density is negligible compared to the inflaton one. During this period the values of parameters p and q are determined, depending on the mechanism of conformal invariance breaking.

In section 3.3.2 the general predictions for the non-linearity parameter f_{NL} are derived in the vector curvaton scenario with $p \neq 0$ and $q \neq 0$. First, we find that $f_{\rm NL}$ has an angular modulation, similarly to the power spectrum. The amplitude of the modulation is parametrized by $\mathcal{G}$ given by

$$f_{\rm NL} = f_{\rm NL}^{\rm iso} \left(1 + \mathcal{G}\, W_{\perp}^{2}\right), \tag{4.2}$$

where $W_{\perp}$ is the projection of the unit vector of the preferred direction onto the plane of vectors $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ which were used to calculate the bispectrum. The preferred direction is determined by the direction of the homogeneous vector field. Therefore, we find that both, the power spectrum and $f_{\rm NL}$ have the same direction of angular modulation. Another important prediction of the vector curvaton scenario is that the magnitude of $f_{\rm NL}$ is correlated with the anisotropy in the power spectrum, i.e. $f_{\rm NL}^{\rm iso} \propto g^2$.

We calculated $f_{\rm NL}$ in the squeezed and equilateral configurations (Eqs. (3.71), (3.72) and (3.75), (3.76)) and found that only the equilateral configuration is sensitive to the parity violating terms in the Lagrangian of the theory. If the theory is parity conserving, isotropic parts of $f_{\rm NL}$ are equal in both configurations. Therefore, the detection of different values of $f_{\rm NL}^{\rm iso}$ in the squeezed and equilateral configurations would indicate parity violation. However, the amplitude of the angular modulation $\mathcal{G}$ is not equal in the squeezed and equilateral configurations for both parity violating and conserving theories. In addition, the anisotropic part of $f_{\rm NL}$ dominates over the isotropic part if $p > 1$. Although presently there are no observational constraints on the values of $\mathcal{G}$ in the squeezed and equilateral configurations, the detection of them would allow a unique determination of p and q, and therefore, would constraint very tightly the possible conformal invariance breaking mechanisms for the vector field during inflation.

If, on the other hand, the particle production is isotropic, i.e. $p = 0$ and $q = 0$, the predictions of the vector curvaton scenario do not differ from the standard scalar curvaton case. However, this offers a possibility to generate the total curvature perturbation in the Universe solely by the vector field, without directly invoking scalar fields at all. But even if the particle production is anisotropic with $|p| \lesssim 0.3$ and any value of q, the vector field can still generate the total ζ with the amount of statistical anisotropy satisfy observational bounds.

To find the values of p and q, we consider two mechanisms of breaking the conformal invariance. In the first one a massive Abelian vector field is non-minimally coupled to gravity through the Ricci scalar, see Eq. (3.90). We calculate the perturbation power spectra for all three polarizations and find that the scale invariance is achieved if the non-minimal coupling constant is equal to $1/6$ and the bare mass of the vector field is much smaller than the Hubble parameter. Because the given Lagrangian is parity conserving, the parity violation parameter is $q = 0$. The other anisotropy parameter is $p = 1$ in this model. As was discussed after Eq. (3.32), because $p > 0.3$ such a vector field cannot produce the total curvature perturbation in the Universe, without violating observational bounds on statistical anisotropy. Therefore, the dominant contribution to ζ must come from some other, statistically isotropic source. In the context of the vector curvaton scenario this means that the vector field must decay before dominating, while the dominant contribution to ζ must be present in the radiation dominated background before the curvaton decay.

We find that, in non-minimally coupled vector curvaton model, the isotropic part of $f_{\rm NL}$ is equal to $2g^2/\Omega_W$, where $\Omega_W < 1$ is the density parameter of the vector field just before its decay. The amplitudes of anisotropic parts are 1 and 9/8 times the isotropic part in the squeezed and equilateral configurations respectively. After taking into account all cosmologically relevant bounds we find that the parameter space for this model is

$$H \gtrsim g\, 10^{10}\,\text{GeV} \iff V^{1/4} \gtrsim g^{1/2}\, 10^{14}\,\text{GeV}, \tag{4.3}$$

where H is the inflationary Hubble parameter and $V^{1/4}$ is the energy scale of the inflation. From this result it is clear that the parameter space is large enough for a successful realization of this scenario in particle physics models.

Another model considered in this thesis is of the vector curvaton with time dependent kinetic function and mass, see Eq. (3.169). As in the previous model we calculate the superhorizon perturbation spectra for all three polarizations and find that they are scale invariant if the mass varies with the scale factor as $m \propto a$ and is smaller than the Hubble parameter when cosmological scales exit the horizon, while the kinetic function scales as $f \propto a^{-1\pm3}$.

We assume that degrees of freedom, which modulate the time dependence of the kinetic function and mass, are stabilized at the end of inflation. Therefore, the vector field mass becomes constant at that moment, i.e. $m = \text{constant} \equiv \hat{m}$. Since any constant value in front of the kinetic function may be absorbed into the definition of the vector field, we may set $f = 1$ at the end of inflation, and the field becomes canonically normalized. As we saw, the scale invariant perturbation spectra may be achieved if the kinetic function f is increasing as well as decreasing. If the vector field is a gauge field, then f is the gauge kinetic coupling. In this case it is inversely proportional to the gauge coupling e as $f \propto e^{-2}$. Therefore, an increasing $f \propto a^2$ (small during inflation) would correspond to the strongly coupled regime, while $f \propto a^{-4}$ would correspond to the weak coupling.

Therefore, only the second case may be realized in the particle physics models.

First we calculate the anisotropy in the particle production for $f \propto a^{-4}$ and find that it depends on the mass of the vector field at the end of inflation $\hat{m}$. Since the Lagrangian of this model has no parity violating terms, $q = 0$. But the value of p depends on $\hat{m}$. If the vector field is light at the end of inflation $p \neq 0$ and is given by $p = (3H/\hat{m})^2 - 1$. If, on the other hand, the vector field is heavy, $p = 0$. Therefore, the light vector field generates the statistically anisotropic curvature perturbation, while the heavy field generates the statistically isotropic one.

The isotropic part of $f_{\rm NL}$ in the $p \gg 1$ case is equal to $f_{\rm NL}^{\rm iso} = \left(2g^2/\Omega_W\right) \cdot (3H/\hat{m})^4$. The amplitude of the $f_{\rm NL}$ angular modulation in this regime is $(3H/\hat{m})^2$ and $\frac{1}{8}\,(3H/\hat{m})^4$ times larger than the isotropic part in the squeezed and equilateral configurations respectively.

If the vector field is heavy at the end of inflation, $p = 0$ and the generated curvature perturbation is statistically isotropic. Such a vector field may generate the total curvature perturbation in the Universe without the need of scalar field contribution. In this regime the standard curvaton scenario predictions for the non-Gaussianity are valid, i.e. if the curvaton decays before domination $f_{\rm NL} \approx 3/\left(2\Omega_W\right)$. In the opposite case, when it decays being dominant, the generated ζ is Gaussian.

For this model, when the vector curvaton is light at the end of inflation, the allowed range of inflationary Hubble parameter and energy scale is

$$H > g^{1/2}\, 10^7 \text{ GeV} \quad \Longleftrightarrow \quad V^{1/4} > g^{1/4}\, 10^{13} \text{ GeV}, \tag{4.4}$$

while the allowed region for the vector field mass at the end of inflation is

$$10 \text{ TeV} \lesssim \hat{m} \lesssim H. \tag{4.5}$$

For the heavy field, and consequently statistically isotropic curvature perturbation, the analogous bounds are

$$H > 10^9 \text{ GeV} \quad \Longleftrightarrow \quad V^{1/4} > 10^{14} \text{ GeV} \tag{4.6}$$

and

$$1 \lesssim \hat{m}/H \lesssim 10^6, \tag{4.7}$$

were we considered that the vector field produces the total curvature perturbation. Although for the statistically isotropic case the parameter space is somewhat reduced, in both cases it is large enough for a successful implementation in realistic particle physics models.

So far we have discussed only the case $f \propto a^{-4}$. In the case of increasing kinetic function with $f \propto a^2$, the same results apply, but the vector field has to be light and can produce only statistically anisotropic ζ.

In the final section 3.6 of this thesis we calculate the non-Gaussianity in the end-of-inflation scenario introduced in Ref. [89]. The conformal invariance of the vector field in this model is broken by the time varying kinetic function, similarly as in the vector curvaton case discussed above. However, in this model the vector field is massless; therefore, it has only two degrees of freedom and particle production is necessarily anisotropic. In the end-of-inflation scenario the vector field is always subdominant. However, it influences the generation of ζ through a coupling to the waterfall field of hybrid inflation. In this way the end of inflation is spatially modulated by the vector field, i.e. the hypersurface of the synchronous end of inflation does not coincide with the uniform density hypersurface. We calculated the non-Gaussianity for this model and found that the amplitude of angular modulation of $f_{\rm NL}$ is larger than the value of the isotropic part, as in the curvaton scenario. However, in contrast to the curvaton scenario, in this model $f_{\rm NL}$ has an additional modulation, proportional to $W_\perp^4$, where the latter is the projection of the preferred direction onto the plane of three $\mathbf{k}$ vectors, used to calculate the bispectrum.

In summary, we have shown that a vector field can influence or even generate the total curvature perturbation in the Universe. If the particle production of the vector field is isotropic, the generated curvature perturbation by such field is statistically isotropic. Then the vector field may generate the total ζ in the Universe without the direct involvement of scalar fields. In this case observational predictions for the curvature perturbation are the same as for models with scalar fields. If, on the other hand, the particle production of the vector field is anisotropic, the generated contribution to ζ by such field is statistically anisotropic. In this case observational signatures, very distinct from the scalar field case, will be present: anisotropic power spectrum and $f_{\rm NL}$, where the magnitude and the preferred direction of the latter is correlated with the anisotropy in the spectrum.

Until recently CMB analyses were performed assuming statistical isotropy of the curvature perturbation. Our results suggest a new observable: statistical anisotropy. Therefore, it is desirable to reanalyze CMB maps without imposing rotational invariance *a priory*. Although current measurements might not be sensitive enough to constraint the statistical anisotropy in the primordial curvature perturbation (see Refs. [38, 43]), with an advent of the Planck data the situation will improve considerably. For example, according to Ref. [115] the lowest detectable value of g from WMAP data is $|g| \gtrsim 0.1$. With an expected performance of the Planck satellite this bound will be reduced to $|g| \gtrsim 0.02$.

Planck measurements will be much more sensitive to non-Gaussianity as well. Current WMAP bound is $|f_{\rm NL}| \lesssim 100$, in case of no detection with Planck data it will be reduced to $|f_{\rm NL}| \lesssim 5$, very close to the cosmic variance limit [42]. Presently there are no observational constraints on the angular modulation of $f_{\rm NL}$. With such increase in sensitivity of measurements in a very near future one expects that anisotropy in the spectrum and bispectrum will be discovered or constrained very tightly. In case of the discovery, with the magnitude and anisotropy of $f_{\rm NL}$ proposed above, it will be a smoking gun for a vector field contribution to the primordial curvature perturbation.

Another important advancement in this direction will be the confirmation or falsification of the presence of the "Axis of Evil", which suggests that low multipoles of the CMB are aligned along one direction [74]. Presently the statistical significance of the discovery of the "Axis of Evil" is still debatable. However, its confirmation will have profound implications: this would prove the existence of the preferred direction in the Universe. Such direction cannot be accounted for by scalar fields, but for vector fields it is natural.

It is necessary for vector field models to be confronted with observations, in addition, the treatment presented in this thesis should be extended in several directions. We have investigated only two mechanisms of breaking the conformal invariance of massless Abelian vector fields: non-minimal coupling to gravity and the time varying kinetic function. In the literature on primordial magnetic fields there are many more mechanisms proposed to brake this invariance. It would be desirable to explore which of them may give scale invariant perturbation spectra from quantum fluctuations of vector fields.

Even more so, one would also like to understand the generation of perturbations from vacuum fluctuations in the anisotropically inflating Universe. In the scalar field dominated Universe it is natural to assume isotropic expansion, provided inflation lasted long enough before cosmological scales exit the horizon, so that according to the no-hair theorem, initial anisotropy was inflated away. In the presence of light vector fields, the backreaction on the expansion of the Universe might not be negligible, generating the large scale anisotropy. We neglected such backreaction in vector curvaton models because the vector field energy density is negligible during inflation. However, the anisotropic expansion can be easily accommodated within these models or it might be obligatory in others.

We considered two scenarios for the generation of the curvature perturbation: vector curvaton and end-of-inflation. However, the developed formalism may be easily extended to include other scenarios that have already been explored for the contribution of the scalar field perturbation.

Three toy models were presented in this thesis to generate the curvature perturbation by vector fields. Ultimately any model of the early Universe must be firmly rooted in realistic particle physics theories. In the context of inflationary model building 'particle physics theories' can mean two things. It may be the realization of the inflationary expansion of the Universe in some fundamental theory, currently the best developed of which is string theory. For the cosmological aspects of string theory one may see Refs. [116, 117] and references therein. Or particle theory may mean an effective field theory, which accurately describes the Nature at the energies when cosmological scales exit the horizon. In this direction an extensive effort exists in explaining the inflationary epoch in the context of supersymmetry and supergravity (for reviews see Refs. [118, 119]).

From this point of view a particularly attractive is the vector curvaton model with time varying kinetic function $f(t)$ and mass $m(t)$, presented in section 3.5. These functions, f and m, cannot have an explicit time dependence but must be modulated by some dynamical DoF during inflation. It might be an inflaton itself, or some other field. A vector field with such Lagrangian is very natural in string theory, where parameters such as masses and kinetic functions are modulated by scalar fields called moduli. Moduli are not fundamental scalar fields, they parametrize the size and shape of the manifold on which extra dimensions are compactified. But from our four dimensional perspective they act as scalar fields. In Ref. [108] it was shown that the modulus field with an exponential potential (which is reasonable for a modulus field) can play a role of a single DoF driving inflation as well as modulating the time dependence of the vector field kinetic function and mass.

From the effective field theory point of view, the time varying kinetic function and mass is very general in supergravity theories. In this case f is the gauge kinetic function, which is a holomorphic function of the scalar fields of the theory. In supergravity the potential of these scalar fields receive a correction from the Kähler potential such that their mass become $m_\varphi^2 \sim H^2$ [120, 121, 122]. In the inflationary model building this is known as the η problem. Therefore, scalar fields fast-roll during inflation and one expects a considerable evolution of the gauge kinetic function, which is modulated by these fields. Indeed, in Ref. [108] it was shown that an expectation of $\dot{f}/f \sim H$ is quite generic. The time dependence of the mass in these theories may be modulated by the same or additional DoF through the Higgs mechanism. In the same work it was shown that the required scaling of the gauge field mass, i.e. $m \propto a$, can be achieved if the mass of the Higgs field is $m_H \sim H$. This, again, is very reasonable due to corrections from the Kähler potential. However, to be a gauge field, the gauge coupling constant of the vector field must be small. As was discussed above, this means that only the kinetic function with $f \propto a^{-4}$ scaling is applicable in this case. Fortunately, this is a case which have the richest phenomenology.

In the context of implementing vector curvaton scenario in the particle physics theories it is important to note that models considered in this thesis involve only Abelian vector fields. However, most of gauge bosons in simple extensions of the Standard Model (SM) are non-Abelian. Therefore, an investigation of the particle production and the generation of the curvature perturbation by non-Abelian vector fields is desirable (a related work can be found in Refs. [85, 86]).

The investigation of the very early Universe is exciting for two reasons. First, it offers a possibility to understand the origin and history of the observable structure in the Universe. Secondly, it serves as the giant laboratory to constrain theories of the fundamental physics. From the second point of view the research in cosmology is complimentary to the research in particle physics which can be tested by large experiments, such as LHC. At the time of writing LHC just started operating and everyone is looking forward with excitement for new discoveries. First of all, the detection of the Higgs boson is expected. This would prove the existence of fundamental scalar fields in Nature. If it is not discovered, the particle physics models without a fundamental Higgs field will become favorable, such as technicolor. But for inflationary model building until very recently only scalar fields were considered for the generation of the curvature perturbation. If such a field is not discovered, alternative models will become more attractive. However, currently the only alternatives being explored in the literature are vector fields.

Another exciting possibility is the discovery of signatures of physics beyond SM. This will have a profound significance for particle physics as well as early Universe theories. If these signatures will be compatible with the supersymmetry, it will be a strong assurance that investigation of supersymmetric or supergravity models of inflation is the fruitful direction.

From the astronomy side a large contribution towards the particle physics theories will be provided by the observations of recently launched Planck satellite. The most relevant questions for this thesis which Planck is expected to answer are: does the primordial curvature perturbation have a detectable level of non-Gaussianity and statistical anisotropy? If non-Gaussianity and anisotropy is detected and if it is of the form suggested in this thesis, it will prove the non-negligible contribution of vector fields to the primordial curvature perturbation. This will provide a new observable allowing to probe the gauge field content of the effective field theory which governs the physics at energies when cosmological scales exit the horizon.

A. Calculation of $\boldsymbol{W}_\perp$ in Equilateral Configuration

First note that in the equilateral configuration the unit vectors $\hat{\mathbf{k}}_a$ satisfy $\hat{\mathbf{k}}_1 + \hat{\mathbf{k}}_2 = -\hat{\mathbf{k}}_3$, where $a = 1, 2, 3$. If we define scalar products of the unit 3-vector, $\hat{\mathbf{A}}$, with each $\hat{\mathbf{k}}_a$ as $W_a \equiv \hat{\mathbf{W}} \cdot \hat{\mathbf{k}}_a$, then in the equilateral configuration $W_1 + W_2 = -W_3$ and

$$\begin{aligned}
&W_1^2 + W_2^2 + W_3^2 = 2\left(W_1^2 + W_1W_2 + W_2^2\right);\\
&W_1W_2 + W_2W_3 + W_3W_1 = -\left(W_1^2 + W_1W_2 + W_2^2\right);\\
&W_1^2W_2^2 + W_2^2W_3^2 + W_3^2W_1^2 = \left(W_1^2 + W_1W_2 + W_2^2\right)^2.
\end{aligned} \tag{A.1}$$

Let us define a vector $\mathbf{W}_\perp$ which is the projection of $\hat{\mathbf{W}}$ to the plane containing vectors $\hat{\mathbf{k}}_1$, $\hat{\mathbf{k}}_2$ and $\hat{\mathbf{k}}_3$ (see Figure A.1). Then the scalar product of these vectors and $\hat{\mathbf{W}}$ is the same as the product with $\mathbf{W}_\perp$:

$$\hat{\mathbf{W}} \cdot \hat{\mathbf{k}}_a = \mathbf{W}_\perp \cdot \hat{\mathbf{k}}_a. \tag{A.2}$$

Without loss of generality we can assume that the angle between $\mathbf{W}_\perp$ and $\hat{\mathbf{k}}_1$ is φ:

$$W_1 \equiv \hat{\mathbf{W}} \cdot \hat{\mathbf{k}}_1 = \mathbf{W}_\perp \cdot \hat{\mathbf{k}}_1 = W_\perp \cos\varphi, \tag{A.3}$$

where $W_\perp = |\mathbf{W}_\perp|$. In equilateral configuration the angle between vectors $\hat{\mathbf{k}}_1$ and $\hat{\mathbf{k}}_2$ is $2\pi/3$, and W_2 becomes

$$W_2 \equiv \mathbf{W}_\perp \cdot \hat{\mathbf{k}}_2 = W_\perp \cos\left(\varphi + \frac{2\pi}{3}\right) = -W_\perp\left(\frac{1}{2}\cos\varphi + \frac{\sqrt{3}}{2}\sin\varphi\right). \tag{A.4}$$

From the last two equations we get

$$W_1^2 + W_1W_2 + W_2^2 = \frac{3}{4}W_\perp^2. \tag{A.5}$$

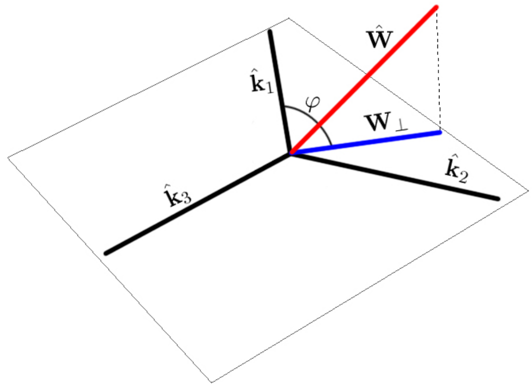

Figure A.1.: $W_\perp$ is the projection of the unit vector $\hat{\mathbf{W}}$ into the plane of three vectors $\hat{\mathbf{k}}_1$, $\hat{\mathbf{k}}_2$ and $\hat{\mathbf{k}}_3$. φ is the angle between $W_\perp$ and $\hat{\mathbf{k}}_1$. In this figure the equilateral configuration of $\hat{\mathbf{k}}_a$ is shown.

Putting this result back into Eq. (A.1) we find

$$\begin{aligned}
&W_1^2 + W_2^2 + W_3^2 = \tfrac{3}{2} W_\perp^2; \\
&W_1 W_2 + W_2 W_3 + W_3 W_1 = -\tfrac{3}{4} W_\perp^2; \\
&W_1^2 W_2^2 + W_2^2 W_3^2 + W_3^2 W_1^2 = \tfrac{9}{16} W_\perp^4.
\end{aligned} \tag{A.6}$$

B. Scale Invariant Perturbation Spectrum of the Vector Field with Time Varying Kinetic Function

In section 3.5.2 it was stated that the vector field with the time varying kinetic and mass terms in Eq. (3.169) acquires a scale invariant spectrum if the kinetic function scales as $f \propto a^{\alpha}$, where $\alpha = -1 \pm 3$, and the mass scales as $m \propto a^{\beta}$, where $\beta = 1$. Here we will prove this result following Ref. [108], where it was derived by Dr. K. Dimopoulos.

The equation of motion of the transverse modes is calculated in Eq. (3.178). For convenience let us rewrite it here using the conformal time

$$w_{+}'' + 2\frac{a'}{a}w_{+}' + \left[-\frac{1}{4}(\alpha+4)(\alpha-2)\,(aH)^2 + (aM)^2 + k^2\right]w_{+} = 0, \tag{B.1}$$

where primes denote derivatives with respect to the conformal time τ. This equation is simpler that the one of the longitudinal mode. Thus we will find the value of α first, and then consider the equation for the longitudinal mode to determine β.

The value of α can be readily deduced by noting that Eq. (B.1) reduces to the equation of motion of a scalar field in Eq. (2.79) if[1]

$$\alpha = -1 \pm 3. \tag{B.2}$$

Then in subsection 2.2.2.5 it was calculated that the scalar field acquires a scale invariant perturbation spectrum if the field is effectively massless and is initially in the Bunch-Davies vacuum state. Therefore, by analogy we conclude that transverse modes of the vector field acquire the scale invariant perturbation spectrum if it is effectively massless, has Bunch-Davies initial conditions and the kinetic function scales as $f \propto a^2$ or $f \propto a^{-4}$.

However, the perturbation spectrum of the longitudinal mode depends not only on the scaling of the kinetic function but on the scaling of the mass as well, i.e. on the value of

[1] In Fourier space Eq. (2.79) becomes $\phi_k'' + 2\frac{a'}{a}\phi_k' + \left[(am)^2 + k^2\right]\phi_k = 0$.

β. To determine this parameter let us rewrite Eq. (3.179) as

$$w_{||}'' - \frac{4-\alpha+2\beta}{\tau} w_{||}' + \left[-\frac{1}{2} (\alpha - 2) (2 - \alpha + 2\beta) \tau^{-2} + k^2 \right] w_{||} = 0, \tag{B.3}$$

where we have also taken into account that the field has to be light for the perturbation spectrum of transverse modes to be scale invariant and we used the substitution $\tau = -(aH)^{-1}$ valid in the de Sitter space-time. This equation can be solved using the vacuum initial conditions. For the longitudinal mode they are

$$\lim_{k\tau \to -\infty} = \gamma \frac{a^{-1}}{\sqrt{2k}} \mathrm{e}^{-ik\tau}, \tag{B.4}$$

where γ is the Lorentz boost factor defined in Eq. (3.184). For the light vector field it is equal to

$$\gamma = \frac{k/a}{M}, \tag{B.5}$$

where M is the mass of the physical vector field and is defined in Eq. (3.171). Solving Eq. (B.3) with initial conditions in Eq. (B.4) we find

$$w_{||} = \frac{k}{aM} \frac{\sqrt{-\tau\pi}}{2a} \frac{\mathrm{e}^{-i\frac{\pi}{2}\left(\nu - \frac{3}{2}\right)}}{\sin(\pi\nu)} \left[J_\nu(-k\tau) - \mathrm{e}^{i\pi\nu} J_{-\nu}(-k\tau) \right], \tag{B.6}$$

where J_ν denotes Bessel function of the first kind, and

$$\nu = \frac{1}{2}\sqrt{2(\alpha-2)(2-\alpha+2\beta) + (5-\alpha+2\beta)^2}. \tag{B.7}$$

At late times, when the mode exits the horizon, the dominant term of the above solution approaches

$$\lim_{k\tau \to -0} w_{||} = -\frac{1}{\Gamma(1-\nu)} \frac{\sqrt{-\tau\pi}}{a} \frac{\mathrm{e}^{i\frac{\pi}{2}\left(\nu + \frac{3}{2}\right)}}{\sin(\pi\nu)} \left(\frac{H}{M}\right) \left(\frac{k}{2aH}\right)^{1-\nu}. \tag{B.8}$$

With this solution we find that the power spectrum is given by

$$\mathcal{P}_{||} = \frac{k^3}{2\pi^2} \left| \lim_{k\tau \to -0} w_{||} \right|^2 = \frac{16\pi}{\sin^2(\pi\nu) \left[\Gamma(1-\nu)\right]^2} \left(\frac{H}{M}\right)^2 \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{2aH}\right)^{5-2\nu}. \tag{B.9}$$

The above expression becomes scale invariant if $\nu = 5/2$, and $\mathcal{P}_{||}$ becomes

$$\mathcal{P}_{||} = 9 \left(\frac{H}{M}\right)^2 \left(\frac{H}{2\pi}\right)^2. \tag{B.10}$$

Using Eqs. (B.7) and (B.2) we find that $\nu = 5/2$ is achieved if

$$\beta = -\frac{1}{2}\left(3 \pm 5\right). \tag{B.11}$$

However, the value $\beta = -4$ must be disregarded. This can be seen using the definition of the mass M in Eq. (3.171)

$$\frac{M}{k/a} \propto a^{-3-\alpha/2}. \tag{B.12}$$

The above expression is a decreasing function of a with any value of α in Eq. (B.2). Thus, with $\beta = -4$ the vector field is massive at early times, which is contradictory to the requirement for scale invariance of the perturbation spectrum of transverse modes. Therefore, only the value $\beta = 1$ is allowed.

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