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CHARACTERIZATIONS OF LOCALLY STRONGLY CONVEX HOMOGENEOUS AFFINE SURFACES

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Let $\mathcal{A}^3$ be the unimodular affine 3-space. A locally strongly convex affine surface M in $\mathcal{A}^3$ is called homogeneous if M is the orbit under the action of a subgroup of the group of affine transformations of $\mathcal{A}^3$. This kind of surfaces has been studied quite intensively the last ten years and different results in this area have been obtained. In this note we give a survey of some characterizations concerning locally strongly convex homogeneous affine surfaces.

1 Local results

Let M be a locally strongly convex affine surface in the affine real 3-space $\mathcal{A}^3$ provided with a volume element given by the usual determinant function, Det, and the flat connection, D. Then there exist a unique choice of transversal vector field ξ such that

$$D_X Y = \overline{\nabla}_X Y + h(X,Y)\xi,$$
$$D_X \xi = -SX,$$
$$\theta(X,Y) = Det(X,Y,\xi) = \sqrt{h(X,X)h(Y,Y) - h(X,Y)^2},$$

for any tangent vector fields X, Y. We call ξ the *affine normal*, $\overline{\nabla}$ the *induced affine connection*, h the *affine metric*, S the *affine shape operator* and θ the *affine volume element*. The *affine mean curvature* and the *affine Gauss-Kronecker curvature* are $H = \frac{1}{2} trace S$ and $\tau = det\, S$, respectively.

If $S = HI$, the affine surface is called *affine sphere*, and in this case either the position vector with respect to a fixed point is parallel to the direction of the affine normal ($H \neq 0$), or the direction of the affine normal is constant along M, ($H = 0$).

Let κ be the intrinsic Gaussian curvature of h and let K be the difference tensor between the induced connection and the Levi-Civita connection for h. The integrability conditions imply the affine theorem egregium,

$$\kappa = H + J.$$

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where $J = \frac{1}{2}h(K,K)$ is the *Pick invariant.*

Guggenheimer in [G] proved that a locally strongly convex homogeneous affine surface is in the following list:

Locally strongly convex homogeneous affine surfaces			
	Affine spheres $S = HI$		*Rank* S *=1*
	$J = 0$	$J = -H > 0$	$J = -\frac{2}{3}H > 0$
$\tau = H^2$ $H > 0$	Ellipsoid $x^2 + y^2 + z^2 = 1$		
$\tau = 0$	Elliptic paraboloid $z = x^2 + y^2$		surface $z = \frac{1}{2}(x^2 + y^{-\frac{2}{3}}),\ y > 0$
$\tau = H^2$ $H < 0$	two-sheeted hyperboloid $x^2 - y^2 - z^2 = 1$	surface $xyz = 1$	

Notice that locally strongly convex homogeneous affine surfaces with $J = 0$ or $J = -H$ are affine spheres, while in the case $J = -\frac{2}{3}H$, the rank of the affine shape operator is one.

Conversely, Kurose, Li-Penn and Magid-Ryan obtained, see [K],[LP],[MR]

Theorem 1 *Every locally strongly convex affine sphere with constant Pick invariant lies on a quadric or it is affinely equivalent to the surface given by* $xyz = 1$.

A generalization of this result is the following, see [DMMSV],

Theorem 2 *A locally strongly convex affine surface* M *has constant Pick invariant and constant affine mean curvature if and only if* M *lies on a homogeneous affine surface or* $J = -\frac{2}{3}H = constant$.

In [M], the second author proves,

Theorem 3 *A locally strongly convex affine surface with constant Pick invariant and vanishing affine Gauss-Kronecker curvature is contained in the graph of* $z = \frac{1}{2}(x^2 + y^{-\frac{2}{3}}),\ y > 0$ *or in an elliptic paraboloid.*

In the proof of this result we consider two cases:
1) If $J = 0$ then it is known that M is contained in an elliptic paraboloid, [B].
2) If $J \neq 0$ then the affine mean curvature never vanishes on an open dense subset M' of M, and, for any point of M', we can find a local orthonormal basis $\{E_1, E_2\}$, such that $SE_1 = 2HE_1$ and $SE_2 = 0$. Using this basis and the structure equations we get that the affine mean curvature satisfies a polynomial equation with constant coefficients, $g(H) = 0$, and H must be a constant. Thus, from the Theorem 2, we obtain that M is contained in the graph of $z = \frac{1}{2}(x^2 + y^{-\frac{2}{3}})$.

Two consequences of this result are:

Corollary 1 *The graph of* $z = \frac{1}{2}(x^2 + y^{-\frac{2}{3}})$ *is the only locally strongly convex affine surface with constant Pick invariant and rank* $S = 1$.

Corollary 2 *Let* M *be a locally strongly convex affine surface in* $\mathcal{A}^3$ *with constant Pick invariant, then* M *lies on a homogeneous affine surface if and only if* M *has vanishig affine Gauss-Kronecker curvature or constant affine mean curvature* $H \neq -\frac{3}{2}J$.

2 Global results

Among locally strongly convex homogeneous affine surfaces, the elliptic paraboloid has an special interest, because it is the only one which satisfies the Euler Lagrange equations for the affine area, (equivalently, it is the unique locally strongly convex homogeneous affine surface with vanishing affine mean curvature).

Concerning the elliptic paraboloid, different global characterizations have been obtained and all of them can be considered as partial solutions to the Affine Bernstein Problem which states, [Ch], [C1], [C2], [C3]:

> *Every locally strongly convex affine surface with complete affine metric and vanishing affine mean curvature is an elliptic paraboloid.*

We wish to emphasize two methods which have been used in the proofs of the results in this area. One of them consists of using the estructure equations of a locally strongly convex affine surface with vanishing affine mean curvature, together with some results for differential inequalities on complete surfaces with non-negative Gauss curvature.

By taking this method we proved, see [MM1], [MM2]:

Theorem 4 *A locally strongly convex affine complete surface with vanishing affine mean curvature and bounded affine Gauss-Kronecker curvature is an elliptic paraboloid.*

The second method consists of studying the affine conormal immersion N of a locally strongly convex affine surface with vanishing affine mean curvature. Using this method, Calabi and Li proved, see [C2], [C3], [L1], [L2],

Theorem 5 *To give a triple of harmonic functions* $N(u, v) = (N_1, N_2, N_3)$ *defined on a simply connected domain* Ω *of* $\mathbf{R}^2$ *and satisfying* $Det(N, N_u, N_v) > 0$ *in* Ω *(where* N_u *and* N_v *are the partial derivatives of* N *with respect to* u *and* v*) is equivalent to give a locally strongly convex affine surface with* $H = 0$ *, affine conormal* N *and affine metric* $ds^2 = Det(N, N_u, N_v)(du^2 + dv^2)$.

It is also know that the affine shape operator is identically zero if and only if a coordinate function of N is constant. In this case the affine surface M is contained in an elliptic paraboloid if and only if the others two coordinates functions are conjugate harmonic. In general we can state the following

Theorem 6 *Let $N : \Omega \longrightarrow \mathbf{R}^3$ be a harmonic map such that $Det(N, N_u, N_v) > 0$ in Ω. If two coordinate functions of N are conjugate harmonic. Then there exists a positive function ϕ on Ω such that*

$$\Delta log\phi \geq 0, \quad and \tag{2.1}$$

$$\Delta\left(\frac{Det(N, N_u, N_v)}{\phi}\right) \leq 0, \tag{2.2}$$

where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$.

Proof. If N_1 and N_2 are conjugate harmonic, then we take

$$\phi = N_{1u}^2 + N_{1v}^2 = N_{2u}^2 + N_{2v}^2$$

and by a straight computation we see that (1) and (2) are satisfied.

Remark 1. If the harmonic map N is a minimal immersion in the Euclidean space $\mathbf{R}^3$, then by taking $\phi = |N_u \wedge N_v|$, (1) and (2) are satisfied, see [HK].

Remark 2. We would like to state that, when in Theorem 6 one takes $\Omega = \mathbf{R}^2$, then from Liouville's Theorem, (1) and (2) imply that the function $\frac{Det(N,N_u,N_v)}{\phi}$ is a positive constant C and from Theorem 5, the affine metric $h = C\phi(du^2 + dv^2)$ has non-positive Gauss curvature $\kappa = -\frac{1}{2C\phi}\Delta log\phi$. Thus, one concludes that the associated locally strongly convex affine surface with vanishing affine mean curvature has flat affine metric and it must be contained in an elliptic paraboloid.

Consequently, one could solve the Affine Bernstein Problem if one solves the following

Problem:

Let $N : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$ be a harmonic map such that $Det(N, N_u, N_v) > 0$ in $\mathbf{R}^2$. Then there exists a positive function ϕ on $\mathbf{R}^2$ such that

$$\Delta log\phi \geq 0, \quad and$$

$$\Delta\left(\frac{Det(N, N_u, N_v)}{\phi}\right) \leq 0.$$

At the moment, we know that this problem is true when two coordinate functions of N are congujate harmonic or N is a minimal immersion in $\mathbf{R}^3$.

Remark 3. The Theorem 6 leads in another way to one result of Calabi about holomorphic curves [C3].

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