Geometry and Topology of Submanifolds, VII. World Scientific Publ. Aparecerá. 95

AFFINE DEFINITE 2-SPHERES IN R^4

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To Professor Katsumi Nomizu on his 70^{th} birthday

1 Definitions and Examples

Let $x : M \longrightarrow \mathbf{R}^4$ be an immersion of a surface M. For any local basis $u = \{E_1, E_2\}$ of M we define the symmetric bilinear form

$$G_u(Y, Z) = \frac{1}{2}\left\{[E_1, E_2, D_Y E_1, D_Z E_2] + [E_1, E_2, D_Z E_1, D_Y E_2]\right\},$$

where D is the usual connection in $\mathbf{R}^4$ and $[.,.,.,.]$ the determinat funtion.

We call a surface *definite* if the symmetric bilinear form G_u is definite. It is clear that this condition is independent of the choice of the local frame u. In that case, by exchanging the last two coordinates of $\mathbf{R}^4$ if it is necessary, we may assume that G_u is definite positive. The class of these surfaces is large (actually, the most of the Euclidean minimal surfaces in $\mathbf{R}^4$ are, locally, affine definite surfaces).

The affine metric g on M is defined

$$g = \frac{G_u}{(Det\, G_u(E_i, E_j))^{1/3}},$$

and it is equiaffinely invariant. Moreover Nomizu-Vrancken introduced an affine invariant normal bundle which preserve the equiaffine structure of $\mathbf{R}^4$ (see [NV]).

Analogously to Riemannian case one can define for each orthonormal frame $\alpha = \{\xi_1, \xi_2\}$ of the affine normal plane, the *affine mean curvature vector*,

$$H = \frac{1}{2}\left((traceS_{\xi_1})\xi_1 + (traceS_{\xi_2})\xi_2\right),$$

which is independent of α.

When $H = 0$, M is called *maximal.* If for any normal vector field ξ, the affine Weingarten operator S_ξ is a multiple of the identity, M is called *umbilical* .

Although there exist others normalizations, one introduce by Burstin-Mayer in 1927, (see [BM]), and another one introduce by Klingenberg in 1951, (see [K1], [K2]), the Nomizu-Vrancken normalization has an interesting advantage, Dillen-Mys-Verstraellen-Vrancken proved (see [DMVV]):

The critical affine definite surfaces for the area integral of the affine metric are the affine

[1]Research partially supported by DGICYT Grant PB91-0731-Co3-02

definite surfaces with vanishing affine mean curvature vector respect to the transversal plane of K. Nomizu and L. Vrancken.

It is easy to see that if all affine normal planes are mutually parallel or in a pencil of planes in $\mathbf{R}^4$, then M is umbilical. Some examples of umbilical definite surfaces are:

Example 1.- The complex paraboloid:

$$M = \{(z, z^2) | z \in \mathbf{C}\}.$$

Example 2.- The surface

$$x(u,v) = \left(\frac{3}{4}vu^{\frac{4}{3}} + \frac{1}{9}v^3, u^{\frac{4}{3}} + \frac{4}{9}v^2, v, \frac{3}{4}u^2\right).$$

Definition 1.- In analogy with the affine theory of hypersurfaces, an affine definite surface in $\mathbf{R}^4$ is called **improper** affine sphere if all their affine normal planes are mutually parallel and **proper** affine sphere if all their affine normal planes pass through a fixed point.

The Examples 1 and 2 are improper affine spheres and it is not difficult to see that

Example 3.- The complex hyperbola:

$$M = \{(z, w) \in \mathbf{C} \times \mathbf{C} | z^2 - w^2 = 1\}$$

is a proper affine sphere.

2 Results

About affine umbilical surfaces, we have the following global result (see [MM]):

Theorem 1.- *An affine complete affine definite surface in* $\mathbf{R}^4$ *is umbilical if and only if it is affine equivalent to the complex paraboloid.*

As an improper affine sphere in $\mathbf{R}^4$ is umbilical with vanishing affine mean curvature vector. Then we have the following result of Bernstein type:

Corollary. *Every definite affine complete improper affine sphere in* $\mathbf{R}^4$ *is affine equivalent to the complex paraboloid.*

We also have the following local characterization of the complex hyperbola:

Theorem 2.- *A definite affine harmonic, proper affine sphere M in $\mathbf{R}^4$ is locally equivalent to the complex hyperbola.*

Proof.- If M is regarded as a Riemannian surface and (u,v) are local isotermal parameters with respect to the affine metric g, then

(1) $$g = F^2(z,\bar{z})|dz|^2$$

where $z = u + iv$ and $dz = du + idv$.

As M is affine harmonic,

(2) $$x_{z\bar{z}} = 0,$$

moreover, the immersion satisfies,

(3) $$-4\,[x_z, x_{\bar{z}}, x_{zz}, x_{\bar{z}\bar{z}}] = F^6,$$

where $x_z = \frac{\partial x}{\partial z}$, $x_{\bar{z}} = \frac{\partial x}{\partial \bar{z}}$, $x_{zz} = \frac{\partial^2 x}{\partial z \partial z}$, etc.

If $\omega = log(F^2)$, then it is easy to see that the direction of the affine normal vector field ξ given by,

(4) $$\xi = x_{zz} - \omega_z x_z,$$

is independent of the choice of the complex parameter z, and x is a *proper* affine sphere if and only if

(5) $$x = \beta\xi + \overline{\beta\xi},$$

for some complex function β.

From (2) and (4), we have,

(6) $$\xi_{\bar{z}} = -\omega_{z\bar{z}} x_z.$$

Now, taking the Cauchy-Riemann operators on (3) and using that M is a proper affine sphere, from (6), we obtain that

$$\left(e^{2\omega}\overline{\beta}\right)_{\bar{z}} = 0,$$

and $h(z) = e^{2\omega}\overline{\beta}$ is a holomorphic function. Let τ be a holomorphic function such that $(\tau_z)^2 = h$. Then, if we introduce τ as a new local complex coordinate, we obtain,

(7) $$x = e^{-2\omega}\{\xi + \overline{\xi}\}.$$

From (6) and (7),

$$\xi_z = e^{2\omega} x_z + \omega_{z\bar{z}} x_{\bar{z}} + 2\omega_z(\xi + \overline{\xi}).$$

As $\xi_{z\bar{z}} = \xi_{\bar{z}z}$, (6) and the above expression give,

$$\begin{aligned} 0 &= \omega_{z\bar{z}z} + \omega_{z\bar{z}}\omega_z + 2\omega_{\bar{z}} e^{2\omega} \\ 0 &= 3\omega_{z\bar{z}} + 4\omega_z\omega_{\bar{z}}. \end{aligned}$$

Taking again the Cauchy-Riemann operators on the last expressions we can conclude that the affine Gauss curvature of g must be identically zero on M. Then the theorem follows from the classification of the affine flat harmonic surfaces in $\mathbf{R}^4$ given by Li (see [L]).

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