Conferencia de Rio 96

On the exterior Plateau problem of a Monge-Ampère equation

L. Ferrer A. Martínez F. Milán

The purpose of this lecture is to present some properties of solutions of the Monge-Ampère equation

$$(\mathbf{P}) \qquad det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = 1 \quad on \quad \Omega\,,$$

where Ω is the unbounded region of $\mathbb{C} \setminus \gamma$, and γ is a simple closed convex curve. Contrary to the case of bounded convex domain, very little is known about solutions of $(\mathbf{P})$ when the domain is unbounded.(We mention a famous result of Jörgens and Pogorelov:

Theorem (Jörgens, Pogorelov).- All solutions of $(\mathbf{P})$ on $\Omega = \mathbb{R}^2$ are quadratic).

We are interesed in two kind of properties:

I.- BEHAVIOUR AT INFINITY (UNICITY OF SOLUTION)

II.- SYMMETRY PROPERTIES.

1 Preliminaries

If $\chi : M \longrightarrow \mathbb{R}^3$ is an oriented immersed locally strongly convex surface in $\mathbb{R}^3$, that is with positive Euclidean Gauss curvature K_e and a unit normal Euclidean vector u_e such that the second fundamental form σ_e of the inmersion is positive definite everywhere, then the Riemannian metric

$$h = K_e^{\frac{-1}{4}} \sigma_e,$$

its element of area

$$dA = K_e^{\frac{1}{4}} dA_e$$

and the transversal vector field

$$\xi = \frac{1}{2} \Delta_h \chi \ ,$$

(where Δ_h is the Laplace-Beltrami operator associated with h) are the most basic unimodular affine invariants of the surface. They are called affine metric, element of affine area and affine normal respectively.

One of the more obvious problems in ADG is to study surfaces which are extremal for the affinely invariant area

$$\int dA = \int K_e^{\frac{1}{4}} dA_e \ ,$$

under interior deformations.

The Euler-Lagrange equation for this variational problem is equivalent to the following system of differential equations:

$$\Delta_h (K_e^{\frac{-1}{4}} u_e) = 0 \, .$$

The immersion $N = K_e^{\frac{-1}{4}} u_e : M \longrightarrow \mathbb{R}^3$ is called the affine conormal map of χ and consequently, χ is extremal if and only if N is a harmonic map.

In 1982, Calabi proved that the second variation of the affine area under interior deformations is negative and then he called this surfaces **affine maximal surfaces**.

2 Parabolic Affine spheres

An important subset of affine maximal surfaces are the "Parabolic affine spheres".

Definition 1 *An oriented immersed locally strongly convex surface* $\chi : M \longrightarrow \mathbb{R}^3$ *is a* **parabolic affine sphere** *if an only if* ξ *is constant.*

This condition is satisfied if an only if χ is affine maximal and N lies in a plane. Thus, up an unimodular affine transformation, we can assume that $\xi = (0,0,1)$, and then the parabolic affine spheres are given "locally" by the graphs of solutions $f : \Omega \longrightarrow \mathbb{R}$ of $(\mathbf{P})$. We will denote M_f the graph of f.

I.- BEHAVIOUR AT INFINITY

Definition 2 *A solution* f *of* $(\mathbf{P})$ *is called* **regular at infinity** *if there exists* $R > 0$ *such that the graph of* f *on* $\Omega_R = \{w \in \mathbb{C} \mid |w| > R\}$ *is a convex surface in* $\mathbb{R}^3$.

For these solutions we prove the following result of uniqueness:

Maximum Principle at infinity. *Let* f *and* g *be solutions of the problem* $(\mathbf{P})$ *on* Ω *which are regular at infinity. If* $f \geq g$ *on* Ω *and there exists a sequence* $\{w_n\}_{n\in\mathbb{N}}$ *in* Ω *with* $\lim_{n\to\infty} |w_n| = \infty$ *and* $\lim_{n\to\infty} |f(w_n) - g(w_n)| = 0$, *then* $f \equiv g$.

Outline of the proof:
If $f : \Omega \longrightarrow \mathbb{R}$ is a solution of $(\mathbf{P})$ which is regular at infinity, then we show that the transformation of Lewy $L_f : \Omega \longrightarrow \mathbb{C}$ given by

$$L_f(w) = w + 2\,\frac{\partial f}{\partial \overline{w}}\,, \tag{1}$$

where $w = x_1 + ix_2$ and by bar we will denote the complex conjugation, is a diffeomorphism on Ω_R for some $R > 0$. Thus the function $F : \widetilde{\Omega} = L_f(\Omega_R) \longrightarrow \mathbb{C}$ given by

$$F(z) = \overline{w} - 2\,\frac{\partial f}{\partial w}\,, \tag{2}$$

where $z = L_f(w)$, is well defined. This function is known as the **Lewy function** of f.

A straight computation using (1) and (2) shows that F is an analytic function on $\widetilde{\Omega}$ and it can be written on $\Omega_{\widetilde{R}} \subset \widetilde{\Omega}$, for some $\widetilde{R} > 0$, as

$$F(z) = \mu z + \nu + \sum_{n=1}^{\infty} \frac{a_n}{z^n}\,, \tag{3}$$

whith $|\mu| < 1$ and $a_1 \in \mathbb{R}$.

Moreover, the function f can be retraited as

$$f(w) = \mathcal{E}(f(w)) - \frac{a_1}{4}\log(|z|^2) + o(1), \tag{4}$$

$$
\begin{aligned}
\mathcal{E}(f(w)) &= \frac{1}{2(1-|\mu|^2)}\left\{\left(1+|\mu|^2-2\mu_1\right)x_1^2+\left(1+|\mu|^2+2\mu_1\right)x_2^2\right\}+ \\
&+ \frac{1}{1-|\mu|^2}\left\{2\mu_2 x_1 x_2 + a x_1 + b x_2 + \frac{\nu_2 b - \nu_1 a}{4}\right\},
\end{aligned}
$$

with $\nu = \nu_1 + i\nu_2$, $\mu = \mu_1 + i\mu_2$,
$a = -\nu_1 + \mu_1\nu_1 + \mu_2\nu_2$ and $b = \nu_2 - \mu_2\nu_1 + \mu_1\nu_2$.

The expression $o(|w|^n)$ will be used to indicate a term which is bounded in absolute value by a constant times $|w|^n$ for $|w|$ large.

When k is a large positive number, the ellipse

$$\mathcal{E}_k \equiv \mathcal{E}\left(f(w)\right) = k,$$

gives the shape of M_f at infinity. The ellipse $\mathcal{E}_k$ will be called **the ellipse at infinity** associated with f.

Conversely, given an analytic function F as above, one can construct a solution f of the Monge-Ampère equation $(\mathbf{P})$ on Ω_R for some $R > 0$.

Figure 1:

Since the behaviour of $f(w)$ at infinity depends on $\mathcal{E}\left(f(w)\right)$, we prove that if $f \geq g$ then $\mu(f) = \mu(g)$ and $\nu(f) = \nu(g)$, that is, the ellipses at infinity of f and g are the same. Consequently, if F and G are the Lewy functions of f and g, respectively. Using (3) expression, F and G can be written as

$$(6) \qquad F(z) = \mu z + \nu + \sum_{n=1}^{\infty} \frac{a_n}{z^n}\,, \qquad G(\hat{z}) = \mu\hat{z} + \nu + \sum_{n=1}^{\infty} \frac{b_n}{\hat{z}^n}\,,$$

where $a_1, b_1 \in \mathbb{R}$, $\nu, \mu \in \mathbb{C}$, $z = L_f(w)$ and $\hat{z} = L_g(w)$. From (1), (2), (4), (5) and (6) we obtain,

$$f(w) - g(w) = \frac{b_1 - a_1}{4}\log(|z|^2) + o(1)\,.$$

Hence by using that $\lim_{n\to\infty}|f(w_n) - g(w_n)| = 0$, one gets that $a_1 = b_1$.

Finally, using the above expression we prove

$$\begin{aligned}
\lambda_1 &= \frac{1}{2(1-q)}\mathcal{P}\left(\frac{b_q - a_q}{z^{q-1}}\right) + o(|z|^{-q})\,,\\
\lambda_2 &= o(|z|^{-2q})\,.
\end{aligned}$$

which is contrary to $f > g$ when $|z|$ is large.

Definition 3 *Let f be a solution of the problem* $(\mathbf{P})$ *which is regular at infinity and let F be the Lewy function of f. The real number a_1 given by the expression (3) is known as the* **logarithmic growth rate** *of f and it is denoted by $a(f)$.*

From this theorem we can deduce the following corollaries.

Corollary 1 *Let f and g be solutions of* $(\mathbf{P})$ *on Ω. Suppose that $f = g$ on $\partial\Omega$ and $f \geq g$ on Ω. If $a(f) = a(g)$, then $f \equiv g$.*

Corollary 2 *Let f and g be solutions of* $(\mathbf{P})$ *on Ω and let F and G be the Lewy functions of f and g, respectively. Suppose that $f = g$ on $\partial\Omega$, $a(f) = a(g)$, $\lim_{|z|\to\infty} F'(z) = \lim_{|\widehat{z}|\to\infty} G'(\widehat{z}) = \mu$, and $\lim_{|z|\to\infty}(F(z) - \mu z) = \lim_{|\widehat{z}|\to\infty}(G(\widehat{z}) - \mu\widehat{z})$, where $z = L_f(w)$, $\widehat{z} = L_g(w)$ and $w \in \Omega$. Then $f \equiv g$.*

II.- SYMMETRY PROPERTIES

We consider the following exterior Plateau problem

$$(\mathbf{Q}) \qquad \begin{cases} det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) &= 1 & on \;\; \Omega_1 \\ f &= const. & on \;\; \partial\Omega_1, \end{cases}$$

where $\Omega_1 = \{w \in \mathbb{C} \mid \; |w| > 1\}$.

After some rotations and aplying the former result to $g(x_1, x_2) = f(x_1, -x_2)$, we can obtain the following

Proposition 1 *Let f be a solution of the problem $(\mathbf{Q})$. Then M_f is invariant by a normal reflection through a plane Σ if and only if $\Sigma \cap \mathbb{C}$ is an axis of the ellipse at infinity of f.*

From here we have

Corollary 3 *Let f be a solution of $(\mathbf{Q})$. Then M_f is invariant by two different normal reflections if and only if $c = (0,0)$ is the center of the ellipse at infinity associated with f.*

Corollary 4 *Let f be a solution of the problem $(\mathbf{Q})$. Then M_f is invariant by three normal reflections if and only if M_f is a revolution surface.*