AFFINE ISOPERIMETRIC PROBLEMS 
AND SURFACES WITH CONSTANT 
AFFINE MEAN CURVATURE 

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The first affine isoperimetric property for a locally strongly convex surface with 
nonzero constant affine mean curvature $H$ (briefly, CH-surface) in $\mathbb{R}^3$ was obtained 
by Blaschke in 1916. He proved the following result:

Among all ovaloids with constant volume the ellipsoid, and only the 
ellipsoid, attains the greatest affine-area.

In the present note we prove that the condition of nonzero affine mean curvature 
on a locally strongly convex surface is equivalent to the fact that the surface is a 
critical point of an "isoperimetric problem". Actually, a locally strongly convex 
surface has stationary affine-area under all deformations in the affine normal direc-
tion that leave constant the volume if and only if it is a CH-surface (see Lemma 
3).

We also prove, by using the same idea as in [3], the following result:

Main Theorem The second variation of the affine-area of a CH-surface with $H < 0$ 
under all deformations in the affine normal direction that leave constant the volume 
is negative definite.

1 Affine variational problems for CH-surfaces

Let $x : M \rightarrow \mathbb{R}^3$ be a locally strongly convex affine surface in the affine real 3-space 
provided with a volume element given by the usual determinant function, $\text{Det}$, and 
the flat connection, $D$. Then there exist a unique choice of transversal vector field $\xi$ 
such that 

(1) 
$D_XX(Y) = x_*\left(\nabla_X Y\right) + h(X,Y)\xi,$ 

(2) 
$D_XX = -x_*(S X),$

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\[ \theta(X, Y) = \det(x_*(X), x_*(Y), \xi) = \sqrt{h(X, X)h(Y, Y) - h(X, Y)^2}, \]

for any tangent vector fields \( X, Y \). We call \( \xi \) the affine normal, \( \nabla \) the induced affine connection, \( h \) the Blaschke metric, \( S \) the affine shape operator and \( \theta \) the affine volume element. The affine mean curvature and the affine Gauss-Kronecker curvature of the immersion are \( H = \frac{1}{2} \text{trace} S \) and \( T = \text{det} S \), respectively.

As \( S \) is diagonalizable, we can take \( \{e_1, e_2\} \) a local orthonormal frame on an open dense subset of \( M \) such that
\[ h(e_1, e_1) = h(e_2, e_2) = 1, \ h(e_1, e_2) = 0, \ S e_1 = (H + \alpha)e_1, \ S e_2 = (H - \alpha)e_2. \]

Moreover if we write
\[ \nabla e_1 e_1 = pe_2, \quad \nabla e_2 e_2 = q e_1, \]
and
\[ K(e_1, e_1) = ae_1 + be_2 \]
where \( \nabla \) is the Levi-Civita connection of \( h \) and \( K = \nabla - \nabla \), then the fundamental equations of the immersion are given by, (see [2], [4])
\[ \begin{align*}
H + 2(a^2 + b^3) &= e_1(q) + e_2(p) - p^2 - q^2 \\
e_1(b) - e_2(a) &= 3(qb - pa) \\
e_1(a) + e_2(b) &= -\alpha + 3(bp + qa) \\
e_2(H + \alpha) &= 2\alpha(p - b) \\
e_1(\alpha - H) &= 2\alpha(a + q).
\end{align*} \]

We shall denote by \( \{\omega_1, \omega_2, \omega\} \) the dual local frame of \( \{x_*(e_1), x_*(e_2), \xi\} \), then
\[ \omega = <N,>, \]
where \( N \) is the affine conormal vector field and \( <,> \) is the canonical scalar product in \( \mathbb{R}^3 \). And if we take \( \eta = <x, N> \) the affine distance function then we have
\[ x = -\nabla \eta + \eta \xi, \quad \Delta \eta = -2 - 2H \eta, \]
where by \( \Delta \) and \( \nabla \) we denote the Laplacian and the gradient with respect to the Blaschke metric \( h \), respectively.

Let \( \Omega \subset M \) be a relatively compact domain of \( M \). Then
\[ A_{\Omega}(x) = \int_{\Omega} \theta \]
is the affine-area of \( x(\Omega) \), that is, the area of \( x(\Omega) \) in the metric \( h \) and we shall call
\[ V_{\Omega}(x) = \frac{1}{3} \int_{\Omega} <x, N > \theta \]
the volume of \( x(\Omega) \). (When \( \Omega \) is small \(|V_{\Omega}(x)|\) is the volume of the cone over \( x(\Omega) \) with vertex at the origin of \( \mathbb{R}^3 \)).

**Definition** A deformation \( x_t : \Omega \to \mathbb{R}^3, \ t \in (-\epsilon, \epsilon) \), is called a deformation in the affine normal direction if
\[ x_t = x + \phi_t \xi, \]
where $\phi_0 = 0$ and $\phi_t$, $t \in (-\epsilon, \epsilon)$, are smooth functions with compact support in $\Omega$.

If $V_0(x_t) = V_0(x)$, $\forall t \in (-\epsilon, \epsilon)$, we call $x_t$ a volume-preserving deformation in the affine normal direction.

Remark. Another kind of deformations of a locally strongly convex surface called normal deformations have been investigated in [3] and [6].

Let $x_t : \Omega \longrightarrow \mathbb{R}^3$, $x_t = x + \phi_t \xi$, be a deformation in the affine normal direction. We set $A_0(t) = A_0(x_t)$, $V_0(t) = V_0(x_t)$ and denote by $\xi_t$, $N_t$ and $h_t$ the affine normal, the affine conormal vector field and the Blaschke metric of the immersion $x_t$, respectively, then,

$$\theta_t(e_1, e_2)N_t = x_t e_1 \wedge x_t e_2,$$

where $\theta_t$ is the volume element of $h_t$, and

$$h_t(e_i, e_j) = < N_t, D_{x_t} e_i e_j >.$$

Moreover, the first variation of the affine-area (see [3], [7]) is given by

$$2A_0'(0) + 3 \int_{\Omega} H \psi \theta = 0,$$

where $\psi = \frac{d}{dt} \phi_t |_{t=0} = \phi'_0$. (From now on we shall denote derivative in $t$ by prime).

Lemma 1 $V_0'(0) = \int_{\Omega} \psi \theta$.

Proof. Using (1), (2) and (3),

$$\theta_t(e_1, e_2) = Det(e_1 + e_1(\phi_t) \xi - \phi_t S e_1, e_2 + e_2(\phi_t) \xi - \phi_t S e_2, \xi_t),$$

thus

$$\theta_0'(e_1, e_2) = -2H \psi + < N, \xi'_0 >,$$

and from (11) and (15) we have

$$3V_0'(0) = \int_{\Omega} \{ \psi + < x, N'_0 > - 2H \psi < x, N > + < x, N > < \xi'_0, N > \} \theta.$$

As $< N, \xi'_0 > + < N'_0, \xi > = 0$ and $e_i(\psi) = - < N'_0, e_i >$, $i = 1, 2$, then the last expression and (9) give

$$V_0'(0) = \frac{1}{3} \int_{\Omega} \{ \psi + h(\nabla \eta, \nabla \psi) - 2H \eta \psi \} \theta.$$

The Lemma follows by using again (9) and Green’s Theorem.

We consider the functional $J_\Omega : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$ given by

$$J_\Omega(t) = 2A_0(t) + 3H V_0(t) = \int_{\Omega} \{ 2 + H < x_t, N_t > \} \theta_t,$$

where $H = \frac{1}{A_0(x)} \int_{\Omega} H \theta$.

From Lemma 1, (14), (16) and using the same method as in [1], we can prove the following results:
Lemma 2 Let $\psi : M \rightarrow \mathbb{R}$ be a smooth function with compact support in $\Omega$ such that $\int_\Omega \psi \theta = 0$. Then there exists a volume-preserving deformation in the affine normal direction $x_t : \Omega \rightarrow \mathbb{R}^3$, $x_t = x + \phi_t \xi$, with $\phi'_0 = \psi$.

Lemma 3 The following statements are equivalent:
1. $x$ is a CH-surface.
2. For each relatively compact domain $\Omega \subset M$ and each volume-preserving deformation in the affine normal direction, $A'_\Omega(0) = 0$.
3. For each relatively compact domain $\Omega \subset M$ and each deformation in the affine normal direction, $J'_\Omega(0) = 0$.

Next we shall prove that the second variation on $J_\Omega$ depends only of $\psi$.

Proposition 1 If $x$ is a CH-surface then

\begin{equation}
J''_\Omega(0) = \frac{-3}{8} \int_\Omega \left\{ (\Delta \psi)^2 + 20H^2 \psi^2 - 4Hh(\nabla \psi, \nabla \psi) - 4h(S \nabla \psi, \nabla \psi) - 8T\psi^2 \right\} \theta.
\end{equation}

Proof. Using (16) we have

\begin{equation}
J''_\Omega(0) = \int_\Omega \left\{ H < x_t, N_t >''(0) + 2H < x_t, N_t >'(0) \theta'_0(e_1, e_2) + (2 + H\eta) \theta''_0(e_1, e_2) \right\} \theta.
\end{equation}

Now by differentiating (12) and evaluating for $t = 0$, we have,

\begin{equation}
< x_t, N_t >'(0) = -\eta \theta'_0(e_1, e_2) + \psi + h(\nabla \eta, \nabla \psi) - 2H\eta \psi,
\end{equation}

\begin{equation}
< x_t, N_t >''(0) = -\eta \theta''_0(e_1, e_2) + 2\eta [\theta'_0(e_1, e_2)]^2 - 4H\psi^2 + 2\eta \psi^2 T + \phi''_0(1 - 2H\eta) + h(\nabla \eta, \nabla \phi''_0) + \theta'_0(e_1, e_2)[4H\eta \psi - 2\psi - 2h(\nabla \eta, \nabla \psi)] + 2Det(\nabla \eta, e_1(\psi) \xi, \psi Se_2) + 2Det(\nabla \eta, \psi Se_1, e_2(\psi) \xi),
\end{equation}

and using Green's Theorem, (9), (18), (19) and (20) we get

\begin{equation}
J''_\Omega(0) = \int_\Omega \left\{ 3H \phi''_0 + 2H \eta \psi^2 T + 2HDet(\nabla \eta, e_1(\psi) \xi, \psi Se_2) - 4H^2 \psi^2 + 2HDet(\nabla \eta, \psi Se_1, e_2(\psi) \xi) + 2\theta''_0(e_1, e_2) \right\} \theta.
\end{equation}

Now, we can compute $\theta'_0(e_1, e_2)$ and $\theta''_0(e_1, e_2)$ in two different ways. In fact, from the first equality of (3) we obtain (15) and

\begin{equation}
\theta'_0(e_1, e_2) = -2H \phi''_0 + 2\psi^2 T + 2Det(e_1(\psi) \xi, e_2, \xi'_0) + 2Det(e_1, e_2(\psi) \xi, \xi'_0) - 4H \psi < N, \xi'_0 > - < N'_0, \xi'_0 > < -2 < N'_0, \xi'_0 >.
\end{equation}

Moreover, from (13) we also have

\begin{equation}
h'_0(e_1, e_2) = -\left( \nabla_e e_j \right)(\psi) + h(e_1, e_2) < N'_0, \xi' > + e_i e_j(\psi) - \psi h(e_1, Se_j),
\end{equation}

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\( h''(e_1, e_2) = -\left(\nabla_e e_1\right) (\phi''_0) + 2 < N, \xi' > \left(\nabla_e e_1\right) (\psi) - \left(S \nabla_e e_1\right) (\psi^2) + \\
+ < N''', \xi > -2e_1 e_2 (\psi) < N, \xi' > + 2 (S e_1) (\psi^2) + \left(\nabla_e S e_1\right) (\psi^2) + \\
+ 2 \psi h(e_1, e_2) < N, \xi' > + e_1 e_2 (\phi''_0) - \phi''_0 h(e_1, e_2). \)

Thus, a straightforward computation with the second equality of (3), (23) and (24) also give us

\[ \theta_0'(e_1, e_2) = \frac{1}{2} \Delta \psi + < N''', \xi > - H \psi, \]

\[ \theta''_0(e_1, e_2) = -\frac{1}{4} (\Delta \psi + 2 < N'', \xi > -2H \psi^2 + < N''', \xi > - H \psi'' + \\
+ \frac{1}{2} (\Delta \phi''_0 - 2 < N', \xi > + 4H \psi < N, \xi' >) + \\
+ \psi \left(\nabla_e S e_1 + \nabla_e e_2\right) (\psi) - \psi S \left(\nabla_e e_1 + \nabla_e e_2\right) (\psi) + \\
+ h''_0(e_1, e_2) h''_0(e_2, e_2) - h''_0(e_1, e_2)^2 + 2h(S \nabla \psi, \nabla \psi), \]

and from (15), (25) and (26) we also get

\[ \theta''_0(e_1, e_2) = -\frac{3}{2} H \psi + \frac{1}{4} \Delta \psi \]

On the other hand, by using (12), (25) and (27), we obtain

\[ < N'', \xi' > = \frac{1}{16} (\Delta \psi)^2 - \frac{1}{4} H^2 \psi^2 - \frac{1}{4} H \psi \Delta \psi + \\
+ \text{Det}(e_1(e_1), e_2, \xi'') \text{Det}(e_1, e_2(\psi), \xi, \xi''). \]

Thus, from (22), (28) and (29),

\[ 2 \theta''_0(e_1, e_2) = -3H \phi''_0 + 2T \psi^2 - \frac{3}{16} (\Delta \psi)^2 - \frac{11}{4} H^2 \psi^2 + \frac{1}{4} H \psi \Delta \psi + \\
+ \frac{1}{2} \Delta \phi''_0 + \psi \left(\nabla_e S e_1 + \nabla_e e_2\right) (\psi) - \psi S \left(\nabla_e e_1 + \nabla_e e_2\right) (\psi) + \\
+ 2h(S \nabla \psi, \nabla \psi) + h''_0(e_1, e_1) h''_0(e_2, e_2) - h''_0(e_1, e_2)^2. \]

After a long but straight computation, Green’s Theorem, (7), (23), (25) and (27), give

\[ \int\Omega \{ h''_0(e_1, e_1) h''_0(e_2, e_2) - h''_0(e_1, e_2)^2 \} \theta = \int\Omega \left\{ -\frac{1}{2} (S \nabla \psi)(\psi) + \\
+ H h(\nabla \psi, \nabla \psi) - \frac{3}{4} H \psi \Delta \psi - \psi \left(\nabla_e S e_1 + \nabla_e e_2\right) (\psi) + \\
+ \psi S \left(\nabla_e e_1 + \nabla_e e_2\right) (\psi) + T \psi^2 + \frac{5}{4} H^2 \psi^2 - \frac{3}{16} (\Delta \psi)^2 \right\} \theta. \]
From (30) and (31), we have

\[ 2\theta_0''(e_1, e_2) = \int_\Omega \left\{ -3H\phi_0'' + \frac{3}{8}(\Delta \psi)^2 - \frac{3}{2}H^2\psi^2 - \frac{1}{2}H\psi\Delta \psi + \frac{3}{2}(S\nabla \psi)(\psi) + Hh(\nabla \psi, \nabla \psi) \right\} \theta, \]

and combining this formula with (21)

\[ J''_0(0) = \int_\Omega \left\{ \frac{11}{2}H^2\psi^2 + 2H Det(\nabla \eta, \psi Se_1, e_2(\psi)\xi) + 2H\eta T\psi^2 + 3T\psi^2 + 2H Det(\nabla \eta, e_1(\psi)\xi, \psi Se_2) - \frac{3}{8}(\Delta \psi)^2 + \frac{3}{2}Hh(\nabla \psi, \nabla \psi) + \frac{3}{2}h(\nabla \psi, \nabla \psi) \right\} \theta. \]

But, using Green’s Theorem, (7) and (9), we can obtain

\[ \int_\Omega \left\{ -2H^2\psi^2 - 2H T\psi^2 \right\} \theta = \int_\Omega \left\{ 2H Det(\nabla \eta, \psi Se_1, e_2(\psi)\xi) + 2H Det(\nabla \eta, e_1(\psi)\xi, \psi Se_2) \right\} \theta, \]

and the Proposition follows from (33) and (34).

From Lemma 2, Lemma 3 and Proposition 1 we have:

**Proposition 2** \( A''_0(0) \leq 0 \) for all volume-preserving deformations in the affine normal direction if and only if the equation (17) is non positive for every smooth function \( \psi \) with compact support in \( \Omega \) such that \( \int_\Omega \psi = 0 \).

### 2 Proof of the Main Theorem

From (5), (6), (7) and using the same method as in [3], we get

\[ \int_\Omega (\Delta \psi)^2 \theta = 2 \int_\Omega \left\{ L_{11}^2 + L_{22}^2 + L_{12}^2 + L_{21}^2 + \kappa h(\nabla \psi, \nabla \psi) - \frac{1}{2}(\Delta \psi)^2 \right\} \theta, \]

and

\[ \int_\Omega h(S\nabla \psi, \nabla \psi) \theta = \int_\Omega \left\{ -H\psi \Delta \psi - \alpha \psi \left( L_{11}(\psi) - \frac{1}{2}\Delta \psi \right) + \alpha \psi \left( L_{22}(\psi) - \frac{1}{2}\Delta \psi \right) - aae_1(\psi^2) - bae_2(\psi^2) \right\} \theta, \]

where \( \kappa = H^2 + 2(a^2 + b^2) \) and \( L_{ij}(\psi) = e_ie_j(\psi) - (\nabla e_i, e_j)(\psi), \ i, j \in \{1, 2\}. \)

Thus, from (35) and (36), we obtain,

\[
-\frac{3}{8} \int_\Omega \left\{ (\Delta \psi)^2 - 4h(S\nabla \psi, \nabla \psi) \right\} \theta = -\frac{3}{8} \int_\Omega \left\{ 2 \left( L_{11}(\psi) - \frac{1}{2}\Delta \psi + \alpha \psi \right)^2 + 4 \left( aae_1(\psi) + bae_2(\psi) + \alpha \psi \right)^2 + 2L_{21}^2 + 2L_{12}^2 - 8a^2\psi^2 + 2 \left( L_{22}(\psi) - \frac{1}{2}\Delta \psi - \alpha \psi \right)^2 + 4 \left( aae_2(\psi) - bae_1(\psi) \right)^2 - 2Hh(\nabla \psi, \nabla \psi) \right\} \theta \leq \leq -\frac{3}{8} \int_\Omega \left\{ 8T\psi^2 - 8H^2\psi^2 - 2Hh(\nabla \psi, \nabla \psi) \right\} \theta.
\]
As we assume $H < 0$, from this inequality and using the Proposition 1, we get

\begin{equation}
J''(0) \leq -\frac{3}{8} \int_{\Omega} \left\{ 12H^2\psi^2 - 6Hh(\nabla\psi, \nabla\psi) \right\} \theta \leq 0,
\end{equation}

and the Main Theorem follows from Proposition 2 and (37).

**Remark.** For an ellipsoid with affine mean curvature $H$, $H > 0$, the equation (17) gives

$$J''(0) = -\frac{3}{8} \int_{\Omega} (\Delta\psi + 2H\psi)(\Delta\psi + 6H\psi)\theta.$$ 

As $2H$ and $6H$ are the first and second eigenvalues of $\Delta$, respectively, we also obtain that the second variation of the affine area of an ellipsoid under all volume-preserving deformation in the affine normal direction is negative definite.

Now, a result of Blaschke, (see [2] and [4]), states that "every CH-surface, $H > 0$, with complete Blaschke metric is an ellipsoid". Thus the Main Theorem is also true for CH-surfaces with $H > 0$ and complete Blaschke metric.

**References**


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