Results in Mathematics
Vol. 27 (1995)

0378-6218/95/040302-06$1.50+0.20/0

A CHARACTERIZATION OF THE COMPLEX PARABOLOID

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To Professor Katsumi Nomizu on his 70^{th} birthday

Abstract. In this paper we classify in a global way the umbilical affine definite surfaces in $\mathbf{R}^4$ with respect to the Nomizu-Vrancken affine normalization introduced in [NV]. We prove that an affine complete affine definite surface in $\mathbf{R}^4$ is umbilical if and only if it is affine equivalent to the complex paraboloid.

Mathematics Subject Classification: 53A15

Key Words: affine surfaces in $\mathbf{R}^4$, affine complete, affine umbilical, complex paraboloid.

1 Introduction

The affine metric of a surface in the unimodular affine 4-space $\mathbf{R}^4$ was first introduced by C. Burstin and W. Mayer in [BM] and recently rediscovered by E. Calabi as is mentioned in [L]. If the affine metric is a Riemannian metric, then the surface is called an *affine definite* surface. The class of these surfaces is large (most of the Euclidean minimal surfaces in $\mathbf{R}^4$ are, locally, affine definite surfaces).

In contrast to the affine metric there exist different approaches in the definition of an equiaffinely invariant normal bundle (see [BM], [K1], [K2], [NV]). Here we shall consider surfaces with the affine normalization introduced by K. Nomizu and L. Vrancken in [NV]. There exist two advantages which support this approach. First this normalization preserves the equiaffine structure of $\mathbf{R}^4$ and second the critical affine definite surfaces for the area integral of the affine metric are the affine definite surfaces with vanishing affine mean curvature vector with respect to the transversal plane of K. Nomizu and L. Vrancken (see [DMVV]).

Locally affine umbilical surfaces only are classified under additional assumptions (constant curvature, $\nabla^{\perp} g^{\perp} = 0$) by [MSV]. In the present note we classify in a global way

[1]Research partially supported by DGICYT Grant PB91-0731-Co3-02

umbilical affine definite surfaces and prove the following result:

Theorem. *An affine complete affine definite surface in* $\mathbf{R}^4$ *is umbilical if and only if it is affine equivalent to the complex paraboloid.*

In analogy with the affine theory of hypersurfaces, an affine definite surface in $\mathbf{R}^4$ is called an affine *improper sphere* if its affine normal planes are mutually parallel. It is not difficult to prove that an affine improper sphere in $\mathbf{R}^4$ is umbilical with vanishing affine mean curvature vector. Then we have the following result of Bernstein type:

Corollary. *Every affine complete affine improper sphere in* $\mathbf{R}^4$ *is affine equivalent to the complex paraboloid.*

2 Basic formulas

In this section we describe the normalization introduced by Nomizu-Vrancken in [NV] in complex coordinates.
Let $x : M \longrightarrow \mathbf{R}^4$ be an affine definite surface with affine metric g in the affine real 4-space provided with a volume element given by the usual determinant function, $Det = [.,.,.,.]$. Then M can be naturally regarded as a Riemannian surface and choosing local isothermal parameters (u,v) with respect to g, the affine metric can be written as

(1) $$g = F^2(z,\overline{z})|dz|^2,$$

where $z = u + iv$ and $dz = du + idv$.

If we use the Cauchy-Riemann operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right),$$

then the immersion satisfies

(2) $$\begin{aligned} [x_z, x_{\overline{z}}, x_{zz}, x_{\overline{z}z}] &= [x_z, x_{\overline{z}}, x_{z\overline{z}}, x_{\overline{z}\overline{z}}] = 0 \\ -4\,[x_z, x_{\overline{z}}, x_{zz}, x_{\overline{z}\overline{z}}] &= F^6, \end{aligned}$$

where $x_z = \frac{\partial x}{\partial z}$, $x_{\overline{z}} = \frac{\partial x}{\partial \overline{z}}$, $x_{zz} = \frac{\partial^2 x}{\partial z \partial z}$, etc.

From (2) and $x_{z\overline{z}} = \overline{x_{z\overline{z}}}$, we get

(3) $$x_{z\overline{z}} = \lambda x_z + \overline{\lambda} x_{\overline{z}}.$$

Moreover, if $\omega = log(F^2)$, then it follows by a straigthforward computation that the direction of ξ,

(4) $$\xi = x_{zz} - \left(\omega_z - \overline{\lambda}\right) x_z,$$

is independent of the choice of the complex parameter z.

Since

$$(5)\qquad [x_z, x_{\overline{z}}, x_{zz}, x_{\overline{zz}}] = \left[x_z, x_{\overline{z}}, \xi, \overline{\xi}\right] = -\frac{1}{4}e^{3\omega},$$

we obtain at any point $p \in M$ an affine invariant transversal plane N_pM given by

$$N_pM = \left\{\eta(p) = a\xi(p) + \overline{a}\overline{\xi}(p), a \in \mathbf{C}\right\}.$$

It is not difficult to check that the normal bundle

$$N(M) = \{(x(p), \eta(p)) | p \in M, \eta(p) \in N_pM\},$$

is the affine normal plane introduced by Nomizu-Vrancken in [NV].

On $N(M)$ there exists a canonical globally defined metric $g^{\perp}$ (see [NV]) given by

$$(6)\qquad g^{\perp}(\xi,\xi) = g^{\perp}(\overline{\xi},\overline{\xi}) = 0, \qquad g^{\perp}(\xi,\overline{\xi}) = \frac{1}{2}e^{2\omega}.$$

Definition. M is called an *umbilical* affine definite surface if x satisfies

$$(7)\qquad \begin{aligned}\left[\xi_{\overline{z}}, x_{\overline{z}}, \xi, \overline{\xi}\right] &= \left[x_z, \xi_z, \xi, \overline{\xi}\right] = 0,\\ \left[x_z, \xi_{\overline{z}}, \xi, \overline{\xi}\right] &= \left[\xi_z, x_{\overline{z}}, \xi, \overline{\xi}\right].\end{aligned}$$

From (3), (4) and using the identity $x_{zz\overline{z}} = x_{z\overline{z}z}$ we have

$$\begin{aligned}\xi_{\overline{z}} &= \left(\lambda x_z + \overline{\lambda} x_{\overline{z}}\right)_z - \left(\omega_{z\overline{z}} - \overline{\lambda}_{\overline{z}}\right) x_z - \left(\omega_z - \overline{\lambda}\right) x_{z\overline{z}}\\ &= \left(\lambda_z + \overline{\lambda}_{\overline{z}} + \lambda\overline{\lambda} - \omega_{z\overline{z}}\right) x_z + \left(\overline{\lambda}_z + \left(2\overline{\lambda} - \omega_z\right)\overline{\lambda}\right) x_{\overline{z}} + \lambda\xi.\end{aligned}$$

We will assume that M is umbilical. Thus, the above expression and (7) give

$$(8)\qquad \omega_{z\overline{z}} = \lambda_z + \overline{\lambda}_{\overline{z}} + \lambda\overline{\lambda},$$

$$(9)\qquad \xi_{\overline{z}} = \lambda\xi + \alpha x_{\overline{z}},$$

where $\alpha = \overline{\lambda}_z + 2\overline{\lambda}^2 - \omega_z\overline{\lambda}$. If we compute $[x_z, x_{\overline{z}}, \xi, \overline{\xi}]_z$, using (3), (4), (5), (9) and $\overline{\xi}_z = \overline{\xi_{\overline{z}}}$, $\overline{x_{\overline{z}}} = x_z$, then we get

$$\left[x_z, x_{\overline{z}}, \xi_z, \overline{\xi}\right] = -\frac{1}{4}e^{3\omega}\left(2\omega_z - \overline{\lambda}\right).$$

Thus we obtain for ξ_z from (5), (7), (9) and the last expression, the representation

$$(10)\qquad \xi_z = \alpha x_z + \left(2\omega_z - \overline{\lambda}\right)\xi + B\overline{\xi}.$$

From (6), (9) and (10), the vector field

$$(11)\qquad H = -2e^{-2\omega}(\alpha\xi + \overline{\alpha}\overline{\xi}),$$

is independent of the choice of the complex parameter z. H is called *affine mean curvature vector* of the immersion and it is equivalent to the definition used in [MSV].

Remarks:

1. If M is affine harmonic (that is, $x_{\bar{z}z} = 0$), then $N(M)$ coincides with the affine invariant normal bundle introduced by Burstin-Mayer in [BM]. If M is affine harmonic and $\nabla g = 0$ (see [NV]), then $N(M)$ coincides with the affine normal introduced by Klingenberg in [K1], [K2].

2. It is clear that the umbilical condition is independent of the choice of the complex parameter. Furthermore it is equivalent to the definition used in [MSV].

3. If all the affine normal planes N_pM through each point $p \in M$ either intersect at one line or else are mutually parallel, then M is umbilical.

4. An example of umbilical affine definite surface which is not affine equivalent to the complex paraboloid is (see [MSV]):

$$x(u,v) = \left(\frac{3}{4}vu^{\frac{4}{3}} + \frac{1}{9}v^3, u^{\frac{4}{3}} + \frac{4}{9}v^2, v, \frac{3}{4}u^2\right).$$

3 Proof of the theorem

As the complex paraboloid is umbilical and affine complete, we only need to prove that an umbilical, affine complete definite surface in $\mathbf{R}^4$ must be affine equivalent to the complex paraboloid.

STEP I: A FORMULA FOR THE GAUSS CURVATURE κ OF g

The identity $\xi_{z\bar{z}} = \xi_{\bar{z}z}$, (9) and (10) give

$$\begin{aligned}
&(12) & \alpha_z &= \alpha\left(2\omega_z - \overline{\lambda}\right) + B\overline{\alpha} \\
&(13) & \lambda\alpha &= \alpha_{\bar{z}} \\
&(14) & \lambda_z &= \left(2\omega_z - \overline{\lambda}\right)_{\bar{z}} + B\overline{B} \\
&(15) & \lambda B &= B_{\bar{z}} + B\left(2\omega_{\bar{z}} - \lambda\right).
\end{aligned}$$

Thus (8) and (14) give,

$$\kappa = -2e^{-\omega}\omega_{z\bar{z}} = 2e^{-\omega}\left(B\overline{B} + \lambda\overline{\lambda}\right) \geq 0. \tag{16}$$

Now, if we take on M the nonnegative function f defined by

$$f = \frac{1}{2}\kappa - \frac{1}{16}g(\Delta x, \Delta x),$$

where by Δ we denote the Laplacian operator of g. Then from (1), (3) and (16) we can write,

$$f = B\overline{B}e^{-\omega}, \tag{17}$$

and using (15) and (17) we have,

$$(18) \qquad \begin{aligned} \frac{1}{4}\Delta f &= e^{-\omega}\left(B\overline{B}e^{-\omega}\right)_{\overline{z}z} = (2\lambda_z - 3\omega_{\overline{z}z})\, B\overline{B}e^{-2\omega} \\ &+ e^{-2\omega}(2\lambda - 3\omega_{\overline{z}})\left(B_z\overline{B} + B\overline{B}_z\right) + B_z\overline{B}_{\overline{z}}e^{-2\omega} \\ &- e^{-2\omega}\omega_z(2\lambda - 3\omega_{\overline{z}})\, B\overline{B} + B\overline{B}_{\overline{z}z}e^{-2\omega} \\ &- e^{-2\omega}\omega_z B\overline{B}_{\overline{z}}. \end{aligned}$$

Since, from (15), $\overline{B}_z = 2\overline{B}\left(\overline{\lambda} - \omega_z\right)$, then from the last expression, (8), (14) and (17), we get

$$(19) \qquad \begin{aligned} \frac{1}{4}\Delta f &= e^{-2\omega}\left\{B\overline{B}\left(3B\overline{B} + \lambda\overline{\lambda}\right) + B_z\overline{B}_{\overline{z}}\right. \\ &+ B\overline{B}(2\lambda - 3\omega_{\overline{z}})\left(2\overline{\lambda} - 3\omega_z\right) \\ &+ \left.(2\lambda - 3\omega_{\overline{z}})\, B_z\overline{B} + B\overline{B}_{\overline{z}}\left(2\overline{\lambda} - 3\omega_z\right)\right\} \\ &\geq e^{-2\omega}\left\{3\left(B\overline{B}\right)^2 + \left|B_z + B\left(2\overline{\lambda} - 3\omega_z\right)\right|^2\right\} \\ &\geq 3f^2. \end{aligned}$$

Since M is affine complete with affine Gauss curvature $\kappa \geq 0$, we conclude from Theorem 8 of [CY] that $f \equiv 0$ on M, and (16) can be written as

$$(20) \qquad \kappa = -2e^{-\omega}\omega_{z\overline{z}} = 2e^{-\omega}\lambda\overline{\lambda}.$$

STEP II: THE AFFINE MEAN CURVATURE VECTOR H OF M

From (12) and (13), $\alpha_z = \alpha\left(2\omega_z - \overline{\lambda}\right)$ and $\alpha_{\overline{z}} = \alpha\lambda$. Thus,

$$\begin{aligned} \left(\alpha\overline{\alpha}e^{-2\omega}\right)_z &= e^{-2\omega}\left(\alpha_z\overline{\alpha} + \alpha\overline{\alpha}_z - 2\alpha\overline{\alpha}\omega_z\right) = \\ &= e^{-2\omega}\alpha\overline{\alpha}\left(2\omega_z - \overline{\lambda} + \overline{\lambda} - 2\omega_z\right) = 0, \end{aligned}$$

and since $\alpha\overline{\alpha}e^{-2\omega}$ is real valued, it must be constant. Thus, from (6) and (11) we conclude that $g^{\perp}(H,H)$ is constant on M. Therefore, H must be zero on M by Theorem 2 in [MSV].

STEP III: ESTIMATION OF $\Delta\kappa$ AND CONCLUSION

As $H = 0$, then $\overline{\alpha} = 0$ and

$$\lambda_{\overline{z}} = -2\lambda^2 + \lambda\omega_{\overline{z}}$$

and from (8) and (20), we have

$$\left(\lambda\overline{\lambda}e^{-\omega}\right)_{\overline{z}z} = e^{-\omega}\left(11\lambda^2\overline{\lambda}^2 + \lambda_z\overline{\lambda}_{\overline{z}}\right).$$

The last expression and (20), give

$$\Delta\kappa = 2\Delta\left(|\lambda|^2 e^{-\omega}\right) \geq 88\left(|\lambda|^2 e^{-\omega}\right)^2 = 22\kappa^2.$$

Now, using again the Theorem 8 in [CY] we conclude $\kappa \equiv 0$ and M is an affine harmonic, affine flat definite surface with $H = 0$ and it has parallel affine normal planes.

The theorem follows from the structure equations, from [MSV], Theorem 1, or [VW], Theorem 1.

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Eingegangen am 28. Oktober 1994