CONVEX AFFINE SURFACES WITH CONSTANT AFFINE MEAN CURVATURE

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An interesting (open) problem in Affine Differential Geometry is, (see [S1]): the classification of all affine complete, noncompact, locally strongly convex surfaces M, with constant affine mean curvature H, in the unimodular real affine 3-space $\mathcal{A}^3$.

The compact case was studied by Blaschke, he could prove: *"Every ovaloid in $\mathcal{A}^3$ with constant affine mean curvature is an ellipsoid".*

Blaschke's assertion holds true for affine complete, locally strongly convex surfaces with positive constant affine mean curvature in $\mathcal{A}^3$, (see [B] and [S2]).

The problem for affine-maximal surfaces, that is, H = 0 on M, is called Affine Bernstein Problem (see [Ch]) and states:

"Any locally strongly convex, affine complete, affine-maximal surface M in $\mathcal{A}^3$ is an elliptic paraboloid".

Partial solutions to this problem have been obtained with additional assumptions involving M (affine sphere, ([C1],[CY2],[J] [P]), global graph, ([C2]), or some conditions in the image of the conormal map, ([C3], [L1]), and Gauss map, ([L2]).

When H=constant<0 there are known results which characterize the hyperboloid and the surface $Q(a,2)=\{(x_1,x_2,x_3)\in\mathcal{A}^3 |\ x_1x_2x_3=a>0,\ x_1>0,\ x_2>0,\ x_3>0\}$ as complete hyperbolic <u>affine</u> <u>spheres</u> with Pick invariant satisfying some additional assumptions, ([LP], [K]).

In this communication, we give a step in the classification of the affine complete, locally strongly convex surfaces in $\mathcal{A}^3$ with <u>constant</u> affine mean curvature. We obtain the following result,

THEOREM .- *Let M be a locally strongly convex, affine complete surface in $\mathcal{A}^3$ with constant affine mean curvature H. Denote by τ, κ and J the affine Gauss-Kronecker curvature of M, the intrinsic Gaussian curvature of the affine metric and the Pick invariant, respectively. If*

(I) $J - cB^2 \geq d$, *for some real numbers c and d,* $c > \frac{2}{5}$, *and*

(II) $3J\kappa + 2HB \geq 0$,

where $B^2 = H^2 - \tau$. *Then M is one of the following surfaces:*

i) an ellipsoid,

ii) an elliptic paraboloid,

iii) an hyperboloid,

iv) an affine image of the surface Q(a,2), a>0.

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Notes:

It is known the affine egregium theorem: $\kappa = J + H$. Then we have

1.- A locally strongly convex, affine complete surface in A^3 with positive constant affine mean curvature has $\kappa \geq H > 0$ and, from Bonnet's Theorem, is compact. Then, assumptions (I) and (II), in the Theorem hold and Blaschke's result is a corollary.

2.- If M is an affine-maximal surface, then assumption (II) in Theorem holds. Thus, we obtain the following partial solution of the Affine Bernstein Problem (see [MM]):

"A locally strongly convex, affine complete, affine-maximal surface in A^3, with $\kappa + c\tau$ bounded from below by a constant, for some real number $c > \frac{2}{5}$, is an elliptic paraboloid".

In particular:

If the affine Gauss-Kronecker curvature is bounded from below, we obtain that M is an elliptic paraboloid.

3.- In the case that $H < 0$ we do not assume that M is an affine sphere (B^2 vanishes identically on M, see [S3]). However, we need to assume some growth conditions for B, (expressions (I) and (II)). Using this Theorem one can obtain the following result concerning Q(a,2):

"Let M be a locally strongly convex, affine complete surface in A^3 with H=constant<0. If the affine Gauss-Kronecker curvature is bounded from below and $3\kappa \geq 2B$, then M is an affine image of the affine sphere Q(a,2)".

Proof of the Theorem.

Let Δ be the Laplacian of the affine metric. If the affine mean curvature is constant, using the integrability conditions and the basic formulas for affine surfaces one gets (see appendix)

(F1) $$\Delta(\tfrac{1}{2}J + B^2) \geq 3J\kappa + 10JB^2 + 4HB^2 - \tfrac{1}{2}J + B^2 - 4B^4,$$

(F2) $$\Delta(\tfrac{1}{2}J + B) \geq 3J\kappa + 2HB.$$

If $H > 0$, then M is compact and, from (F2), one gets $J = 0$ and $B=0$, consequently M is an ellipsoid.

If $H \leq 0$, then, from (II), either $J = 0$ (and we have a quadric) or $\kappa \geq 0$. Assume $H\leq 0$ and $\kappa\geq 0$ on M. Then, from (I) and (F1) one gets,

$$\Delta(\tfrac{1}{2}J+B^2) \geq 3J^2 + 3JH + 10JB^2 + 4HB^2 - \tfrac{1}{2}J + B^2 - 4B^4 =$$
$$= \frac{10c-4}{1+c}(\tfrac{1}{2}J+B^2)^2 + \left(\frac{28d}{c(1+c)} + 6H-1\right)(\tfrac{1}{2}J+B^2) +$$

$$+\left(3-\frac{10c-4}{4(1+c)}\right)J^2+\left(10-\frac{10c-4}{1+c}\right)JB^2-$$
$$-\left(4+\frac{10c-4}{1+c}\right)B^4-\frac{14d}{c(1+c)}J+\left(2-2H-\frac{28d}{c(1+c)}\right)B^2\geq$$
$$\geq\frac{10c-4}{1+c}\left(\tfrac{1}{2}J+B^2\right)^2+\left(\frac{28d}{c(1+c)}+6H-1\right)\left(\tfrac{1}{2}J+B^2\right)+$$
$$+\left(\frac{14}{1+c}J-\frac{14c}{1+c}B^2\right)B^2-\frac{14d}{c(1+c)}J\geq$$
$$\geq\frac{10c-4}{1+c}\left(\tfrac{1}{2}J+B^2\right)^2+\left(\frac{28d}{c(1+c)}+6H-1\right)\left(\tfrac{1}{2}J+B^2\right)-\frac{14d^2}{c(1+c)}$$

that is, $\Delta v \geq f(v)$, where $v = \frac{1}{2}J + B^2$ and $f:\mathbb{R}\longrightarrow\mathbb{R}$ is a polynomial of degree 2 with positive principal coefficient. Using Theorem 8 of [CY1], we conclude that $\frac{1}{2}J + B^2$ is bounded from above by a constant, and $\frac{1}{2}J + B$ must be bounded from above also.

One can supposes that M is simply connected (otherwise we may pass to the universal covering surface of M). As M is affine complete with $\kappa \geq 0$ and there is no compact affine surface in A^3 with $H \leq 0$ (see [CY2]), then M is conformally equivalent to $\mathbb{C}$.

From (II) and (F2), $\frac{1}{2}J + B$ is a bounded subharmonic function on the Riemann surface $M\equiv\mathbb{C}$ which implies that $\frac{1}{2}J + B$ is constant, and $3J\kappa + 2HB = 0$. Thus, the intrinsic Gaussian curvature κ is a nonnegative constant. As M is not compact, one gets $\kappa = 0$ on M, $J = -H$ and $HB = 0$. Therefore, if $H=0$, then $J = 0$ and M is an elliptic paraboloid and if $H < 0$, then $B=0$ and M is an affine sphere with $J = -H > 0$ that is, an affine image of the surface $Q(a,2)$, $a>0$, (see [LP]).

Appendix.

Let M be an oriented, connected and convex affine surface immersed in A^3. If ξ is the affine normal of the immersion and we denote by $\overline{\nabla}$, h and S the induced connection, the affine metric and the affine Weingarten operator associated to ξ, respectively, then the equations of the immersion are given by:

(1) $\overline{R}(X,Y)Z = h(Y,Z)SX-h(X,Z)SY$ (Gauss)

(2) $(\overline{\nabla}h)(X,Y,Z) = (\overline{\nabla}h)(Y,X,Z)$ (Codazzi)

(3) $(\overline{\nabla}S)(X,Y) = (\overline{\nabla}S)(Y,X)$ (Codazzi)

(4) $h(SX,Y) = h(X,SY)$ (Ricci)

for any X, Y, Z tangent vector fields to M, $(X,Y,Z \in TM)$, where $\overline{R}$ is the curvature tensor of $\overline{\nabla}$.

Let $\hat{\nabla}$ be the Levi-Civita connection for the affine metric h.

If we denote the difference tensor between $\bar{\nabla}$ and $\hat{\nabla}$ by K, then

(5) $$K(X,Y) = \bar{\nabla}_X Y - \hat{\nabla}_X Y, \qquad X,Y\in TM$$

and one obtains the following relations

(6) $$h(K(X,Y),Z) = -(1/2)(\bar{\nabla}h)(X,Y,Z), \qquad X,Y,Z\in TM$$

(7) $$\text{trace } K_X = 0 \qquad X\in TM$$

where $K_X Y = K(X,Y)$ for any $X,Y\in TM$.

From (1), (5) and (7) it follows that the intrinsic Gaussian curvature κ of the affine metric h is given by

(8) $$\kappa h(X,Y) = h(X,Y)H + \text{trace}(K_X K_Y), \qquad X,Y\in TM$$

where

$$H = \frac{1}{2}\,\text{trace}S$$

is the <u>affine</u> <u>mean</u> <u>curvature</u> of the immersion.

Let $\{E_1,E_2\}$ be an orthonormal frame with respect to the affine metric and parallel at a point $x\in M$. One writes:

(9) $$\hat{\nabla}_{E_1} E_1 = pE_2 , \quad \hat{\nabla}_{E_2} E_2 = qE_1$$
$$K(E_1,E_1) = aE_1 + bE_2$$
$$SE_1 = (H+\alpha)E_1 + \beta E_2$$

for some functions p, q, a, b, α, and β defined on a neighbourhood of x, then from (2), (4), (6) and (7) one gets

(10) $$p(x) = q(x) = 0,$$
$$K(E_1,E_2) = bE_1 - aE_2 , \qquad K(E_2,E_2) = -aE_1 - bE_2$$
$$SE_2 = \beta E_1 + (H-\alpha)E_2.$$

From (1), (8), (9) and (10),

(11) $$\kappa = H + 2(a^2+b^2) = q_1 + p_2 - p^2 - q^2, \qquad J = 2(a^2+b^2),$$

(12) $$b_1 - a_2 = -\beta - 3(pa-qb), \qquad a_1 + b_2 = -\alpha + 3(bp+qa),$$

where by $(\ )_1$ and $(\ )_2$ we denote the covariant derivatives respect to E_1 and E_2 respectively.

In the rest we suppose that <u>H</u> <u>is</u> <u>constant</u>. Then from (3), (9) and (10),

$$(13)\qquad \beta_1 - \alpha_2 = 2(\alpha b-\beta a+q\beta-p\alpha), \qquad \beta_2 + \alpha_1 = 2(\beta b+\alpha a+p\beta+q\alpha).$$

If we denote by Δ and ∇ the Laplacian and the Gradient of the affine metric h, respectively, then (making the calculations at the point $x\in M$), (11) and (12) gives

$$(14)\quad a\Delta a+b\Delta b = a(a_{11}+a_{22}) + b(b_{11}+b_{22}) = 3(a^2+b^2)\kappa + a(\beta_2-\alpha_1)-b(\beta_1+\alpha_2)$$

and

$$(15)\qquad \alpha^2+\beta^2 = |\nabla a|^2 + |\nabla b|^2 - 2b_1a_2 + 2a_1b_2.$$

From (11), (14) and (15) one has

$$(16)\quad \tfrac{1}{2}\Delta J = 3J\kappa + (\alpha^2+\beta^2) + (a_1-b_2)^2 + (b_1+a_2)^2 + 2a(\beta_2-\alpha_1)-2b(\beta_1+\alpha_2).$$

Now, using (11) and (13)

$$(17)\qquad \alpha\Delta\alpha + \beta\Delta\beta = \alpha(\alpha_{11}+\alpha_{22}) + \beta(\beta_{11}+\beta_{22}) =$$
$$= 8(a^2+b^2)B^2 + 2HB^2 + 4\beta\alpha(b_1+a_2) + 2(\alpha^2-\beta^2)(a_1-b_2)$$

where $B^2 = (\alpha^2+\beta^2) = H^2 - \det S$, thus adding the squares in (13)

$$4(a^2+b^2)B^2 = |\nabla\alpha|^2 + |\nabla\beta|^2 - 2\beta_1\alpha_2 + 2\beta_2\alpha_1$$

and

$$(18)\qquad B^{-1}(|\nabla\alpha|^2 + |\nabla\beta|^2) - B^{-3}|\alpha\nabla\alpha + \beta\nabla\beta|^2 =$$
$$= (a^2+b^2)B + \tfrac{1}{4}B^{-1}[(\alpha_1-\beta_2)^2 + (\beta_1+\alpha_2)^2]-$$
$$- B^{-3}[\tfrac{1}{2}(\alpha^2-\beta^2)(|\nabla\alpha|^2-|\nabla\beta|^2) + 2\alpha\beta(\alpha_1\beta_1+\alpha_2\beta_2)],$$

and from (17) and (18) one gets

$$(19)\qquad \Delta B = 9(a^2+b^2)B + B^{-1}\{4\beta\alpha(b_1+a_2)+2(\alpha^2-\beta^2)(a_1-b_2)\}-$$
$$-B^{-3}\{\tfrac{1}{2}(\alpha^2-\beta^2)(|\nabla\alpha|^2-|\nabla\beta|^2) + 2\alpha\beta(\alpha_1\beta_1+\alpha_2\beta_2)\} + 2HB +$$
$$+ \tfrac{1}{4}B^{-1}[(\alpha_1-\beta_2)^2+(\beta_1+\alpha_2)^2],$$

Furthermore from (13)

$$(20)\qquad \begin{aligned} b &= \tfrac{1}{2}(\alpha^2+\beta^2)^{-1}[\alpha(\beta_1-\alpha_2)+\beta(\beta_2+\alpha_1)],\\ a &= \tfrac{1}{2}(\alpha^2+\beta^2)^{-1}[\alpha(\beta_2+\alpha_1)-\beta(\beta_1-\alpha_2)],\end{aligned}$$

and one gets

$$(21)\quad (\alpha_1-\beta_2)[a(\alpha^3-3\alpha\beta^2)+b(3\alpha^2\beta-\beta^3)] + (\beta_1+\alpha_2)[a(3\alpha^2\beta-\beta^3)-b(\alpha^3-3\alpha\beta^2)] =$$

$$= \tfrac{1}{2}(\alpha^2-\beta^2)(|\nabla\alpha|^2-|\nabla\beta|^2) + 2\alpha\beta(\alpha_1\beta_1+\alpha_2\beta_2).$$

Using (16), (19), (20) and (21), one has

$$(F2)\quad \Delta(\tfrac{1}{2}J + B) = 3J\kappa + 2HB + [B^{-1}2\beta\alpha+(b_1+a_2)]^2 + [B^{-1}(\alpha^2-\beta^2)+(a_1-b_2)]^2 +$$

$$+ \tfrac{1}{3}B^{-1}\left\{[2^{-1/2}(\alpha_1-\beta_2)-3a2^{1/2}B]^2 + [2^{-1/2}(\beta_1+\alpha_2)-3b2^{1/2}B]^2\right\} +$$

$$+ \tfrac{1}{6}B^{-1}\left\{\left[2^{-1/2}(\alpha_1-\beta_2) - 3B^{-2}2^{1/2}[a(\alpha^3-3\alpha\beta^2) + b(3\alpha^2\beta-\beta^3)]\right]^2 +\right.$$

$$\left.+ \left[2^{-1/2}(\beta_1+\alpha_2)-3B^{-2}2^{1/2}[a(3\alpha^2\beta-\beta^3)-b(\alpha^3-3\alpha\beta^2)]\right]^2\right\} \geq$$

$$\geq 3J\kappa + 2HB .$$

In a similar way, from (13), (16) and (17), one can gets

$$(F1)\quad \Delta(\tfrac{1}{2}J + B^2) = 3J\kappa + B^2 + 10JB^2 + 4HB^2 - 4B^4 - \tfrac{1}{2}J +$$

$$+ \left[4\beta\alpha+(b_1+a_2)\right]^2+\left[2(\alpha^2-\beta^2)+(a_1-b_2)\right]^2+\left[a+(\beta_2-\alpha_1)\right]^2+\left[b-(\beta_1+\alpha_2)\right]^2 \geq$$

$$\geq 3J\kappa + B^2 + 10JB^2 + 4HB^2 - 4B^4 - \tfrac{1}{2}J.$$

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This paper is in final form and no version will appear elsewhere.