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ON THE PICK INVARIANT, THE AFFINE MEAN CURVATURE AND THE GAUSS CURVATURE OF AFFINE SURFACES

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INTRODUCTION

The Affine Theorema Egregium for nondegenerate surfaces M^2 in $\mathbb{R}^3$ states that

$$\kappa = H + J,$$

where κ is the Gauss curvature of the affine metric, H is the affine mean curvature and J is the Pick invariant. In this paper we study affine surfaces such that κ, H, and J are constant.

The first result in this direction probably was obtained by J. Radon [R] in 1918. He classified all surfaces with $J = 0$ and H constant. In particular, if the Pick invariant J vanishes and the affine metric is indefinite, then there is an open dense subset U of M^2 such that every connected component of U is ruled. If moreover H is constant, then a classification of such ruled surfaces is given, see also [B], [MM] and [SS]. In section 4 we will give an alternative description and show that M^2 is globally ruled. Note that if the affine metric is definite and $J = 0$, then the Berwald theorem states that M^2 is a part of a quadric.

If M^2 is an affine sphere, then H is automatically constant. Affine spheres with affine metric of constant curvature are classified by U. Simon in [S], hereby generalizing results of M. Magid and P. Ryan [MR1], A.-M. Li and G. Penn [LP], T. Kurose [K] and J. Radon [R]. Here it should be remarked that one can investigate the analoguous problem in higher dimensions. In fact, it is shown in [VLS] that a locally strongly convex hypersphere of $\mathbb{R}^{n+1}$ has affine metric of constant sectional curvature if and only if it is either a part of a quadric or a part of the hypersurface with equation $x_1x_2\ldots x_{n+1} = c$. For hypersurfaces with indefinite metric, only few is known up to today [MR2].

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The complete list of (2-dimensional) affine spheres with constant curvature metric (or equivalently, constant Pick invariant) will be recalled in section 2. Obviously it contains all quadrics. Further in this section we will give some examples of affine surfaces which are not affine spheres but whose affine metric is flat. One of those examples has definite affine metric which moreover is complete. This provides a counterexample for a conjecture of A.-M. Li.

In section 3 we will assume that the Pick invariant J is a nonzero constant. We will show that such a surface with constant Gauss curvature such that $3\kappa \neq H$ has to be an affine sphere. However, the situation changes drastically if $3\kappa = H$. Indeed, there are examples of affine surfaces with constant κ and H, satisfying $3\kappa = H$, which are not affine spheres. We will give a complete classification. Some examples will be given by their affine metrics and induced connections. An application of Radon's theorem ensures the existence.

Originally, the results contained in this paper were obtained independently by some of the authors separately. At the meeting in Oberwolfach we learned that we had been studying the same subject, and obtained similar results. We decided that it would be better not to scatter our results around in different papers, so we agreed to write one larger paper, containing all results. Therefore sometimes we will give two proofs, if we believe the different techniques both could be used to tackle other problems in this field. Since some of the original papers already had been submitted, and accepted, for publication, we hereby apologize to the editors and the referees of the journals concerned for the inconvenience caused by our decision to withdraw from publication.

§1. Preliminaries

We will give a short introduction to the classical affine differential geometry of surfaces. For more detailed information the reader is refered to [N]. Let M be a nondegenerate surface in the 3-dimensional real affine space $\mathbb{R}^3$. We denote the usual affine connection on $\mathbb{R}^3$ by D, and we fix a parallel volume form ω on $\mathbb{R}^3$.

Let ξ be any transversal vector field, defined in the neighbourhood of an arbitrary point. If X and Y are tangent vector fields to M, then we can decompose $D_X Y$ into a tangent part and a part in the direction of ξ, in the following way

$$D_X Y = \nabla_X Y + h(X, Y)\xi.$$

Then ∇ is a torsion free affine connection and h is a nondegenerate symmetric bilinear form on M. We can define a volume form ω_ξ on M by

$$\omega_\xi(X, Y) = \omega(X, Y, \xi).$$

Then we can choose ξ uniquely, up to sign, satisfying the following properties.

(1.1) $\qquad \nabla \omega_\xi = 0$

(1.2) $\qquad \omega_\xi = \nu_h$ (ν_h is the metric volume form of h)

Condition (1.1) means that $D_X\xi$ is tangent to M for all X, such that we can define a (1,1)-tensor field S, called the affine shape operator, on M by $SX = -D_X\xi$. Condition

(1.2) implies that

$$\text{trace}_h \{(X,Y) \mapsto (\nabla h)(Z,X,Y)\} = 0 \tag{1.3}$$

for all vectors Z, where trace_h denotes the trace with respect to h. The symmetric 2-form h is called the affine metric. It is known that h is definite if and only if M^2 is locally strongly convex. If h is definite, we can (and will) always suppose that h is positive definite. The first equation of Codazzi states that the (0,3)-tensor field ∇h is symmetric in all three components:

$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z).$$

Condition (1.3) is called the apolarity condition. The second equation of Codazzi states that

$$(\nabla_X S)(Y) = (\nabla_Y S)(X),$$

and the equation of Gauss states that

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY,$$

where R is the Riemann-Christoffel curvature tensor of ∇.

Since the affine metric h is nondegenerate, h has a Levi Civita connection $\widehat{\nabla}$. Let κ be the Gaussian curvature of $\widehat{\nabla}$. Then

$$\kappa = H + J,$$

where $H = \frac{1}{2}\,\text{trace}\, S$ is called the affine mean curvature and J is called the Pick invariant and is defined by $J = \frac{1}{8}h(\nabla h, \nabla h)$. We also define the affine Gauss-Kronecker curvature τ by $\tau = \det S$.

The difference tensor K is a (1,2)-tensor field defined by

$$K(X,Y) = K_X Y = \nabla_X Y - \widehat{\nabla}_X Y.$$

It is related to ∇h in the following way.

$$h(K(X,Y),Z) = -\frac{1}{2}(\nabla h)(X,Y,Z).$$

Hence the apolarity condition can also be expressed by

$$\text{trace}\, K_X = 0.$$

The equation of Gauss and the second equation of Codazzi change to

$$\widehat{R}(X,Y)Z = \tfrac{1}{2}\left(h(Y,Z)SX - h(X,Z)SY + h(SY,Z)X - h(SX,Z)Y\right) - [K_X, K_Y]\,Z$$

and

$$(\widehat{\nabla}_X S)(Y) - (\widehat{\nabla}_Y S)(X) = K(SX,Y) - K(X,SY),$$

where $\widehat{R}$ is the curvature tensor of $\widehat{\nabla}$.

Now we take a local orthonormal frame $\{E_1, E_2\}$ on M^2, that is

$$h(E_1, E_1) = 1, \qquad h(E_1, E_2) = 0, \qquad h(E_2, E_2) = \epsilon,$$

where $\epsilon = 1$ if h is definite and $\epsilon = -1$ if h is indefinite. Then there are functions p and q such that

$$\begin{aligned} \widehat{\nabla}_{E_1} E_1 &= pE_2, & \widehat{\nabla}_{E_2} E_2 &= qE_1, \\ \widehat{\nabla}_{E_1} E_2 &= -\epsilon p E_1, & \widehat{\nabla}_{E_2} E_1 &= -\epsilon q E_2, \end{aligned} \tag{1.4}$$

so that

$$\kappa = \epsilon q_1 + p_2 - \epsilon p^2 - q^2, \tag{1.5}$$

where $(\ )_1$ and $(\ )_2$ denote the covariant derivatives with respect to E_1 and E_2. From the symmetry properties of K and the apolarity, we can conclude that there are functions A and B such that

$$\begin{aligned} K(E_1, E_1) &= AE_1 + BE_2, \\ K(E_1, E_2) &= \epsilon BE_1 - AE_2, \\ K(E_2, E_2) &= -\epsilon AE_1 - \epsilon BE_2, \end{aligned} \tag{1.6}$$

and therefore

$$J = 2(A^2 + \epsilon B^2). \tag{1.7}$$

The Ricci equation implies that there are functions α and β such that

$$SE_1 = (H + \alpha)E_1 + \beta E_2, \qquad SE_2 = \epsilon\beta E_1 + (H - \alpha)E_2. \tag{1.8}$$

Using these expressions, we can compute the equation of Gauss and the second equation of Codazzi and obtain

$$\begin{aligned} \epsilon B_1 - A_2 &= -\epsilon\beta - 3(\epsilon pA - qB), \\ A_1 + B_2 &= -\alpha + 3(\epsilon pB + \epsilon qA), \\ \epsilon\beta_1 - (H + \alpha)_2 &= 2(\epsilon\alpha B - \epsilon\beta A + q\beta - \epsilon p\alpha), \\ \beta_2 + (\alpha - H)_1 &= 2(\epsilon\beta B + \alpha A + \epsilon p\beta + \epsilon q\alpha). \end{aligned} \tag{1.9}$$

Affine spheres with constant Gaussian curvature. M^2 is called an affine sphere if S is proportional to the identity, so if $S = HI$. If moreover κ is constant, then either M^2 is a quadric or M^2 is affine equivalent to one of the following surfaces.

(1) The surface with equation $xyz = 1$, which has definite affine metric, $H < 0$ and $\kappa = 0$.
(2) The surface with equation $x(y^2 + z^2) = 1$, which has indefinite affine metric, $H \neq 0$ and $\kappa = 0$.
(3) The surface with equation $z = xy + f(y)$, where f is any function. This surface has indefinite affine metric, $H = 0$ and $\kappa = 0$.

Clearly a surface of the third kind is ruled.

Ruled surfaces with constant mean curvature. If M^2 is a ruled surface, then the Pick invariant J vanishes. So any ruled surface with constant mean curvature has affine metric of constant curvature. In §4 we will show that a ruled surface with constant mean curvature H is given by

$$x(u, v) = uf(v) + g(v), \tag{2.1}$$

where f and g are curves satisfying the following two conditions:

$$\det[fg'f'] = 1 \text{ and } \det[ff'f''] = H. \tag{2.2}$$

Surfaces with $3\kappa = H$. Surfaces with $3\kappa = H$ play a fundamental role in the study of surfaces with constant mean curvature and constant Pick invariant. From [V] we get that the surfaces

$$(u, v, \frac{1}{2}(u^2 + \epsilon v^{-\frac{2}{3}})), \qquad \epsilon = \pm 1 \tag{2.3}$$

have H and τ constant, and moreover $3\kappa = H$. The main result of [V] is a classification of all surfaces with constant H and τ. From this classification and from the calculations in the paper, it follows immediately that the surfaces (2.3) are the only ones which also have constant Pick invariant $J \neq 0$, and which are not affine spheres.

In §3, we will give other examples of surfaces with constant Pick invariant and constant mean curvature, satisfying $3\kappa = H$. These surfaces will be given by their affine metric and induced connection, such that the fundamental equations are satisfies. By the fundamental theorem of Radon, see [DNV] and [D], the surfaces can be realized as surfaces in $\mathbb{R}^3$ with that particular affine metric and induced connection. One of the examples still can be given explicitly, but it seems difficult to give an analytical description of the other.

A complete flat locally strongly convex surface. One could ask what happens if we weaken the condition to surfaces having affine metric of constant Gauss curvature. There are a lot of surfaces with flat affine metric. The following four examples are affine rotation surfaces in the sense of [SS, p. 193].

$$x_1(u,v) = (\cos(3u)^{\frac{1}{3}}\cos(\sqrt{3}v), \cos(3u)^{\frac{1}{3}}\sin(\sqrt{3}v), \int_0^u \cos(3t)^{-\frac{2}{3}}\,dt),$$
$$x_2(u,v) = (\cosh(3u)^{\frac{1}{3}}\cos(\sqrt{3}v), \cosh(3u)^{\frac{1}{3}}\sin(\sqrt{3}v), \int_0^u \cosh(3t)^{-\frac{2}{3}}\,dt),$$
$$x_3(u,v) = (\cos(3u)^{\frac{1}{3}}\cosh(\sqrt{3}v), \cos(3u)^{\frac{1}{3}}\sinh(\sqrt{3}v), \int_0^u \cos(3t)^{-\frac{2}{3}}\,dt),$$
$$x_4(u,v) = (\cosh(3u)^{\frac{1}{3}}\cosh(\sqrt{3}v), \cosh(3u)^{\frac{1}{3}}\sinh(\sqrt{3}v), \int_0^u \cosh(3t)^{-\frac{2}{3}}\,dt).$$

The last surface x_4 is locally strongly convex, and its affine metric h is complete and flat. This gives a counterexample to a conjecture of A.M. Li, which stated that a locally strongly convex surface with complete flat affine metric is a paraboloid or the surface with equation $xyz = 1$–we learned about this conjecture during the talk of Wang at the conference in Oberwolfach.

§3 Surfaces with nonzero Pick invariant

In this section we suppose that the Pick invariant J is nowhere zero. In this case, we can suppose that J is a positive function (if J is negative, then h is indefinite and we can change the sign of the affine normal, so that J becomes positive). Later in this section, we will assume that J is constant.

Surfaces with $3\kappa \neq H$.

Let M^2 be an affine surface in $\mathbb{R}^3$ and let $\{E_1, E_2\}$ be a local orthonormal basis with respect to the affine metric h, like in §1. First we show that we can always choose $\{E_1, E_2\}$ such that the function B vanishes.

Lemma 3.1. *Let M^2 be a surface with $J > 0$, then in the neighbourhood of every point there is a local orthonormal basis $\{E_1, E_2\}$ such that*

$$h(K(E_1, E_1), E_2) = 0.$$

Proof. We make the following usual definitions:

$$c(\theta) = \begin{cases} \cos(\theta) & \text{if } \epsilon = 1 \\ \cosh(\theta) & \text{if } \epsilon = -1 \end{cases}, \qquad s(\theta) = \begin{cases} \sin(\theta) & \text{if } \epsilon = 1 \\ \sinh(\theta) & \text{if } \epsilon = -1 \end{cases}.$$

If we put

$$E_1(\theta) = c(\theta)E_1 + \epsilon s(\theta)E_2, \qquad E_2(\theta) = -s(\theta)E_1 + c(\theta)E_2,$$

then $\{E_1(\theta), E_2(\theta)\}$ is again a local orthonormal basis.

We can introduce functions A and B like in §1 for $\{E_1, E_2\}$ and functions $A(\theta)$ and $B(\theta)$ for $\{E_1(\theta), E_2(\theta)\}$ for every θ. It can be computed immediately that

$$A(\theta) = c(3\theta)A + s(3\theta)B, \qquad B(\theta) = -\epsilon s(3\theta)A + c(3\theta)B. \tag{3.1}$$

Since we assume that $J \neq 0$, A and B do not vanish simultaneously we can suppose that $A \neq 0$. Moreover, since $J > 0$, we get that $1 > -\frac{B^2}{\epsilon A^2}$. If $\epsilon = -1$, this last inequality implies that we can find θ_0 such that $\tanh(3\theta_0) = -\frac{B}{A}$. If $\epsilon = 1$, then we can always find θ_0 such that $\tan(3\theta_0) = \frac{B}{A}$. From (3.1), we get that $B(\theta_0) = 0$. □

From now on we assume that H and J are constant. We choose a basis such that $B = 0$. Then, since $J = 2A^2$, A is a nonzero constant. From the equations (1.9) we get that

$$\begin{aligned} \alpha &= 3\epsilon qA, \qquad \beta = -3pA, \\ p_1 + q_2 &= 4\epsilon pq - 2pA, \\ q_1 - \epsilon p_2 &= -2p^2 + 2\epsilon q^2 + 2Aq. \end{aligned} \tag{3.2}$$

These equations imply

$$\begin{aligned} 2\epsilon q_1 &= \kappa + 2\epsilon Aq + 3q^2 - \epsilon p^2, \\ 2p_2 &= \kappa - 2\epsilon Aq - q^2 + 3\epsilon p^2. \end{aligned} \tag{3.3}$$

Differentiating gives us, where $(\ )_{12}$ means $E_2E_1(\ )$,

$$\begin{aligned} p_{11} + q_{21} &= 4\epsilon p_1 q + 4\epsilon p q_1 - 2p_1 A, \\ q_{12} &= Aq_2 + 3\epsilon q q_2 - pp_2. \end{aligned} \tag{3.4}$$

Since ∇ is torsion free, we obtain from (1.4) that

$$f_{21} - f_{12} = (\nabla_{E_1} E_2) f - (\nabla_{E_2} E_1) f = -\epsilon p f_1 + \epsilon q f_2, \tag{3.5}$$

for any function f. Using this fact for q and combining it with the other equations, we get

$$p_{11} = 2pA^2 - Ap_1 + 4\epsilon q p_1 + 3p\kappa + 7pq^2 - \epsilon p^3 - 4\epsilon q q_2.$$

Now we again take derivatives and get

$$p_{112} = 2p_2A^2 - Ap_{12} + 4\epsilon q_2 p_1 + 4\epsilon q p_{12} + 3p_2\kappa + 7p_2q^2 + 14pqq_2 - 3\epsilon p^2 p_2 - 4\epsilon q_2^2 - 4\epsilon q q_{22}.$$

Deriving (3.2) and (3.3) we also obtain

$$\begin{aligned} p_{12} + q_{22} &= -2p_2 A + 4\epsilon p_2 q + 4\epsilon p q_2, \\ p_{21} &= -\epsilon A q_1 - q q_1 + 3\epsilon p p_1. \end{aligned}$$

Using (3.5) for p, a straightforward calculation gives us

$$p_{112} = \frac{3}{2}A^2\kappa - \epsilon A^3 q - \frac{31}{2}A^2q^2 + \frac{5}{2}\epsilon A^2p^2 - 2\epsilon Aq\kappa + 20qAp^2 - 6\epsilon Aq^3 - 4\epsilon App_1 + 4\epsilon q_2p_1$$
$$- 14q^2\kappa - 20\epsilon q^2p^2 - \frac{7}{2}q^4 + 32qpp_1 + \frac{3}{2}\kappa^2 + 3\epsilon p^2\kappa - 2pqq_2 - \frac{9}{2}p^4 - 4\epsilon q_2^2.$$

On the other hand, we can apply (3.5) for $f = p_1$, and get another expression for p_{112}:

$$p_{112} = -2q^2\kappa + 36\epsilon p^2q^2 - \frac{7}{2}q^4 + 14qpp_1 - 2\epsilon Aq\kappa - \frac{7}{2}A^2q^2 - 6\epsilon Aq^3 + \frac{21}{2}\epsilon A^2p^2 - 4\epsilon App_1$$
$$+ 15\epsilon p^2\kappa - \frac{9}{2}p^4 - \frac{1}{2}\kappa^2 - 20pqq_2 + 4\epsilon p_1^2 - \frac{1}{2}A^2\kappa - \epsilon A^3 q.$$

Comparing both expressions for p_{112}, we can prove the following formula:

$$3(p_1 - q_2)^2 = (A^2 + \kappa)(2\epsilon\kappa - 12(p^2 + \epsilon q^2)). \tag{3.6}$$

Lemma 3.2. *Let M^2 be an affine surface with constant mean curvature and constant Pick invariant $J > 0$, then the affine Gauss-Kronecker curvature τ is constant, or else $3\kappa = H < 0$.*

Proof. We proceed with the same notations as above. First we observe that $3\kappa = H$ if and only if $\kappa + A^2 = 0$. Hence we assume that $\kappa + A^2 \neq 0$. Then we can compute p_1 and q_2 from (3.2) and (3.6), and obtain that

$$\begin{aligned} 2p_1 &= -2pA + 4\epsilon pq + c\left(\frac{2}{3}\epsilon\delta\kappa - 4\delta R\right)^{\frac{1}{2}}, \\ 2q_2 &= -2pA + 4\epsilon pq - c\left(\frac{2}{3}\epsilon\delta\kappa - 4\delta R\right)^{\frac{1}{2}}, \end{aligned} \tag{3.7}$$

where $\delta = 1$ if $A^2 + \kappa > 0$ and $\delta = -1$ if $A^2 + \kappa < 0$, $c^2 = \delta(A^2 + \kappa)$ and $R = p^2 + \epsilon q^2$. Since $\tau = H^2 - 9\epsilon A^2 R$, we see that τ is constant if and only if R is constant. Having expressions for p_1 and p_2, we can compute p_{12} and p_{21}, and obtain

$$2(p_{12} - p_{21}) = \frac{1}{2}c\left(\frac{2}{3}\epsilon\delta\kappa - 4\delta R\right)^{-\frac{1}{2}}(-8\delta pp_2 - 8\delta\epsilon qq_2) - 2p_2A$$
$$+ 4\epsilon p_2q + 4\epsilon pq_2 + 2\epsilon Aq_1 + 2qq_1 - 6\epsilon pp_1.$$

Applying (3.5) for p as before, we find another expression for $p_{12} - p_{21}$, and we can substitute the expressions for p_1, p_2, q_1, q_2, and obtain the following equation, after using the initial assumption $c \neq 0$:

$$0 = 6\epsilon qc\left(\frac{2}{3}\epsilon\delta\kappa - 4\delta R\right)^{\frac{1}{2}} - 6p\kappa + 18\epsilon p^3 + 18q^2p + 8\epsilon qpA. \tag{3.8}$$

Similarly, we can start from the expressions for q_1 and q_2, compute q_{12} and q_{21}, and obtain

$$0 = -6pc\left(\frac{2}{3}\epsilon\delta\kappa - 4\delta R\right)^{\frac{1}{2}} - 6q\kappa + 18\epsilon qp^2 + 18q^3 + 4p^2A - 4\epsilon q^2A. \tag{3.9}$$

Adding p times (3.8) to ϵq times (3.9) gives us

$$3R\kappa - 9\epsilon R^2 = 4\epsilon p^2qA + 2qA(\epsilon p^2 - q^2). \tag{3.10}$$

Adding q times (3.8) to $-p$ times (3.9) we find

$$3Rc\left(\frac{2}{3}\epsilon\delta\kappa - 4\delta R\right)^{\frac{1}{2}} = -4\epsilon q^2pA + 2pA(p^2 - \epsilon q^2). \tag{3.11}$$

Now, by adding the square of (3.10) to ϵ times the square of (3.11), we obtain

$$0 = R^2(81R^2 - (90\epsilon\kappa + 40\epsilon A^2)R + 15\kappa^2 + 6\kappa A^2),$$

which clearly implies that R is a constant. □

It is now easy to prove the first theorem.

Theorem 3.1. *Let M^2 be an affine surface in $\mathbb{R}^3$ with constant mean curvature and constant Pick invariant $J > 0$, then either $3\kappa = H < 0$ or else M^2 is a flat affine sphere.*

Perhaps it is interesting to state one special case in the following corollary.

Corollary 3.1. *Let M^2 be a locally strongly convex affine surface in $\mathbb{R}^3$ with constant mean curvature and constant Pick invariant. If $\kappa > 0$, then M^2 is an ellipsoid.*

Surfaces with $3\kappa = H$.

If $3\kappa = H$, then from (1.5), (3.3) and (3.6) we have that functions p and q given in (1.4) satisfy the following system of differential equations

$$(*)\qquad \begin{aligned} p_1 &= q_2 = 2\epsilon pq - pA,\\ 2\epsilon q_1 &= -A^2 + 2\epsilon Aq + 3q^2 - \epsilon p^2,\\ 2p_2 &= -A^2 - 2\epsilon Aq - q^2 + 3\epsilon p^2. \end{aligned}$$

Conversely, a straightforward computation shows the following lemma.

Lemma 3.3. *Let (M,h) be a semi-Riemannian surface with Levi-Civita connection $\hat{\nabla}$. We suppose that there exists a local orthonormal frame for h, $\{E_1, E_2\}$ such that if we write*

$$\hat{\nabla}_{E_1}E_1 = pE_2, \qquad \hat{\nabla}_{E_1}E_2 = -\epsilon pE_1, \qquad \hat{\nabla}_{E_2}E_2 = qE_1, \qquad \hat{\nabla}_{E_2}E_1 = -\epsilon qE_2,$$

the functions p and q satisfy

$$\begin{aligned} E_1 p &= E_2 q = 2\epsilon pq - pA, \\ 2\epsilon E_1 q &= -A^2 + 2\epsilon Aq + 3q^2 - \epsilon p^2, \\ 2E_2 p &= -A^2 - 2\epsilon Aq - q^2 + 3\epsilon p^2, \end{aligned}$$

for some positive constant A. Then the connection ∇ and the (1,1)-tensor field S given by

$$\begin{aligned} \nabla_{E_1} E_1 &= AE_1 + pE_2, & \nabla_{E_1} E_2 &= -\epsilon p E_1 - AE_2, \\ \nabla_{E_2} E_1 &= -(A + \epsilon q)E_2, & \nabla_{E_2} E_2 &= (q - \epsilon A)E_1, \end{aligned}$$

$$\begin{aligned} SE_1 &= (-3A^2 + 3\epsilon qA)E_1 - 3pAE_2, \\ SE_2 &= -3\epsilon pAE_1 - (3A^2 + 3\epsilon qA)E_2. \end{aligned}$$

satisfy the equations of Gauss, Codazzi and Ricci given in §1 and the apolarity condition (1.3).

Using this lemma and Radon's Theorem we have the following corollary.

Corollary 3.2. *Under the same hypothesis of Lemma 3.3 there exists a Blaschke immersion of (M, h) in $\mathbb{R}^3$ which satisfies $3\kappa = H = -3A^2$.*

Let M^2 be a nondegenerate affine surface in $\mathbb{R}^3$ with $3\kappa = H$. We again consider the functions p and q which satisfy (*). Let D be the function defined by

$$D = p_1 q_2 - p_2 q_1 = 3p^4 + 3(2\epsilon A^2 - 8qA + 2\epsilon q^2)p^2 - A^4 + 6q^2A^2 + 8\epsilon q^3 A + 3q^4,$$

and let $U = \{x \in M | D(x) \neq 0\}$.

Case 1 : $U \neq \emptyset$.

If $U \neq \emptyset$, then (p, q) can be considered as local coordinates on each connected component of U. In these coordinates, the frame $\{E_1, E_2\}$ and the affine metric h have the following expression.

$$\begin{aligned} E_1 &= (2\epsilon pq - pA)\partial_p + \tfrac{1}{2}(-\epsilon A^2 + 2Aq + 3\epsilon q^2 - p^2)\partial_q, \\ E_2 &= \tfrac{1}{2}(-A^2 - 2\epsilon Aq - q^2 + 3\epsilon p^2)\partial_p + (2\epsilon pq - pA)\partial_q, \\ h(\partial_p, \partial_p) &= D^{-2}\left((2\epsilon pq - pA)^2 + \tfrac{1}{4}\epsilon(-A^2 + 2\epsilon Aq + 3q^2 - \epsilon p^2)^2\right), \\ h(\partial_p, \partial_q) &= -D^{-2}\left((2\epsilon pq - pA)(\epsilon p^2 + q^2 - A^2)\right), \\ h(\partial_q, \partial_q) &= D^{-2}\left(\epsilon(2\epsilon pq - pA)^2 + \tfrac{1}{4}(-A^2 - 2\epsilon Aq - q^2 + 3\epsilon p^2)^2\right). \end{aligned}$$

Conversely, if we define h and $\{E_1, E_2\}$ like this on each connected component of $\{(p, q) \in \mathbb{R}^2 | D(p, q) \neq 0\}$, where D is defined like above, then one can check that $\{E_1, E_2\}$ is orthonormal with respect to h and that the Levi Civita connection $\hat{\nabla}$ of h satisfies

$$\hat{\nabla}_{E_1} E_1 = pE_2, \qquad \hat{\nabla}_{E_1} E_2 = -\epsilon p E_1, \qquad \hat{\nabla}_{E_2} E_2 = qE_1, \qquad \hat{\nabla}_{E_2} E_1 = -\epsilon q E_2.$$

Moreover, then the conditions of Corollary 3.2 are satisfied, such that such an immersion exists and is unique on every connected component.

Case 2 : $U = \emptyset$ and $q_2(x) = 2\epsilon pq - pA \equiv 0$ on an open subset $V \subseteq M$.

If $U = \emptyset$ and $q_2(x) \equiv 0$ on an open subset $V \subseteq M$, then we have that either $p \equiv 0$ on V or $2q\epsilon = A$ on V. From the system $(*)$ we then obtain that $\tau = \det S$ must be constant on V and in this case we have one of the surfaces given in (2.3).

Case 3 : $U = \emptyset$ and $F = \{x \in M | q_2(x) = 0\}$ has no interior points.

In that case, on $M \setminus F$ we have that p is a function of q, $(p = p(q))$ and we can suppose

$$\begin{array}{llll} p = p(q) & \text{with} & -A < q < A/2 & \text{if} \quad \epsilon = 1, \\ p = p(q) & \text{with} & 3q > -A & \text{if} \quad \epsilon = -1. \end{array}$$

The integrability condition for the system of differential equations

$$u_1 = (A - 2\epsilon q)^{\frac{1}{2}},$$
$$u_2 = 0.$$

is satisfied, and moreover, if u is a solution of this system, then $u_1 q_2 - u_2 q_1$ is nowhere zero. So (u, q) can be considered as local coordinates and we find the following expression for the frame $\{E_1, E_2\}$ and the affine metric h

$$\begin{aligned} E_1 &= (A - 2\epsilon q)^{\frac{1}{2}} \partial_u + \tfrac{1}{2}(-\epsilon A^2 + 2Aq + 3\epsilon q^2 - p^2)\partial_q, \\ E_2 &= (2\epsilon pq - pA)\partial_q, \\ h(\partial_u, \partial_u) &= (A - 2\epsilon q)^{-1} + \frac{\epsilon(-\epsilon A^2 + 2Aq + 3\epsilon q^2 - p^2)^2}{4(A - 2\epsilon q)(2\epsilon pq - pA)^2}, \\ h(\partial_u, \partial_q) &= \frac{A^2 - 2\epsilon Aq - 3q^2 + \epsilon p^2}{2(A - 2\epsilon q)^{\frac{1}{2}}(2\epsilon pq - pA)^2}, \\ h(\partial_q, \partial_q) &= \epsilon(2\epsilon pq - pA)^{-2}. \end{aligned}$$

Conversely, if we define h and $\{E_1, E_2\}$ like this on the open subset

$$\tilde{D}(\epsilon) = \begin{cases} \{(u, q) | -A < q < A/2\} & \text{if } \epsilon = 1 \\ \{(u, q) | 3q > -A\} & \text{if } \epsilon = -1 \end{cases},$$

then one can check that $\{E_1, E_2\}$ is orthonormal with respect to h and that the Levi Civita connection $\hat{\nabla}$ of h satisfies

$$\hat{\nabla}_{E_1} E_1 = pE_2, \qquad \hat{\nabla}_{E_1} E_2 = -\epsilon p E_1, \qquad \hat{\nabla}_{E_2} E_2 = qE_1, \qquad \hat{\nabla}_{E_2} E_1 = -\epsilon q E_2.$$

Applying Corollary 3.2 like in Case 1, we again obtain that such an immersion exists and is unique.

Summarizing all cases, we have proved the following theorem.

Theorem 3.2. *Let M^2 be a nondegenerate affine surface in $\mathbb{R}^3$ with κ and H constant and satisfying $3\kappa = H \neq 0$. Then there is an open dense subset of M^2 such that each connected component is affinely equivalent to an open part of (2.3) or of one of the examples given in Case 1 and Case 3.*

An alternative proof if M^2 is locally strongly convex.

In this section we give an alternative treatment if M^2 is locally strongly convex. The main difference between the two approaches is the choice of local frame. From now on we assume that M^2 is locally strongly convex with constant mean curvature and constant Pick invariant.

Since h is positive definite in this case, the equation of Ricci implies that we can diagonalize S at any point, so we can find a local orthonormal basis of eigenvectors of S in the neighbourhood of every point, at least if the multiplicities of the eigenvalues of S do not change at that point. So from now on in this section, we will implicitly assume that M^2 is not an affine sphere and even more restrictive that we are working in the neighbourhood of a point where the multiplicities of the eigenvalues of S do not change. Clearly the set of those points is dense in M^2. Then we can choose a frame $\{E_1, E_2\}$ in this way such that moreover E_1 corresponds to the largest eigenvalue of S.

Using the notations of §1, this means that we have a local orthonormal frame with $\beta = 0$ and $\epsilon = 1$. We put $c = 2H$ and $d^2 = \frac{1}{2}J$, where c and d are constants and $d > 0$. Since $A^2 + B^2 = d^2$, we can introduce a function t such that $A = d\cos(t)$ and $B = -d\sin(t)$. Putting $a = B - p$ and $b = -A - q$, we obtain the following expression for ∇

$$\begin{aligned} \nabla_{E_1}E_1 &= d\cos(t)E_1 - (a + 2d\sin(t))E_2, \\ \nabla_{E_1}E_2 &= aE_1 - d\cos(t)E_2, \\ \nabla_{E_2}E_1 &= -d\sin(t)E_1 + bE_2, \\ \nabla_{E_2}E_2 &= -(b + 2d\cos(t))E_1 + d\sin(t)E_2. \end{aligned} \tag{3.12}$$

If we define the function $\lambda = H + \alpha$, then λ satisfies the following system of differential equations

$$\begin{aligned} \lambda_1 &= (c - 2\lambda)b, \\ \lambda_2 &= (c - 2\lambda)a. \end{aligned} \tag{3.13}$$

A straightforward computation shows that the Gauss equation is equivalent to the following system of differential equations.

$$\begin{aligned} b_1 + a_2 + 6d^2 + a^2 + b^2 + \lambda &= 6d^2\sin^2 t + 2d\sin(t)t_1 + da\sin(t) - 5db\cos(t), \\ \cos(t)t_1 - \sin(t)t_2 &= -3a\cos(t) - 3b\sin(t) - 6d\sin(t)\cos(t), \\ b_1 + a_2 + c + a^2 + b^2 - \lambda &= -6d^2\sin^2 t - 2d\cos(t)t_2 - 5da\sin(t) + db\cos(t). \end{aligned} \tag{3.14}$$

Subtracting the first and third equation of (3.14) gives

$$2d\sin(t)t_1 + 2d\cos(t)t_2 = -(c - 2\lambda) - 6da\sin(t) + 6db\cos(t) + 6d^2(1 - 2\sin^2(t)).$$

Solving t_1 and t_2 from these two equations then proves the following lemma.

Lemma 3.4. *The function t defined on U satisfies the following system of differential equations.*

$$t_1 = -\frac{1}{2d}\sin(t)(c+6d^2-2\lambda)-3a,$$
$$t_2 = -\frac{1}{2d}\cos(t)(c-6d^2-2\lambda)+3b.$$

Then the integrability condition for the function λ and for the function t imply the following two equations

$$b_2 - a_1 = da\cos(t) - db\sin(t),$$

$$(c-2\lambda)(4d^2\sin^2(t) - 8da\sin(t) + 8db\cos(t) - c - 2d^2) + 12d^2(b_1+a_2)+ \\ 60d^3(a\sin(t)+b\cos(t)) + 48d^4 + 12d^2(a^2+b^2) - 4\lambda^2 = 0.$$

Combining this last equation with (3.14) then gives

$$(3.16)\quad (c-2\lambda)(8d^2\sin^2(t) + 8da\sin(t) - 8db\cos(t) + c - 2\lambda) + 2cd^2 + 24d^4 + 8d^2\lambda = 0.$$

Now, we want to prove that the constants c and d are related by $c + 6d^2 = 0$, which is equivalent to proving that $3\kappa = H$. In order to do so, we have to consider two different cases. First, we assume that $\sin(t)$ is identically zero. Hence t is a constant and $\cos(t) = \epsilon$, where $\epsilon = \pm 1$. But then it follows from Lemma 3.4 that the function a is identically zero and that the function b satisfies

$$b = \frac{\epsilon(c-6d^2-2\lambda)}{6d}.$$

Substituting b in (3.16) then gives

$$-4\lambda^2 + (4c - 24d^2)\lambda + (-c^2 + 30d^2c + 72d^4) = 0.$$

Hence λ is constant and then we are done. Therefore, we restrict to an open set where $\sin(t)$ is nowhere zero. Then, from (3.16) we obtain that

$$(3.17)\qquad a = \frac{8db\cos(t) - c + 2\lambda - 8d^2\sin^2(t)}{8d\sin(t)} - \frac{2cd^2 + 24d^4 + 8d^2\lambda}{8d(c-2\lambda)\sin(t)}.$$

From (3.14) and (3.16), we then deduce that the function b satisfies the following system of differential equations.

$$b_1 = -\big(\sin^4 t(8c^4 + 48c^3d^2 - 64c^3\lambda + 384c^2d^4 - 128c^2d^2\lambda + 192c^2\lambda^2 - 1536cd^4\lambda \\ -64cd^2\lambda^2 - 256c\lambda^3 + 1536d^4\lambda^2 + 256d^2\lambda^3 + 128\lambda^4) + \sin^2(t)\cos(t)b(128c^3d \\ -64c^2d^3 - 768c^2d\lambda + 256cd^3\lambda + 1536cd\lambda^2 - 256d^3\lambda^2 - 1024d\lambda^3) \\ +\sin^2(t)(-12c^4 - 24c^3d^2 + 96c^3\lambda - 288c^2d^4 - 320c^2d^2b^2 - 288c^2\lambda^2 + 1152cd^4\lambda \\ +1280cd^2b^2\lambda + 288cd^2\lambda^2 + 384c\lambda^3 - 1152d^4\lambda^2 - 1280d^2b^2\lambda^2 - 384d^2\lambda^3 - 192\lambda^4) \\ +\cos(t)b(-48c^3d - 96c^2d^3 + 288c^2d\lambda - 1152cd^5 - 192cd^3\lambda - 576cd\lambda^2 + 2304d^5\lambda \\ +768d^3\lambda^2 + 384d\lambda^3) + (3c^4 + 12c^3d^2 - 24c^3\lambda + 156c^2d^4 + 192c^2d^2b^2 + 72c^2\lambda^2 \\ +288cd^6 - 480cd^4\lambda - 768cd^2b^2\lambda - 144cd^2\lambda^2 - 96c\lambda^3 + 1728d^8 + 1152d^6\lambda \\ +768d^4\lambda^2 + 768d^2b^2\lambda^2 + 192d^2\lambda^3 + 48\lambda^4)\big)\,/(8d\sin(t)(c-2\lambda))^2,$$

$$b_2 = -\left(\sin^2(t)\cos(t)(c^3 + 2c^2d^2 - 6c^2\lambda + 56cd^4 + 12cd^2\lambda + 12c\lambda^2 - 112d^4\lambda - 32d^2\lambda^2 \right.$$
$$-8\lambda^3) + \sin^2(t)b(-16c^2d + 48cd^3 + 64cd\lambda - 96d^3\lambda - 64d\lambda^2) + \cos(t)(-c^3 + 2c^2d^2$$
$$+6c^2\lambda - 16cd^4 - 40cd^2b^2 - 20cd^2\lambda - 12c\lambda^2 + 96d^6 + 80d^4\lambda + 80d^2b^2\lambda + 32d^2\lambda^2 + 8\lambda^3)$$
$$\left. +b(13c^2d - 22cd^3 - 52cd\lambda + 120d^5 + 104d^3\lambda + 52d\lambda^2)\right)/(8d^2(c - 2\lambda)\sin(t)).$$

The integrability condition of this system of differential equations is given by

(3.18)
$$0 = \sin^2(t)\left(c^4 - 4c^3d^2 - 8c^3\lambda + 28c^2d^4 + 32c^2d^2\lambda + 24c^2\lambda^2 + 336cd^6 + 32cd^4\lambda\right.$$
$$\left.-80cd^2\lambda^2 - 32c\lambda^3 + 1152d^8 + 384d^6\lambda + 64d^4\lambda^2 + 64d^2\lambda^3 + 16\lambda^4\right) + b\cos(t)\left(16c^3d\right.$$
$$\left.+32c^2d^3 - 96c^2d\lambda + 384cd^5 + 64cd^3\lambda + 192cd\lambda^2 - 768d^5\lambda - 256d^3\lambda^2 - 128d\lambda^3\right)$$
$$+\left(-c^4 - 4c^3d^2 + 8c^3\lambda - 52c^2d^4 - 64c^2d^2b^2 - 24c^2\lambda^2 - 96cd^6 + 160cd^4\lambda + 256cd^2\lambda b^2\right.$$
$$\left.+48cd^2\lambda^2 + 32c\lambda^3 - 576d^8 - 384d^6\lambda - 256d^4\lambda^2 - 256d^4\lambda^2b^2 - 64d^2\lambda^3 - 16\lambda^4\right).$$

Differentiating (3.17) with respect to E_1 and eliminating the function b, using again (3.18) then gives the following equation in t and λ

$$0 = (c + 6d^2)f_1(\lambda, t), \tag{3.19}$$

where f_1 is given by

$$f_1 = \cos^2(t)(-144\lambda^4 + \lambda^3(192d^2 + 288c) + \lambda^2(3392d^4 + 432cd^2 - 216c^2) + \lambda(-1152d^6$$
$$-3680cd^4 - 576c^2d^2 + 72c^3) + (-9c^4 + 156c^3d^2 + 452c^2d^4 - 4176cd^6 - 10368d^8))$$
$$+(6d^2 + c)d^2(288(-\lambda^2 + c\lambda) - 72(c^2 - 3cd^2 - 12d^4)).$$

Similarly, by differentiating (3.17) with respect to E_2 and eliminating the function b, using again (3.18) we obtain that

$$0 = (c + 6d^2)f_2(\lambda, t), \tag{3.20}$$

where f_2 is given by

$$f_2 = \sin^2(t)(144\lambda^4 + \lambda^3(192d^2 - 288c) + \lambda^2(-3392d^4 - 1008cd^2 + 216c^2) + \lambda(-1152d^6$$
$$+3104cd^4 + 864c^2d^2 - 72c^3) + (9c^4 - 204c^3d^2 - 164c^2d^4 + 5328cd^6 + 10368d^8))$$
$$-(6d^2 + c)d^2(288d^2(-\lambda^2 + c\lambda) - 72(c^2 - 3cd^2 - 12d^4)).$$

Now we put

$$g_1(\lambda) = -144\lambda^4 + \lambda^3(192d^2 + 288c) + \lambda^2(3392d^4 + 432cd^2 - 216c^2) + \lambda(-1152d^6$$
$$-3680cd^4 - 576c^2d^2 + 72c^3) + (-9c^4 + 156c^3d^2 + 452c^2d^4 - 4176cd^6 - 10368d^8))$$

and

$$g_2(\lambda) = 144\lambda^4 + \lambda^3(192d^2 - 288c) + \lambda^2(-3392d^4 - 1008cd^2 + 216c^2) + \lambda(-1152d^6 + 3104cd^4 + 864c^2d^2 - 72c^3) + (9c^4 - 204c^3d^2 - 164c^2d^4 + 5328cd^6 + 10368d^8)).$$

Hence if $c + 6d^2 \neq 0$, we obtain by combining (3.19) and (3.20) that

$$\begin{aligned} 1 &= \sin^2 t + \cos^2 t \\ &= (6d^2 + c)d^2(288d^2(-\lambda^2 + c\lambda) - 72(c^2 - 3cd^2 - 12d^4))(-1/g_1(\lambda) + 1/g_2(\lambda)). \end{aligned}$$

Hence λ is a constant and we are done again, actually we obtain a contradiction, since in this case $c + 6d^2 = 0$.

So, we have given an alternative proof of Lemma 3.2, at least if M^2 is locally strongly convex. But we can proceed now to determine all locally strongly convex surfaces with κ and H constant and satisfying $3\kappa = H$, but which are not affine spheres. Note that the assumptions that E_1 belongs to the largest eigenvalue and that $c + 6d^2 = 0$, implies that $\lambda > -3d^2$.

If $\sin(t)$ vanishes nowhere, then we can express a and b as functions of t and λ as follows, using $c + 6d^2 = 0$, (3.17) and (3.18)

$$\begin{aligned} a &= \sin(t)\frac{\lambda(\lambda + 4d^2)}{4d(3d^2 + \lambda)}, \\ b &= -\cos(t)\frac{(6d^2 + \lambda)(2d^2 + \lambda)}{4d(3d^2 + \lambda)}. \end{aligned}$$

It is easy to see that both formulas also are valid if $\sin(t) = 0$.

From (3.12), we then see that the connection is completely determined by the functions t and λ . Lemma 3.4 and (3.13) then give the following first order partial differential equations for these two functions:

$$\begin{aligned} \lambda_1 &= \cos(t)\frac{(6d^2 + \lambda)(2d^2 + \lambda)}{2d}, \\ \lambda_2 &= -\sin(t)\frac{\lambda(\lambda + 4d^2)}{2d}, \\ t_1 &= \sin(t)\frac{\lambda^2}{4d(3d^2 + \lambda)}, \\ t_2 &= \cos(t)\frac{(6d^2 + \lambda)^2}{4d(3d^2 + \lambda)}. \end{aligned}$$

First case. If λ is identically zero, then M^2 is equivalent to (2.3), so we assume that λ is nowhere zero.

Second case. If $\lambda_1 t_2 - \lambda_2 t_1$ is identically zero, we immediately obtain that

$$\cos^2(t)(6d^2 + \lambda)^3(2d^2 + \lambda) + \sin^2(t)\lambda^3(4d^2 + \lambda) = 0.$$

Since $\lambda > -3d^2$, it immediately follows from the previous equation that $(2d^2+\lambda)\lambda < 0$. Hence $-2d^2 \leq \lambda < 0$. If $\lambda = -2d^2$ on a non-empty open set, we obtain a contradiction from [V]. So from now on, we resrict ourselves to an open subset and assume that $-2d^2 < \lambda < 0$. Then we can compute $\cos(t)$ and $\sin(t)$ as follows

$$\sin(t) = \frac{(6d^2+\lambda)}{4d(3d^2+\lambda)}\left(\frac{(6d^2+\lambda)(2d^2+\lambda)}{3d^2+\lambda}\right)^{\frac{1}{2}},$$
$$\cos(t) = -\frac{\lambda}{4d(3d^2+\lambda)}\left(\frac{-\lambda(\lambda+4d^2)}{3d^2+\lambda}\right)^{\frac{1}{2}}.$$

Now we consider the following system of differential equations:

$$u_1 = \left(\frac{-\lambda(\lambda+3d^2)}{\lambda+4d^2}\right)^{\frac{1}{2}},$$
$$u_2 = 0.$$

A straightforward computation shows that the integrability condition for the function u is satisfied. Also $u_1\lambda_2 - u_2\lambda_1$ is nowhere zero. So we can consider u and λ as local coordinates. We have

$$E_1 = u_1\partial_u + \lambda_1\partial_\lambda,$$
$$E_2 = \lambda_2\partial_\lambda.$$

Using these relations and (3.12), we find the following expressions for the connection and the affine metric:

$$h(\partial_u,\partial_u) = \frac{-8d^2}{\lambda(\lambda+6d^2)},$$
$$h(\partial_u,\partial_\lambda) = \frac{8d^2(\lambda+3d^2)}{\lambda(\lambda+6d^2)^2(\lambda+4d^2)},$$
$$h(\partial_\lambda,\partial_\lambda) = \frac{64d^4(\lambda+3d^2)^3}{\lambda^2(\lambda+6d^2)^3(\lambda+4d^2)^2(\lambda+2d^2)},$$

$$\nabla_{\partial_u}\partial_u = \frac{2d^2-\lambda}{6d^2+\lambda}\partial_u - 2\frac{(\lambda+2d^2)(\lambda+4d^2)}{(\lambda+3d^2)}\partial_\lambda,$$

$$\nabla_{\partial_u}\partial_\lambda = \nabla_{\partial_\lambda}\partial_u = -\frac{8d^2(\lambda+3d^2)}{(\lambda+6d^2)^2(\lambda+4d^2)}\partial_u - \frac{2d^2-\lambda}{6d^2+\lambda}\partial_\lambda,$$

$$\nabla_{\partial_\lambda}\partial\lambda = \frac{-4(12d^4-\lambda^2)d^2(\lambda+3d^2)}{(\lambda+6d^2)^3(\lambda+4d^2)^2(\lambda+2d^2)}\partial_u - \frac{1296d^{10}+1944d^8\lambda+1104d^6\lambda^2+312d^4\lambda^3+46d^2\lambda^4+3\lambda^5}{(\lambda+6d^2)^2(\lambda+4d^2)(\lambda+3d^2)(\lambda+2d^2)\lambda}\partial_\lambda.$$

Defining ∇ and h like this on the open subset $\{(u,\lambda)| -2d^2 < \lambda < 0\}$ of $\mathbb{R}^2$, one can check that the equations of Gauss, Codazzi and Ricci are satisfied. Applying Radon's theorem shows that such an immersion exists and is unique. Moreover, one can integrate this system and obtain that the immersion is given by

$$(e^{2u}\frac{4d^2+\lambda}{(6d^2+\lambda)^3}, p_1(\lambda)e^{-u}, p_2(\lambda)e^{-u}),$$

where p_1 and p_2 are two independent solutions of the following ordinary differential equation :

$$p''(\lambda) = -\frac{4d^2(3d^2+\lambda)}{(6d^2+\lambda)(4d^2+\lambda)(2d^2+\lambda)\lambda}p(\lambda) - \frac{3\lambda^4+44\lambda^3d^2+224\lambda^2d^4+480\lambda d^6+360d^8}{\lambda(\lambda+2d^2)(\lambda+3d^2)(\lambda+4d^2)(\lambda+6d^2)}p'(\lambda).$$

Third case. If $t_1\lambda_2 - t_2\lambda_1$ is nowhere zero, then we can consider t and λ as local coordinates. We have

$$E_1 = t_1\partial_t + \lambda_1\partial_\lambda,$$
$$E_2 = t_2\partial_t + \lambda_2\partial_\lambda.$$

Using these relations and Lemma 3.2, we find the following expressions for the connection and the affine metric. First, we put

$$D = 1/(16d^2(3d^2+\lambda)^3\sin^2(t) - (\lambda+6d^2)^3(\lambda+2d^2))^2.$$

Then

$$h(\partial_t,\partial_t) = 16Dd^2(\lambda+3d^2)((\lambda+6d^2)^2(\lambda+2d^2)^2\cos^2(t) + \lambda^2(4d^2+\lambda^2)\sin^2(t)),$$
$$h(\partial_t,\partial_\lambda) = 64Dd^4\lambda(\lambda+3d^2)^2(\lambda+6d^2)^2\sin(t)\cos(t),$$
$$h(\partial_\lambda,\partial_\lambda) = 4Dd^2(\lambda^4\sin^2(t) + (\lambda+6d^2)^4\cos^2(t)),$$

$$\nabla_{\partial_t}\partial_t = -128Dd^4(3d^2+\lambda)^3\sin(t)\cos(t)\left((24d^4+12d^2\lambda+2\lambda^2)(\lambda+3d^2)\sin^2(t) - (6d^2+\lambda)^2(2d^2+\lambda)\right)\partial_t - 4D(3d^2+\lambda)\left(\sin^4(t)(24d^8(2d^2+\lambda)(6d^2+\lambda)(36d^4+24d^2\lambda+5\lambda^2)) - \sin^2(t)(20736d^{16}+24192d^{14}\lambda+2592d^{12}\lambda^2-8448d^{10}\lambda^3-5272d^8\lambda^4-1368d^6\lambda^5-168d^4\lambda^6-8d^2\lambda^7) + (10368d^{16}+10368d^{14}\lambda-4320d^{12}\lambda^2-10752d^{10}\lambda^3-6456d^8\lambda^4-1944d^6\lambda^5-322d^4\lambda^6-28d^2\lambda^7-\lambda^8)\right)\partial_\lambda,$$

$$\nabla_{\partial_t}\partial_\lambda = \nabla_{\partial_\lambda}\partial_t = D(24d^2(\lambda+3d^2)^3(\lambda+6d^2)(48d^6+24d^4\lambda+6d^2\lambda^2+\lambda^3)\sin^2(t) - 1152d^8(\lambda+3d^2)^4\sin^4(t) - (6d^2+\lambda)^5(2d^2+\lambda)(6d^4+4d^2\lambda+\lambda^2))/(\lambda+3d^2)\partial_t - 32Dd^2(3d^2+\lambda)^3\sin(t)\cos(t)\left(d^2(48d^4+48d^2\lambda+8\lambda^2)(\lambda+3d^2)\sin^2(t) - (6d^2+\lambda)^2(2d^2+\lambda)^2\right)\partial_\lambda,$$

$$\nabla_{\partial_\lambda}\partial_\lambda = 8Dd^2(3d^2+\lambda)\sin(t)\cos(t)\left(24d^2(18d^4+6d^2\lambda+\lambda^2)(\lambda+3d^2)\sin^2(t)\right.$$
$$\left.-(6d^2+\lambda)^4\right)\partial_t - 2D(\lambda+3d^2)(64d^4(3d^2+\lambda)^2(3d^2+2\lambda)(9d^2+2\lambda)\sin^4(t)$$
$$-4d^2(3d^2+\lambda)(6d^2+\lambda)(432d^6+408d^4\lambda+106d^2\lambda^2+7\lambda^3)\sin^2(t)$$
$$+(6d^2+\lambda)^5(2d^2+\lambda))\partial_\lambda.$$

Defining ∇ and h like this on each connected component of the open subset

$$\{(t,\lambda)|(16d^2(3d^2+\lambda)^3\sin^2(t)-(\lambda+6d^2)^3(\lambda+2d^2))^2\neq 0\}$$

of $\mathbb{R}^2$, one can check that the equations of Gauss, Codazzi and Ricci are satisfied. Applying Radon's theorem shows that such an immersion exists and is unique on each connected component.

All arguments used above are local in the neighbourhood of a point satisfying some condition. It is easy to see that one can summarize everything in the following theorem, which is of course merely a reformulation of Theorem 3.2 under the extra convexity condition.

Theorem 3.3. *Let M^2 be a locally strongly convex affine surface in $\mathbb{R}^3$ with κ and H constant and satisfying $3\kappa = H \neq 0$. Then there is an open dense subset of M^2 such that each connected component is affine equivalent to an open part of (2.3) or of one of the two previous examples.*

§4 Surfaces with vanishing Pick invariant

A locally strongly convex surface with $J = 0$ obviously is quadric, so we assume that h is indefinite. The best choice of coordinates in this case is to choose an asymptotic parametrization, in the sense of [B, page 121] and [SS, page 144]. This means that we take a parametrization $x(u,v)$ such that

$$h(x_u,x_u) = h(x_v,x_v) = 0 \text{ and } h(x_u,x_v) = F, \tag{4.1}$$

where F is a positive function. From [SS, §28] it follows that $(\nabla h)(x_u,x_u,x_v) = 0$ and $(\nabla h)(x_u,x_v,x_v) = 0$. We can define functions A and B by $A = -\frac{1}{2}(\nabla h)(x_u,x_u,x_u)$ and $B = -\frac{1}{2}(\nabla h)(x_v,x_v,x_v)$. We then have

$$\widehat{\nabla}_{x_u}x_u = F^{-1}F_u x_u,\ \widehat{\nabla}_{x_u}x_v = 0,\ \widehat{\nabla}_{x_v}x_v = F^{-1}F_v x_v, \tag{4.2}$$

and also

$$\begin{aligned} \kappa F &= -(\log F)_{uv}, \\ JF^3 &= AB, \\ FH_u &= F^{-2}AB_u - (F^{-1}A_v)_v, \\ FH_v &= F^{-2}BA_v - (F^{-1}B_u)_u. \end{aligned} \tag{4.3}$$

So, if we assume that $J = 0$, then at any point either $A = 0$ or $B = 0$. Note that, if A vanishes on an open set, then the u-lines are straight lines, so M^2 is locally ruled. We now show that, if $J = 0$ and H is constant, then A or B vanish identically for any local asymptotic parametrization.

To see this, we observe that from $AB = 0$, it follows that

$$A_u B = A_v B = AB_u = AB_v = 0.$$

Hence the formulas (4.3) imply that $(F^{-1}A_v)_v = (F^{-1}B_u)_u = 0$. Therefore, if P and Q are functions such that $P_u = Q_v = F$, then

$$A(u,v) = A_1(u)Q(u,v) + A_2(u),$$
$$B(u,v) = B_1(v)P(u,v) + B_2(v).$$

Now assume that A is not identically zero. Hence either A_1 or A_2 is not identically zero. If A_1 is not identically zero, then there exists a number u_0 such that $A_1(u_0) \neq 0$. Since $0 = BA_v = BA_1F$, we obtain that $B(u_0,v) = 0$ for any v. Hence

$$B(u,v) = B_1(v)\left(P(u,v) - P(u_0,v)\right).$$

Now if B_1 is identically zero, then B is identically zero, and we are done in this case. So we assume that there exists a number v_0 such that $B_1(v_0) \neq 0$, and try to obtain a contradiction.

Since $BA_1 = 0$, we have that

$$\left(P(u,v_0) - P(u_0,v_0)\right)A_1(u) = 0.$$

Deriving this last equation gives us

$$F(u,v_0)A_1(u) + \left(P(u,v_0) - P(u_0,v_0)\right)A_1'(u) = 0,$$

so we get that $F(u,v_0)\left(A_1(u)\right)^2 = 0$ for any u. This clearly contradicts our assumption $A_1(u_0) \neq 0$.

In case A_1 is identically zero, but A_2 is not, one can show similarly that B vanishes identically.

Now we can prove the following theorem.

Theorem 4.1. *Let M^2 be an affine surface in $\mathbb{R}^3$ with h indefinite, $J = 0$ and H constant. Then for every point p of M^2 there are curves f and g in $\mathbb{R}^3$ satisfying*

$$\det[fg'f'] = 1 \text{ and } \det[ff'f''] = H,$$

such that a neighbourhood of p is contained in the ruled surface given by

$$x(u,v) = uf(v) + g(v).$$

Proof. We take any asymptotic parametrization $x(u,v)$, and we use the same formalism as above. We can assume that the function A is identically zero. Then the formula of Gauss can be written as

$$Fx_{uu} = F_u x_u, \tag{4.4}$$
$$x_{uv} = F\xi, \tag{4.5}$$
$$Fx_{vv} = Bx_u + F_v x_v. \tag{4.6}$$

If P is a function such that $P_u = F$, then from (4.4) we obtain that

(4.7) $$x(u,v) = f(v)P + g(v),$$

for some functions $f, g : I \subset \mathbb{R} \mapsto \mathbb{R}^3$. A short computation gives us

$$\det[x_u, x_v, x_{uv}] = \det[fg'f']F^2.$$

The apolarity condition then implies

$$\det[fg'f'] = 1.$$

Further, we have that

$$\xi_v = f''(v) + f'(v)F^{-1}F_v + f(v)(F^{-1}F_v)_v,$$

and on the other hand, (4.5) and (4.7) imply that

$$\begin{aligned}\xi_v &= (F^{-1})_v x_{uv} + F^{-1}x_{vvu} = F^{-2}B_u x_u - Hx_v \\ &= F^{-1}B_u f(v) - Hf'(v)P - Hf(v)P_v - Hg'(v)\end{aligned}$$

so that

$$-Hg' = f'' + f'\left(F^{-1}F_v + HP\right) + f\left((F^{-1}F_v)_v + HP_v - F^{-1}B_u\right).$$

Finally, this gives us

$$\det[ff'f''] = H.$$

Then taking $\overline{u} = P$ as new first coordinate gives the desired parametrization for the surface.

Given f and g, it is clear that there is no restriction on the range of u, so the surface $x(u,v) = uf(v) + g(v)$ contains a whole straight line, and every point of M has a neighbourhood which is a part of the surface with this equation. □

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