IMPROPER AFFINE HYPERSPHERES

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The purpose of this work is to present some results about improper affine hyperspheres (in short IA-hyperspheres) in the unimodular affine real $(n+1)$-space $\mathcal{A}^{n+1}$. The study of IA-hypersheres is locally equivalent (see [C1],[C2]) to the study of convex solutions of the Monge-Ampère equation

$$(\mathbf{P}) \qquad det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = 1,$$

on a domain in $\mathbb{R}^n$.

In this paper we show some properties of existence and uniqueness of compact IA-hyperspheres in $\mathcal{A}^{n+1}$ (Sect. 2) and we apply these properties to the study of the solutions of $(\mathbf{P})$ on a ring-shaped domain.

In Sect. 3 we tackle the case of non compact IA-spheres with compact boundary in $\mathcal{A}^3$. We introduce a special class of such spheres which are said regular at infinity and give a Maximum Principle at infinity for them.

1 Some Notations

Throughout M will be a smooth locally strongly convex IA-hypersphere in $\mathcal{A}^{n+1}$ with a C^2-boundary B, that is, M is smooth in the interior and C^2 at the boundary B. We shall denote by $(x_1, x_2, ..., x_{n+1})$ a rectangular coordinate system in $\mathcal{A}^{n+1}$ and by $\{e_1, e_2, ..., e_{n+1}\}$ the canonical base of $\mathbb{R}^{n+1}$.

We observe that by an unimodular affine transformation we can assume that the affine normal vector of M is $\xi = e_{n+1}$. If $\Pi_k \equiv \{x_{n+1} = k\}$, then the projection on Π_0 parallel to ξ, $p_\xi : M \longrightarrow \Pi_0$, is an immersion and so M is, locally, the graph of a strictly convex function $f : \Omega \longrightarrow \mathbb{R}$, which is a solution of the Monge-Ampère equation $(\mathbf{P})$ on a domain Ω in Π_0 (see [LSZ]).

Conversely, the graphs of convex solutions of $(\mathbf{P})$ on a domain Ω in Π_0, are IA-hyperspheres with affine normal vector field $\xi = e_{n+1}$.

Furthermore, if f and g are two convex solutions of $(\mathbf{P})$ on Ω, the function $u = f - g$ satisfies $Lu = 0$, where L is a linear elliptic operator (see [B]). Using this linear elliptic operator, we can give the following **Maximum Principle**:

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Proposition 1. *Let M_1 and M_2 be compact IA-hyperspheres with boundaries B_1 and B_2, respectively, and with the same affine normal vector field $\xi_1 = \xi_2 = e_{n+1}$.*

(a) *Suppose p is an interior point of both M_1 and M_2. If $M_1 \geq M_2$ near p, then $M_1 = M_2$ in a neighbourhood of p.*

(b) *Suppose p is an interior point of B_1 and B_2 such that*

i) $T_pM_1 = T_pM_2$.

ii) B_1 and B_2 have the same euclidean conormal vector in p.

iii) $M_1 \geq M_2$ near of p.

Then $M_1 = M_2$ in a neighbourhood of p.

2 The Compact case

The aim of this section is to generalize the results obtained by the authors in [FMM] for compact IA-spheres in $\mathcal{A}^3$. Because of the similarity of the proofs we shall omit most of them.

We observe that if M is an IA-hypersphere and B is a compact $(n-1)$-hypersurface of M such that $B \subset \Pi$ for some Π hyperplane of $\mathcal{A}^{n+1}$, then B lies on the boundary of a convex set of Π.

If we assume that $B \subset \Pi_k$, and we denote by $I(B)$, $E(B)$ the bounded and nonbounded regions of $\Pi_k - B$, then one can prove that the tangent hyperplane to M at every point of B is always transversal to Π_k. Moreover we have:

(A) If $M \geq \Pi_k$ near of B, then in a neighbourhood U of B in M, we have $p_\xi(U) \subset p_\xi(\overline{E(B)})$.

(B) If $\Pi_k \geq M$ near of B, then in a neighbourhood U of B in M, we have $p_\xi(U) \subset p_\xi(\overline{I(B)})$,

where by bar we indicate the closure of the corresponding subset.

Hence, using a topological argument, we obtain the following description of a compact IA-hypersphere.

Proposition 2. *Let M be a compact IA-hypersphere with affine normal vector $\xi = e_{n+1}$ and with boundary B. If $B = B_1 \cup B_2$, with $B_1 \subset \Pi_{k_1}$ and $B_2 \subset \Pi_{k_2}$, $k_1 > k_2$. Then B_1 and B_2 must be connected and $p_\xi(\overline{I(B_2)}) \subset p_\xi(I(B_1))$. Moreover M is globally the graph of the function f defined on the ring-shaped bounded domain given by*

$$\Omega = p_\xi(I(B_1)) - \overline{p_\xi(I(B_2))}$$

which satisfies

$$\begin{aligned} det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) &= 1, \quad on\ \Omega, \\ f &= k_i, \quad in\ p_\xi(B_i), \quad i = 1, 2. \end{aligned}$$

Remark 1. It is easy to see from here that if M is as in Proposition 2, then M lies on the boundary of a convex body. Therefore the shape of these hyperspheres is strongly determinated.

Since (**P**) is invariant under unimodular linear transformations of $\{x_1, x_2, ..., x_n\}$, using Proposition 2 and the Maximum Principle we have the following basic properties of symmetry.

Corollary 1. *Let B_1 and B_2 be compact $(n-1)$-hypersurfaces lying on two different parallel hyperplanes Π_1 and Π_2. We assume there is a hyperplane Σ and a unit vector $\vec{\gamma} \in \vec{\Pi}_1$ (which is transversal to Σ) such that $B = B_1 \cup B_2$ is invariant by the reflection through Σ parallel to $\vec{\gamma}$. If $\xi \in \vec{\Sigma}$ is a fixed vector transversal to Π_1, then any compact IA-hypersphere M with boundary B and with affine normal vector ξ must be invariant by the reflection through Σ parallel to $\vec{\gamma}$.*

Remark 2. The Corollary 1 says that M inherits the symmetry of its boundary. In particular, if B_1 and B_2 are two $(n-1)$-hyperspheres contained in two parallel hyperplanes we have

Corollary 2. *If B_1 and B_2 are two $(n-1)$-hyperspheres lying on two different parallel hyperplanes Π_1 and Π_2, then any compact IA-hypersphere with boundary $B_1 \cup B_2$ and with affine normal vector in the direction of the line joining the centers of the two hyperspheres is affinely equivalent to an IA-hypersphere of rotation.*

Now we are going to describe IA-hyperspheres of rotation with affine normal vector $\xi = e_{n+1}$. We denote by $\mathcal{S}^{n-1}$ the euclidean sphere of radius one with local parameters $u = (u^1, u^2, ..., u^{n-1})$ and position vector $\omega = \omega\,(u^1, u^2, ..., u^{n-1})$. With this notation we can parametrized affine hyperspheres of rotation in the following way

$$x(u, R) = (R\,\omega(u), g(R))$$

where $u \in \mathcal{S}^{n-1}$ and $R > 0$.

The affine metric for these affine hypersurfaces is given by

$$G = \left(\begin{array}{c|c} g'\,R\,h_{ij} & 0 \\ \hline 0 & g'' \end{array}\right)$$

where by prime we denote the derivative with respect to R and h_{ij} are the components of the euclidean metric on $\mathcal{S}^{n-1}$. If we impose on M the condition of IA-hypersphere with affine normal vector $\xi = e_{n+1}$, we find that g must be a curve of the family $\mathcal{G} = \{g_0\} \cup \{\widetilde{g}_c, c > 0\} \cup \{g_c, c > 0\}$, where

$$\begin{aligned} g_0(R) &= \frac{R^2}{2}, \quad R > 0, \\ g_c(R) &= \int_0^R (t^n + c^n)^{1/n} dt, \quad R > 0, \quad c > 0, \\ \widetilde{g}_c(R) &= \int_c^R (t^n - c^n)^{1/n} dt, \quad R > c > 0. \end{aligned}$$

Remark 3. The curve g_0 generates the elliptic paraboloid. The curves g_c satisfy $g'_c(0) = c > 0$, thus these curves generate IA-hyperspheres of rotation with a vertex. Finally, we remark that $\lim_{R\to c} \widetilde{g}'_c(R) = 0$ but $\widetilde{g}_c$ is not C^2 in $[c, \infty)$, thus $\widetilde{g}_c$ generate IA-hyperspheres of rotation with C^1-boundary.
Then we have in a similar way to Theorem 2 in [FMM] the following result:

Theorem 1 (Existence and Elasticity). *Let R_1, R_2 and r be positive real numbers with $R_1 < R_2$. Then there exists a curve g in $\mathcal{G}$ with $g(R_2) - g(R_1) = r$, if and only if $r > d(R_1, R_2)$, where*

$$d(R_1, R_2) = \int_{R_1}^{R_2} (t^n - R_1^n)^{1/n}\, dt$$

Remark 4. Theorem 1 says that for any positive real numbers R_1, R_2, r, with $R_1 < R_2$ and $r > d(R_1, R_2)$, we can always find an IA-hypersphere of rotation with affine normal vector $\xi = e_{n+1}$ which is bounded by the $(n-1)$-hyperspheres of radios R_1 and R_2 lying on parallel hyperplanes separated to a distance r. Furthermore, we observe that the affine volume of these hypersurfaces is

$$Vol(\mathcal{S}^{n-1})\, \frac{(R_2^2 - R_1^2)}{n}$$

which is independent of r. This property states an important difference between euclidean minimal surfaces and IA-hyperspheres.

Corollary 3. *In the ring-shaped domain Ω given by*

$$\Omega = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n | R_1^2 \le x_1^2 + x_2^2 + ... + x_n^2 \le R_2^2\},$$

there is a C^2-solution f of the problem

$$\begin{array}{lll} det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = 1 & on & \Omega \\ f = k_i & in & x_1^2 + x_2^2 + ... + x_n^2 = R_i^2, \;\; i = 1, 2\ , \end{array}$$

where k_1 and k_2 are positive constants, if and only if $k_1 < k_2$ and $k_2 - k_1 > d(R_1, R_2)$. In this case we have that f must be a radial function.

The last result in this part is a theorem about non existence of certain IA-hyperspheres whose boundary are not necessarily two $(n-1)$-hyperspheres.

Theorem 2 (Non existence). *Let B_1 and B_2 be two compact $(n-1)$-hypersurfaces such that $B_1 \subset \Pi_0 = \{x_{n+1} = 0\}$ and $B_2 \subset \Pi_0^+ = \{(x_1, x_2, ..., x_{n+1}) \mid x_{n+1} > 0\}$. Assume that R is a positive number such that B_1 is contained in $I(S_R)$, where S_R is the $(n-1)$-hypersphere with center at $(0, 0, ..., 0)$ and radius R in Π_0 and that B_2 is contained in the exterior of the IA-hypersphere of rotation M_R generated by the curve $\widetilde{g}_R$ (see Fig. 1). Then every compact IA-hypersphere M with affine normal $\xi = e_{n+1}$ and boundary $B_1 \cup B_2$ must be disconnected.*

Figure 1:

Proof. Suppose that M is connected. Then, for each $r > 0$ we are going to consider $\mathcal{T}_{r\xi} : \mathcal{A}^{n+1} \longrightarrow \mathcal{A}^{n+1}$ the translation of vector $r\xi$ and we denote $\mathcal{L}_r = \mathcal{T}_{r\xi}(M_R \cup \overline{I(\mathcal{S}_R)})$. It is clear that the set $Q = \{r > 0 | \mathcal{L}_r \geq M\}$ is closed and non empty. Thus there exists r_0 minimum of Q and $\mathcal{L}_{r_0} \cap M$ is not empty.

If we take $p \in \mathcal{L}_{r_0} \cap M$, since M lies on a convex body (see Remark 1) we have only two posibilities:

(a) p is interior to M and to $\mathcal{T}_{r_0\xi}(M_R)$.

(b) p is in $\mathcal{T}_{r_0\xi}(\mathcal{S}_R)$ and it is an interior point of M.

If (a) happens, from Proposition 1, M coincides with $\mathcal{T}_{r_0\xi}(M_R)$, which contradicts the assumption on B_2 .

If we have (b), then from Remark 2, T_pM must be an horizontal hyperplane which contradicts Remark 1 and the proof is finished.

3 Non compact IA-spheres with compact boundary

We refer the reader to [FMM] for proofs and more details about this section.

Definition 1. *A non compact IA-sphere M with compact boundary ∂M is said to be* **regular at infinity** *if there exists a compact subset K of M such that $M - K$ lies on the boundary of a convex set in $\mathcal{A}^3$.*

If M is regular at infinity it is easy to prove that M is the graph of a convex function f on Ω, where Ω is the exterior of a closed curve in Π_0. We are going to denote by M_f the graph of a function f and $\widetilde{\Omega}_R = \{z \in \mathbb{C} \mid |z| > R\}$, that is, the exterior of a complex disk of radius R.

We are going to define global isothermal coordinates for the affine metric of M_f. For it we consider the following transformation $L_f : \Omega \longrightarrow \mathbb{C}$, known as the Lewy transformation, given by

$$L_f(x_1,x_2) = (u,v) = \left(x_1 + \frac{\partial f}{\partial x_1}\right) + \mathrm{i}\left(x_2 + \frac{\partial f}{\partial x_2}\right).$$

Since M is locally strongly convex we can supose that $\frac{\partial^2 f}{\partial x_i \partial x_i}$ are positive functions for $i = 1, 2$. Then the Jacobian of L_f has determinant

$$1 + \sum_{i=1}^{2} \frac{\partial^2 f}{\partial x_i \partial x_i} + det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) > 2,$$

and so L_f is an immersion.

Using that M_f lies on the boundary of a convex set in $\mathcal{A}^3$ we can prove that L_f is distance increasing and so, $L_f : \Omega \longrightarrow \widetilde{\Omega}$ is a diffeomorphism, where $L_f(\Omega) = \widetilde{\Omega}$.

Moreover it is easy to prove that there exists $R > 0$ such that $\widetilde{\Omega}_R \subset \widetilde{\Omega}$. It allows us to define the function $F : \widetilde{\Omega}_R \longrightarrow \mathbb{C}$, given by

$$F(z) = \left(x_1 - \frac{\partial f}{\partial x_1}\right) + \mathrm{i}\left(-x_2 + \frac{\partial f}{\partial x_2}\right),$$

where $z = u + \mathrm{i}v$.

We have the following expression which relates f and F :

$$f(w) = \frac{1}{8}|z|^2 - \frac{1}{8}|F(z)|^2 + \frac{1}{4}\Re(zF(z)) - \frac{1}{2}\Re\int_{z_0}^{z} F(\zeta)d\zeta, \tag{1}$$

where $w = x_1 + \mathrm{i}x_2$ and $\Re$ denotes real part.

From this expression and the definition of F we have that F is an holomorphic function satisfying

(A) $|F'(z)| < 1, \quad z \in \widetilde{\Omega}_R$

(B) $\lim_{|z|\to\infty} F(z) = \mu, \quad \lim_{|z|\to\infty} F(z) - \mu\, z = \nu, \ \mu, \nu \in \mathbb{C}$

(C) $Residue[F, \infty]$ is a real number.

Moreover, after an affine transformation we can get that F verifies:

$$(\mathbf{B}') \qquad \lim_{|z|\to\infty} F(z) = \lim_{|z|\to\infty} F'(z) = 0.$$

We shall call F **the Lewy function** of f.

Conversely, if we have an holomorphic function $F : \widetilde{\Omega}_R \longrightarrow \mathbb{C}$ satisfying (**A**),(**B'**) and (**C**) we can define the transformation $T_F : \widetilde{\Omega}_R \longrightarrow \mathbb{C}$ given by

$$2T_F(z) = z + \overline{F(z)}, \qquad z \in \widetilde{\Omega}_R.$$

The expression (1) gives us a function f such M_f is an IA-sphere with affine normal vector $\xi = (0, 0, 1)$ and which is regular at infinity.

In this way we have an identification between this special type of IA-spheres and holomorphic functions on the exterior of a disk satisfying (**A**),(**B'**) and (**C**).

In the general case, if (**A**),(**B**) and (**C**) happen, F writes as:

$$F(z) = \mu z + \nu + \sum_{n=1}^{\infty} \frac{a_n}{z^n}, \qquad z \in \widetilde{\Omega}_S,$$

where $a_1 \in \mathbb{R}$.

For IA-spheres of rotation this function F is the simplest that we can hope. If M is generated by g_c, then the expression of F is:

$$F_c(z) = \frac{-c^2}{z}, \qquad z \in \widetilde{\Omega}_c,$$

and if M is generated by $\widetilde{g}_c$, then the expression of F is:

$$F_c(z) = \frac{c^2}{z}, \qquad z \in \widetilde{\Omega}_\lambda.$$

We have seen that IA-spheres which are regular at infinity are given by the graph of a convex function on the whole exterior of a disk. Therefore, if M is regular at infinity, M can not be asymptotic to an horizontal plane and it can not tend to infinity in a finite time.

After some calculations f can be expressed:

$$\begin{aligned} f(w) &= Q(f(w)) + \frac{1}{1-|\mu|^2}\left\{ax_1 + bx_2 + \frac{\nu_2 b - \nu_1 a}{4}\right\} - \\ &- \frac{a_1}{4}\log(|z|^2) + o(1), \end{aligned}$$

where

$$\begin{aligned} Q(f(w)) &= \frac{1}{2(1-|\mu|^2)}\left\{\left(1+|\mu|^2 - 2\mu_1\right)x_1^2\right\} + \\ &+ \frac{1}{2(1-|\mu|^2)}\left\{\left(1+|\mu|^2 + 2\mu_1\right)x_2^2 + 4\mu_2 x_1 x_2\right\} \quad , \end{aligned}$$

with $\nu = \nu_1 + \mathrm{i}\nu_2$, $\mu = \mu_1 + \mathrm{i}\mu_2$, $a = -\nu_1 + \mu_1\nu_1 + \mu_2\nu_2$, $b = \nu_2 - \mu_2\nu_1 + \mu_1\nu_2$ and $o(1)$ denotes a term which tends to a constant when $|w|$ tends to infinity.

By using this expression and a similar technique to [LR] we can prove (see [FMM]), the following results

Theorem 3 (Maximum Principle at infinity). *Let f_1 and f_2 be convex solutions of* (**P**) *on $\widetilde{\Omega}_R$ with $f_1 = f_2$ in $\partial\widetilde{\Omega}_R$. Suppose that the graphs M_{f_1} and M_{f_2} of f_1 and f_2, respectively, are regular at infinity and $f_1 \geq f_2$ on $\widetilde{\Omega}_S$ for some $S > R$. If there exists a sequence $\{w_n\}_{n\in\mathbb{N}}$ in $\widetilde{\Omega}_R$ with $\lim_{n\to\infty}|w_n| = \infty$ and*

$$\lim_{n\to\infty}|f_1(w_n) - f_2(w_n)| = 0,$$

then $f_1 = f_2$.

Let $\mathcal{G}$ be as in Sect. 2 and $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ the function given by $\psi(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$. Then one has,

Theorem 4. *Let f a convex solution of* $(\mathbf{P})$ *on* $\widetilde{\Omega}_R$. *Suppose that the graph M_f of f is regular at infinity and $f \geq g\psi$ on $\widetilde{\Omega}_S$ for some $S > R$, where $g \in \mathcal{G}$. If there exists a sequence $\{w_n\}_{n\in\mathbb{N}}$ in $\widetilde{\Omega}_R$ with* $\lim_{n\to\infty} |w_n| = \infty$ *and*

$$\lim_{n\to\infty} |f(w_n) - g\psi(w_n)| = 0,$$

then $f = g\psi$.

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