Flat fronts in hyperbolic 3-space with prescribed singularities [1]

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Abstract

The paper deals with the study of flat fronts in the hyperbolic 3-space, $\mathbb{H}^3$. We characterize when an analytic curve of $\mathbb{H}^3$ is in the singular set of some flat front with prescribed cuspidal edges and swallowtails singularities. We also prove every complete flat front with a non degenerate analytic planar singular set must be rotational.

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1 Introduction

Many results about Monge-Ampère equations have important geometric applications, see the recent survey [17]. In particular, the unimodular Hessian equation

$$\phi_{xx}\phi_{yy} - \phi_{xy}^2 = 1$$

admits a complex resolution in terms of holomorphic data. This fact can be applied to obtain conformal representations of two classical families of surfaces (maybe with some kind of singularities): flat surfaces (fronts) in $\mathbb{H}^3$ and improper affine spheres (maps) in $\mathbb{R}^3$, see [4, 6] (and [13, 14]).

In both cases, the surfaces can be represented by using a pair of holomorphic 1-forms, (ω, θ) and the points where $|\theta| = |\omega| \neq 0$ give non-degenerate singularities that, locally, can be parameterized by smooth curves. Generically, the image of these curves are singular curves with cuspidal edges and swallowtails,

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see [11, 12] and the KRSUY criterion in the next section. However, the image can be constant and provide isolated singularities. Conversely, as an analytic application of the conformal representations, one can solve the corresponding Cauchy problem and prove that the singular set may determine uniquely the corresponding surfaces, see [1, 8].

The paper is organized as follows. In Section 2 we review the theory of flat surfaces (fronts) in $\mathbb{H}^3$ together with their associated conformal representations in [6, 13], and the criterion for singular points in [12].

In Section 3 we characterize when an analytic curve in $\mathbb{H}^3$ is a singular curve of some flat front with prescribed cuspidal edges and swallowtails.

In Section 4 we classify embedded complete flat fronts with a non degenerate analytic planar singular set and complete flat fronts whose singular set is a finite number of isolated singularities.

Finally, in Section 5 we show how, using the singular set, it is possible to describe, explicitly, a local correspondence between flat fronts and improper affine maps.

2 Conformal representation of flat fronts in $\mathbb{H}^3$

We consider a 2-manifold Σ and a flat immersion $\psi : \Sigma \longrightarrow \mathbb{H}^3$. Then, from the Gauss equation, the second fundamental form $d\sigma^2$ is definite and so Σ is orientable and it inherits a canonical Riemann surface structure. This fact provides a conformal representation for the immersion ψ that let to recover any flat surface in $\mathbb{H}^3$ in terms of holomorphic data (see [6] and [13] for the details).

Actually, for any $p \in \Sigma$, there exist $G(p), G_*(p) \in \mathbb{C}_\infty$ distinct points in the ideal boundary such that the oriented normal geodesic at $\psi(p)$ is the geodesic in $\mathbb{H}^3$ starting from $G_*(p)$ towards $G(p)$. The maps $G, G_* : \Sigma \longrightarrow \mathbb{C}_\infty$ are called the *hyperbolic Gauss maps* and it is proved in [6] that they are holomorphic when we regard $\mathbb{C}_\infty$ as the Riemann sphere.

Later, Kokubu, Umehara and Yamada obtained in [13] how to recover flat immersions with some admissible singularities (flat fronts) in terms of the hyperbolic Gauss maps.

Here, we explain this representation in the Hermitian model of the hyperbolic space. That is, we identify the Lorentz-Minkowski 4-space $\mathbb{L}^4$ with the set of 2×2 Hermitian matrices by means of

$$(x_0, x_1, x_2, x_3) \in \mathbb{L}^4 \longleftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \in Herm(2). \tag{2.1}$$

Thus, the Minkowski metric

$$\langle , \rangle = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2, \tag{2.2}$$

is identified to $\langle X, X \rangle = -det(X)$ for $X \in Herm(2)$, and $\mathbb{H}^3$ can be realized as

$$\begin{aligned} \mathbb{H}^3 &= \{X \in Herm(2) \ : \ det(X) = 1, trace(X) > 0\} \\ &= \{\Phi\Phi^* \ : \ \Phi \in \mathbf{SL}(2, \mathbb{C})\}, \end{aligned} \tag{2.3}$$

where $\Phi^* = \overline{\Phi}^t$.

We say that a map $\psi : \Sigma \longrightarrow \mathbb{H}^3$ is a flat front if the curvature of ψ vanishes identically at every regular point and there exists a map $\eta : \Sigma \longrightarrow \mathbb{S}^3_1 = \{X \in Herm(2); det(X) = -1\}$, called the Gauss map of ψ, such that

$$\langle \psi, \eta \rangle = 0 = \langle d\psi, \eta \rangle \qquad and \qquad \langle d\psi, d\psi \rangle + \langle d\eta, d\eta \rangle \neq 0 \tag{2.4}$$

at every point.

On the other hand, we say that a holomorphic map $F : \Sigma \longrightarrow \mathbf{SL}(2, \mathbb{C})$ is a Legendrian curve if there are holomorphic 1-forms θ, ω on Σ such that

$$F^{-1}dF = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix}. \tag{2.5}$$

Theorem 2.1 ([6], [13]). *If $F : \Sigma \longrightarrow \mathbf{SL}(2, \mathbb{C})$ is a holomorphic Legendrian curve, then $\psi = FF^* : \Sigma \longrightarrow \mathbb{H}^3$ is a flat front whose Gauss map is given by*

$$\eta = F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F^*. \tag{2.6}$$

Conversely, every simply-connected flat front can be described in this way and F is given by the hyperbolic Gauss maps as

$$F = \begin{pmatrix} G/\xi & \xi G_*/(G - G_*) \\ 1/\xi & \xi/(G - G_*) \end{pmatrix}, \qquad \xi = c \exp\Big(\int \frac{dG}{G - G_*}\Big), \tag{2.7}$$

$c \in \mathbb{C} \setminus \{0\}$. In particular, from (2.5) and (2.6), one has

$$d\psi = F \begin{pmatrix} 0 & \theta + \overline{\omega} \\ \omega + \overline{\theta} & 0 \end{pmatrix} F^*, \qquad d\eta = F \begin{pmatrix} 0 & -\theta + \overline{\omega} \\ \omega - \overline{\theta} & 0 \end{pmatrix} F^* \tag{2.8}$$

and the fundamental forms of ψ can be written as:

$$\begin{array}{rcl} ds^2 = \langle d\psi, d\psi \rangle & = & (\theta + \overline{\omega})(\omega + \overline{\theta}), \\ d\sigma^2 = -\langle d\psi, d\eta \rangle & = & |\theta|^2 - |\omega|^2 \geq 0, \\ \langle d\psi, d\psi \rangle + \langle d\eta, d\eta \rangle & = & 2(|\theta|^2 + |\omega|^2) > 0. \end{array} \tag{2.9}$$

Consequently, from (2.4) and (2.9), if we take a conformal parameter z respect $d\sigma^2$, then

$$\langle \psi_z, \eta_z \rangle = 0 = \langle \psi_z, \psi_z \rangle + \langle \eta_z, \eta_z \rangle \tag{2.10}$$

and the following expression holds:

$$\psi_z = i\psi \times \eta \times \eta_z, \tag{2.11}$$

where $\langle \psi \times \eta \times \eta_z, x \rangle$ is the determinant $[\psi, \eta, \eta_z, x]$ for all $x \in \mathbb{L}^4$.

Hence, the second fundamental form is given by

$$d\sigma^2 = 2\rho dz d\overline{z}, \quad \rho = -\langle \psi_z, \eta_{\overline{z}} \rangle = -i[\psi, \eta, \eta_z, \eta_{\overline{z}}] = -i[\psi, \eta, \psi_z, \psi_{\overline{z}}] \geq 0 \tag{2.12}$$

and $z_0 \in \Sigma$ is a non-degenerate singular point of ψ if and only if

$$\rho(z_0) = 0, \qquad (|\theta(z_0)| = |\omega(z_0)| \neq 0), \qquad d\rho\big|_{z_0} \neq 0. \tag{2.13}$$

In this case, the singular set of ψ around z_0 becomes a regular curve $\gamma : I \subset \mathbb{R} \longrightarrow \Sigma$ and one has the following criterion for the singular curve $\beta = \psi \circ \gamma : I \longrightarrow \mathbb{H}^3$.

Theorem 2.2 ([12]). *If ν is a vector field along γ, with $\nu(s) \neq 0$ in the kernel of $d\psi_{\gamma(s)}$ for any s in the interval I, then the following hold.*

1. *$\gamma(0) = z_0$ is a cuspidal edge if and only if $det(\gamma'(0), \nu(0)) \neq 0$, where det denotes the determinant of 2×2 matrices and prime indicates differentiation with respect to s.*
2. *$\gamma(0) = z_0$ is a swallowtail if and only if $det(\gamma'(0), \nu(0)) = 0$ and*

$$\frac{d}{ds}\Big|_{s=0} det(\gamma'(s), \nu(s)) \neq 0.$$

3 Prescribed singular curves

In this section we characterize when an analytic curve in $\mathbb{H}^3$ is a non-degenerate singular curve of some flat front and its points are either cuspidal edge or swallowtail singularities.

Motivated by (2.4), we say that a pair of analytic maps $\beta : I \longrightarrow \mathbb{H}^3$ and $V : I \longrightarrow \mathbb{S}^3_1$ is a pair of initial data if

$$\langle \beta, V \rangle = \langle \beta', V \rangle = 0 \qquad \textit{and} \qquad (\beta', V') \neq (0, 0) \tag{3.1}$$

hold on the interval I. These necessary conditions are also sufficient in the following sense:

Theorem 3.1 ([8]). *If $\{\beta, V\}$ is a pair of initial data, then there exists a unique flat front $\psi = FF^* : \Omega \subset \mathbb{C} \longrightarrow \mathbb{H}^3$, such that $\beta(s) = \psi(s, 0)$ and $V(s) = \eta(s, 0)$ for all $s \in I \subset \Omega$. Moreover, F is given by (2.7) by taking*

$$G(z) = \frac{\mathcal{N}_1(z) + i\mathcal{N}_2(z)}{\mathcal{N}_0(z) - \mathcal{N}_3(z)}, \qquad G_*(z) = \frac{\mathcal{L}_1(z) + i\mathcal{L}_2(z)}{\mathcal{L}_0(z) - \mathcal{L}_3(z)}, \tag{3.2}$$

where $\mathcal{N}(z)$ and $\mathcal{L}(z)$ are holomorphic extensions of $\mathcal{N}(s) = \beta(s) + V(s)$ and $\mathcal{L}(s) = \beta(s) - V(s)$.

As in [8] the real parameter $s \in I$ extends to the complex parameter $z = s + it \equiv (s, t) \in \Omega$ and $\gamma(s) = (s, 0)$, $\forall s \in I$. Thus, the criterion in Theorem 2.2 only depends of the second coordinate of $\nu(s)$ and a natural question is when the flat front determined by $\{\beta, V\}$ has β as a singular curve with prescribed cuspidal edges and swallowtails.

Of course, from (2.10) and (2.12), we obtain

$$-2\rho = \langle \psi_s, \eta_s \rangle = \langle \psi_t, \eta_t \rangle, \qquad \langle \psi_s, \eta_t \rangle = 0 = \langle \psi_t, \eta_s \rangle \tag{3.3}$$

in Ω. Then, $\beta(I)$ is contained in the set of singularities of ψ if and only if

$$\langle \beta', V' \rangle = 0 \tag{3.4}$$

along I. Equivalently, from (2.8) and (2.13), we get

$$\psi_z\, dz = \frac{+\theta}{\overline{\omega}}\, \psi_{\overline{z}}\, d\overline{z} = \frac{+\omega}{\overline{\theta}}\, \psi_{\overline{z}}\, d\overline{z}, \qquad \eta_z\, dz = \frac{-\theta}{\overline{\omega}}\, \eta_{\overline{z}}\, d\overline{z} = \frac{-\omega}{\overline{\theta}}\, \eta_{\overline{z}}\, d\overline{z}$$

in $I = \{z \in \Omega; z = \overline{z}\}$ and

$$i(\omega - \overline{\theta})\psi_s = (\omega + \overline{\theta})\psi_t, \qquad i(\omega + \overline{\theta})\eta_s = (\omega - \overline{\theta})\eta_t. \tag{3.5}$$

Moreover, from (2.4),

$$(\beta', V') = (\psi_s, \eta_s) \neq (0, 0). \tag{3.6}$$

Proposition 3.2. *If $\beta(I)$ lies in a hyperbolic plane (or equivalently, V is a constant η_0), then β must be regular in I and every non-degenerate point of ψ in $\beta(I)$ is a cuspidal edge.*

Proof. Because $V(.) = \eta(., 0) = \eta_0$ is a constant, then

$$\eta_s(s, 0) = 0 = \psi_t(s, 0), \qquad \text{in I},$$

and from (3.6), $\beta : I \longrightarrow \mathbb{H}^3$ must be a regular curve.

Moreover, the kernel of $d\psi$ at $\gamma(s) = (s, 0)$ is spanned by $\nu(s) = (0, 1)$ and, from Theorem 2.2, all the non-degenerate singularities of ψ in $\beta(I)$ are cuspidal edges. □

Remark 1. In the hypothesis of the above Proposition we have that $\theta = \overline{\omega}$ along I, and the Hopf differential, $Q = \omega\theta$, may be written as

$$Q = \omega\overline{\omega} = \theta\overline{\theta} \qquad \text{in I.}$$

In general, we get the following non-degenerate condition for regular curves.

Theorem 3.3. *A regular analytic curve $\beta : I \longrightarrow \mathbb{H}^3$ is a non-degenerate singular curve of some flat front ψ if and only if*

$$(\beta \times \beta' \times \beta'')(s) \neq 0, \qquad \forall s \in I. \tag{3.7}$$

Moreover, all the points of $\beta(I)$ must be cuspidal edges.

Proof. As in [8], we take a complex parameter $z = s + it \in \Omega$, such that $\beta(s) = \psi(s, 0)$, for all $s \in I \subset \Omega$.

Now, if $\beta(I)$ is contained in the set of singularities of ψ, then, from (3.1) and (3.4), we have

$$\beta \times \beta' \times \beta'' = \|\beta \times \beta' \times \beta''\| V, \tag{3.8}$$

where $V(s) = \eta(s,0)$, $s \in I$.

From the regularity of β, (2.11), (3.3) and (3.5), we obtain

$$\psi_t(s,0) = f(s)\psi_s(s,0), \qquad \eta_s(s,0) = -f(s)\eta_t(s,0) = f(s)\beta(s) \times V(s) \times \beta'(s)$$

with $f : I \longrightarrow \mathbb{R}$ an analytic function and the non-degenerate condition is written as

$$\begin{aligned} 0 \neq \frac{d}{dt}\Big|_{(s,0)}(-2\rho) &= \frac{d}{dt}\Big|_{(s,0)}\langle \psi_s, \eta_s\rangle = \big(\langle \psi_{ts}, \eta_s\rangle - \langle \psi_{ss}, \eta_t\rangle\big)(s,0) \\ &= -(f^2+1)\langle \psi_{ss}, \eta_t\rangle = -(f^2+1)\|\beta \times \beta' \times \beta''\|. \end{aligned}$$

Moreover, from Theorem 2.2, we conclude that $\beta(s)$ is a cuspidal edge, because the kernel of $d\psi$ at $\gamma(s) = (s,0)$ is spanned by $\nu(s) = (-f(s), 1)$, for all $s \in I$.

Conversely, from (3.7), (3.8) and Theorem 3.1, it is clear that the flat front ψ is determined by β. Actually, V is given by the following expression:

$$V = \frac{\beta \times \beta' \times \beta''}{\|\beta \times \beta' \times \beta''\|}.$$

□

Corollary 3.4. *Let $\beta : I \longrightarrow \mathbb{H}^3$ be a non-constant analytic curve contained in a hyperplane through the origin orthogonal to $\eta_0 \in \mathbb{S}^3_1$.*

If β is a non-degenerate singular curve of a flat front ψ, then β must be a locally convex regular curve, its points are cuspidal edges and the Gauss map η of ψ along β is the binormal η_0 of β.

Proof. From the above theorem, it is clear that $\beta \times \beta' \times \beta'' \neq 0$ and

$$\eta = \frac{\beta \times \beta' \times \beta''}{\|\beta \times \beta' \times \beta''\|} = \eta_0,$$

at the regular points of β, that is, in a dense subset of I. Hence, $\eta(s,0) = \eta_0$, for all $s \in I$ and the proof follows from (3.6). □

As a consequence, we also know that swallowtails only appear when β' has isolated zeros. Actually, we may prove the following result:

Theorem 3.5. *Let $\beta : I \longrightarrow \mathbb{H}^3$ an analytic curve satisfying*

$$(\beta \times \beta' \times \beta'')(s) \neq 0, \qquad \forall s \in I - \{0\}.$$

Then, β is a non-degenerate singular curve of some flat front ψ and $\beta(0)$ is a swallowtail if and only if $0 \in I$ is a zero of β', $\|\beta \times \beta' \times \beta''\|$ and $[\beta, \beta', \beta'', \beta''']$ of order 1, 2 and 3, respectively.

Proof. As in Theorem 3.3, V is given by (3.8) and ψ is determined by β. In particular, we obtain

$$[\beta, \beta', \beta''', \beta''] = \|\beta \times \beta' \times \beta''\| \langle V', \beta''\rangle. \tag{3.9}$$

However, if $\beta(0)$ is a swallowtail, then $\beta'(0) = 0$ and we have, around $s = 0$,

$$\psi_s(s,0) = g(s)\psi_t(s,0), \qquad g(s)\eta_s(s,0) = -\eta_t(s,0) = \beta(s) \times V(s) \times \beta'(s)$$

and

$$g\langle V', \beta''\rangle = [\beta, V, \beta', \beta''] = \|\beta \times \beta' \times \beta''\|, \tag{3.10}$$

where $g :]-\varepsilon, +\varepsilon[\subset I \longrightarrow \mathbb{R}$ is an analytic function, with $g(0) = 0$ and $\varepsilon > 0$.

Thus, the non-degenerate condition gives

$$\begin{aligned} 0 \neq \frac{d}{dt}\Big|_{(s,0)}(-2\rho) &= \frac{d}{dt}\Big|_{(s,0)}\langle \psi_s, \eta_s\rangle = \big(\langle \psi_{ts}, \eta_s\rangle - \langle \psi_{ss}, \eta_t\rangle\big)(s,0) \\ &= -\Big(\frac{1}{g^2} + 1\Big)\langle \psi_{ss}, \eta_t\rangle = -\Big(\frac{1}{g^2} + 1\Big)\|\beta \times \beta' \times \beta''\|. \end{aligned}$$

Now, using that the kernel of $d\psi$ at $\gamma(s) = (s, 0)$ is spanned by $\nu(s) = (1, -g(s))$ and Theorem 2.2, g and β' have a zero of order 1 at 0. Thus, from the above expression $\|\beta \times \beta' \times \beta''\|$ must have a zero of order 2 at 0.

From (3.9) and (3.10), we conclude that 0 is a zero of $\langle V', \beta''\rangle$ and $[\beta, \beta', \beta'', \beta''']$ of order 1 and 3, respectively.

Conversely, we can apply Theorem 3.1, with $\beta'(0) = 0$, because the hypothesis give $V'(0) \neq 0$. In fact, we have $\beta'(s) = s\ b'(s)$, for an analytic curve $b : I \longrightarrow \mathbb{L}^4$, with $[\beta, b', b'', b'''](0) \neq 0$ and $\langle V'(0), b''(0)\rangle \neq 0$. □

Remark 2. As in [5, 16], if we consider an analytic curve with singular curvature

$$\kappa(s) = \frac{\|\beta \times \beta' \times \beta''\|}{\|\beta'\|^3}(s) \in \mathbb{R} - \{0\}, \qquad \forall s \in I - \{0\},$$

and singular torsion

$$\tau(s) = \frac{[\beta, \beta', \beta'', \beta''']}{\|\beta \times \beta' \times \beta''\|^2}(s) \in \mathbb{R}, \qquad \forall s \in I - \{0\},$$

then $\beta(0)$ is a swallowtail of some flat front if and only if 0 is a zero of order 1 of β' and

$$lim_{s\to 0}\|\beta'\|\kappa \neq 0, \qquad lim_{s\to 0}\|\beta'\|\tau \neq 0.$$

Remark 3. By using Ribaucour transformations on rotational flat fronts, see [3], one can describe explicitely many examples of complete flat fronts with any even number of swallowtails as in Figures 1 and 2.

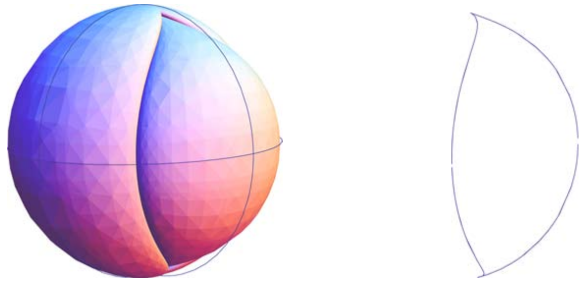

Figure 1: Flat front with only two swallowtails and its singular set.

Figure 2: Flat front with only four swallowtails and its singular set.

Remark 4. When the holomorphic data (ω, θ) satisfy

$$\int \omega = \frac{\theta}{\omega},$$

ψ can be described using Airy's functions. In this case, see also [9], we have a flat front with three ends and three swallowtails (see Figure 3).

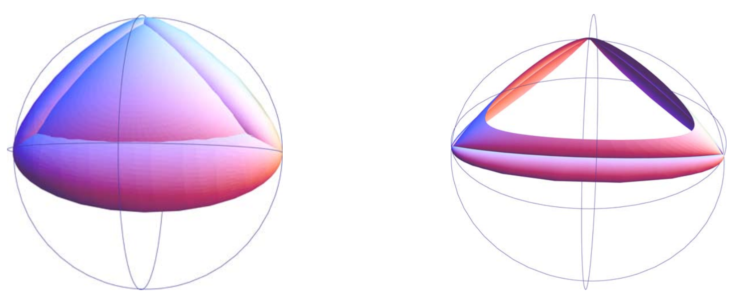

Figure 3: Flat front with only three swallowtails.

Remark 5. *From (3.5) and (3.6), we observe that* $\theta = -\overline{\omega}$ *along* I*, if and only if* $\beta(I) = a \in \mathbb{H}^3$ *and* $V : I \longrightarrow \mathbb{S}^3_1$ *is a regular analytic curve. In this case, the Hopf differential* $Q = \omega\theta$ *verifies*

$$Q = -\omega\overline{\omega} = -\theta\overline{\theta}.$$

For the case of isolated singularities we obtain a new proof of the following result in [8]:

Theorem 3.6. *Let* $V : I \longrightarrow \mathbb{S}^3_1 \cap \{a\}^\perp$ *be a regular analytic curve satisfying*

$$(V \times V' \times V'')(s) \neq 0, \qquad \forall s \in I.$$

Then there exists a unique flat front ψ*, such that* $\psi(s,0) = a$ *is an isolated singularity and* $\eta(s,0) = V(s)$*, for all* $s \in I$*.*

Proof. It follows because the pair of initial data $\{a, V\}$ determines a flat front ψ, with

$$\psi_s(s,0) = 0 = \eta_t(s,0), \qquad \psi_t(s,0) = -a \times V(s) \times V'(s)$$

and

$$\frac{d}{dt}\Big|_{(s,0)}(-2\rho) = \frac{d}{dt}\Big|_{(s,0)}\langle\psi_s, \eta_s\rangle = -\langle\psi_t, \eta_{ss}\rangle(s,0) = \|V \times V' \times V''\|(s) \neq 0.$$

□

4 Global Results

The aim of this section is to determine the global behavior of embedded complete flat fronts such that any connected component of its singular set is mapped on a hyperplane through the origin and those with only a finite number of isolated singularities.

4.1 Embedded complete flat fronts with finitely many isolated singularities.

In [2, 7] is proved the existence of embedded complete flat surfaces with any finite number of isolated singularities, see Figure 4. The situation is totally different for complete flat fronts, in fact from the generalized symmetry principle one has, [8, Corollary 16], any flat front must be symmetric with respect to point reflection in $\mathbb{H}^3$ through any isolated embedded singularity.

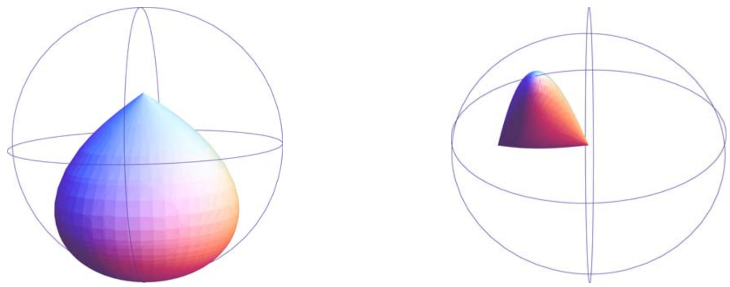

Figure 4: Complete flat surfaces with isolated singularities

As immediate consequence we have

Theorem 4.1. *Any embedded complete flat front whose singular set is a finite number of isolated singularities must be rotational, see Figure 5.*

Proof. An easy application of the Maximum Principle let us to see that any embedded complete flat front with only one isolated singularity must be rotational (another proof of this fact can be found in [7]). Consequently, it is enough to prove that if a complete flat front $\psi : \Sigma \longrightarrow \mathbb{H}^3$ has two different isolated singularities p_1 and p_2, then it has infinitely many isolated singularities.

In fact, having in mind that $\psi(\Sigma)$ is symmetric with respect the reflections, s_1 and s_2, in $\mathbb{H}^3$ through the points p_1 and p_2, respectively, we get that

$$s_1(p_2), s_2(p_1), s_2 \circ s_1(p_2), s_1 \circ s_2(p_1), s_1 \circ s_2 \circ s_1(p_2), \cdots$$

also are isolated singularities of the front. $\square$

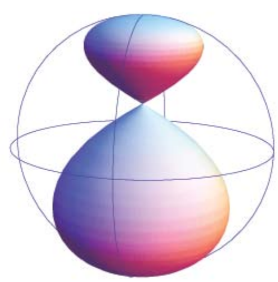

Figure 5: Hourglass flat front

4.2 Embedded complete flat fronts with a planar singular set

We shall prove the following result:

Theorem 4.2. *Let $\psi : \Sigma \longrightarrow \mathbb{H}^3$ be an embedded complete flat front with a non-degenerate analytic singular set $\mathcal{S} \subset \Sigma$ such that $\psi(\mathcal{S})$ lies in a hyperplane through the origin of $\mathbb{L}^4$. Then ψ is a snowman rotational flat front (see Figure 6)*

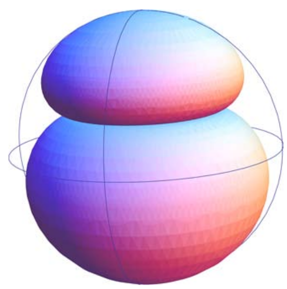

Figure 6: Snowman flat front.

First, we recall some well-known facts about the Beltrami-Klein model. The standard Euclidean 3-space $\mathbb{E}^3$ may be realized as

$$\mathbb{E}^3 = \{y \in \mathbb{L}^4 : < y, e >= 0\}, \qquad e = (1, 0, 0, 0),$$

with the induced metric from $\mathbb{L}^4$ and the Beltrami-Klein model of the hyperbolic 3-space is given by the open unit ball $\mathbb{B}^3$ of $\mathbb{E}^3$ via the diffeomorphism

$$\mathfrak{K} : \mathbb{H}^3 \longrightarrow \mathbb{B}^3, \qquad \mathfrak{K}(x) := -\frac{x+ < x, e > e}{< x, e >}. \tag{4.1}$$

Geometrically, $\mathfrak{K} = \pi \circ \mathcal{P}_e$, where $\pi : \mathbb{L}^4 \longrightarrow \mathbb{E}^3$, is the usual vertical projection $\pi(x) = x+ < x, e > e$ and $\mathcal{P}_e : \mathbb{H}^3 \longrightarrow \Pi_e$ is the central projection from the origin onto the hyperplane $\Pi_e = \{x \in \mathbb{L}^4 \ : < x, e >= -1\}$. Moreover, $\mathfrak{K}$ extends in a smooth way to the ideal boundary $\mathbb{S}^2_\infty$, which is mapped onto the spherical boundary of the ball.

Since the geodesics of the hyperboloid model are the intersection with planes through the Minkowski origin, they are mapped into straight lines in $\mathbb{B}^3$, that is, $\mathfrak{K}$ is a geodesic map and it preserves convexity.

Thus, in the hypothesis of Theorem 4.2, $\mathfrak{K} \circ \psi(\Sigma)$ is locally convex (also in the Euclidean sense) at every regular point.

Up to congruence in $\mathbb{H}^3$, we may assume that $\mathfrak{K} \circ \psi(\mathcal{S}) \subset \Pi_0 = \{(y_1, y_2, y_3) \in \mathbb{B}^3 \ : \ y_3 = 0\}$ and, from Corollary 3.4, $\mathfrak{K} \circ \psi(\mathcal{S})$ is a finite number of disjoint convex Jordan curves in Π_0. Moreover Π_0 is the tangent plane $\mathfrak{K} \circ \psi(\Sigma)$ along $\mathfrak{K} \circ \psi(\mathcal{S})$.

Lemma 4.3. *Consider Σ^+ a connected component of $\{\mathfrak{K}\circ\psi(\Sigma)\}\setminus\{\mathfrak{K}\circ\psi(\mathcal{S})\}$. Then Σ^+ is an embedded complete surface lying on the boundary of a convex body in $\mathbb{B}^3$ and its boundary, $\partial\Sigma^+$, is a convex Jordan curve in Π_0.*

Proof. Let $\mathcal{K}\subset\Sigma$ be a compact containing $\mathcal{S}$ in its interior. Thanks to a classical result of Huber, [10], $\Sigma\setminus\overset{\circ}{\mathcal{K}}$ is conformally a compact Riemann surface with compact boundary and finitely many points removed which are the ends of ψ.

But ψ is an embedding, then each end has to be regular and asymptotic to one of rotational type (see [6, 18]). Thus, the ends go to a finite number of points in the boundary $\mathbb{S}^2$ of $\mathbb{B}^3$ and $\mathfrak{K}\circ\psi(\Sigma)$ is extends to these points as a locally convex surface.

From Corollary 3.4, adding to $\Sigma^+\cup\partial\Sigma^+$ the ends and the planar bounded regions determined by the convex Jordan curves of its boundary, we get a globally convex surface $\widetilde{\Sigma}^+$ homeomorphic to a sphere which lies in the boundary of a convex body in $\mathbb{B}^3\cup\mathbb{S}^2$. Which concludes the proof. □

Lemma 4.4. *Σ^+ is a rotational flat surface.*

Proof. It is clear than Σ^+ has at least one end, otherwise adding its reflexion respect the plane Π_0 we get a compact flat front without boundary, which is impossible by Proposition 3.6 in [13].

Consider (ω_+,θ_+) the holomorphic data associated to Σ^+ and denote by Σ^+_* the corresponding flat surface with holomorphic data $(-\omega_+,\theta_+)$, then Σ^+_* has the following properties:

1. From (3.5), (3.6), Remark 1, Remark 5 and Lemma 4.3, the boundary of Σ^+_* is a singular point $a\in\mathbb{H}^3$.
2. Having in mind that any embedded complete end is of rotational type, see [6], any end of Σ^+_* is also embedded and complete, moreover Σ^+_* has the same number of ends as Σ^+.

In other words, $\Sigma^+_*\cup\{a\}$ is a non compact complete flat surface with only one isolated singularity. An easy application of the Maximum Principle says it must be rotational and, consequently, Σ^+ is also rotational. □

Proof. (Theorem 4.2). It follows directly from Lemmas 4.3 and 4.4. □

Flat fronts with a planar singular set also are symmetric. Actually, we have

Proposition 4.5. *Any flat front containing an analytic singular curve lying in a hyperbolic plane Π must be symmetric with respect to the plane Π in $\mathbb{H}^3$.*

Proof. The proof can be done as application of the generalized symmetry principle ([8, Theorem 7]), we show here how it works.

Let $\psi:\Sigma\longrightarrow\mathbb{H}^3$ the flat front determined by the pair of initial data $\{\beta,V\}$ and assume $\beta(I)\subset\Pi_0\equiv\{x_3=0\}$, that is,

$$\beta(s)=(\beta_0(s),\beta_1(s),\beta_2(s),0),\qquad V(s)=(0,0,0,1),\quad s\in I.$$

From (3.2), the hyperbolic Gauss maps of ψ may be written as

$$G(z) = \frac{\beta_1 + i\beta_2}{\beta_0 - 1}(z), \qquad G_*(z) = \frac{\beta_1 + i\beta_2}{\beta_0 + 1}(z)$$

in $\Omega_\epsilon = \{z \in \mathbb{C} \;:\; \mathrm{Re}(z) \in I,\; -\epsilon \leq \mathrm{Im}(z) \leq \epsilon\}$. Thus, from the Riemann-Schwarz symmetry principle and (2.7), we have the following symmetry properties must be satisfy

$$\overline{G(\overline{z})}G_*(z) = 1, \qquad \overline{\xi(\overline{z})} = \frac{\xi_*(z)}{G_*(z)}, \quad z \in \Omega_\epsilon, \tag{4.2}$$

where

$$\xi\xi_* = G - G_*.$$

From Theorem 2.1 and by a straightforward computation , we conclude $\psi(\Omega_\epsilon)$ is symmetric respect to the plane Π_0 in $\mathbb{H}^3$. □

Using this fact, and in a similar way to the proof of Theorem 4.1, we can generalize the Theorem 4.2 as follows:

Theorem 4.6. *Let $\psi : \Sigma \longrightarrow \mathbb{H}^3$ be an embedded complete flat front with a non-degenerate analytic singular set $\mathcal{S} \subset \Sigma$ such that any connected component of $\psi(\mathcal{S})$ lies in a hyperplane through the origin of $\mathbb{L}^4$. Then ψ is a snowman rotational flat front.*

5 The correspondence with improper affine maps

It is well-known that flat fronts in $\mathbb{H}^3$ and definite improper affine maps in $\mathbb{R}^3$ are related with the Hessian one equation. Actually, their conformal representations can be obtained because one can solve this equation in terms of holomorphic data, see [4, 6].

In fact, we have an interesting correspondence between flat fronts with a prescribed curve $\beta : I \longrightarrow \mathbb{H}^3$ of singularities and improper affine maps with the same kind of singularities.

In particular, from Theorem 3.1, if β is an analytic convex curve in $\mathbb{H}^3$ contained in the hyperbolic plane $x_3 = 0$, we deduce the corresponding flat front ψ, with $\psi(s, 0) = \beta(s)$ and $\eta(s, 0) = (0, 0, 0, 1)$, can be described by the hyperbolic Gauss maps

$$G(z) = \frac{\beta_1 + i\beta_2}{\beta_0 - 1}(z) \quad \textit{and} \quad G_*(z) = \frac{\beta_1 + i\beta_2}{\beta_0 + 1}(z), \quad z = s + it.$$

That is, ψ is determined by the holomorphic extension of β and it inherits the symmetry properties described in (4.2). Moreover, from Section 2, the holomorphic data θ and ω are given by

$$\omega = -\frac{dG}{\xi^2}, \qquad \theta = \frac{dG_*}{\xi_*^2}, \tag{5.1}$$

and from (4.2) and (5.1), they must satisfy the following symmetry condition:

$$\overline{\omega(\overline{z})} = \theta(z), \quad z \in \Omega_\epsilon. \tag{5.2}$$

Now, from [4, 14], we remind that for any holomorphic curve, $(f, g) : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}^2$, the data $(\omega = dg, \theta = df)$ define a definite improper affine map Φ given by

$$\Phi = \left(g + \overline{f}, \frac{|g|^2 - |f|^2}{2} + \mathrm{Re}(\int g df - f dg)\right)$$

whose affine conormal map, N, and its affine metric, h, may be written as

$$N = \left(\overline{f} - g, 1\right), \qquad h := |\omega|^2 - |\theta|^2, \tag{5.3}$$

respectively.

From the above, if $\psi : \Omega_\epsilon \longrightarrow \mathbb{H}^3$ is the flat front whose singular set contains the analytic curve $\beta : I \longrightarrow \mathbb{H}^3$, $\beta(I) \subset \{x_3 = 0\}$, then the holomorphic data (ω, θ) satisfies (5.2) and, from (5.3), the affine conormal is constant along the curve $\alpha(s) = \Phi(s)$, $s \in I$, that is, α is also a planar curve in $\mathbb{R}^3$ which determines uniquely the improper affine map (see [15]).

Example 5.1. *The flat front determined by the analytic curve $\beta : \mathbb{R} \longrightarrow \mathbb{H}^3$, given by*

$$\beta(s) = (a, b\cos(s), b\sin(s), 0), \qquad a \in \mathbb{R} \setminus \{0\}, \quad a^2 - b^2 = 1,$$

is a snowman (see Figure 6) and the corresponding improper affine map is rotational (see Figure 7). It is determined by the curve:

$$\alpha(s) = \frac{b}{a}(\cos(as), -\sin(as), 0).$$

Figure 7: Rotational improper affine map with cuspidal edges singularities.

Example 5.2. *If the holomorphic data (ω, θ) satisfy*

$$\int \omega = \frac{\theta}{\omega},$$

then, the flat front has three swallowtails, see Figure 3. The corresponding improper affine map also have a singular set with three swallowtails, (see Figure 8), determined by the analytic curve:

$$\alpha(s) = \Big(\cos(s) + \frac{1}{2}\cos(2s),\ -\sin(s) + \frac{1}{2}\sin(2s),\ \frac{1}{6}\cos(3s)\Big)$$

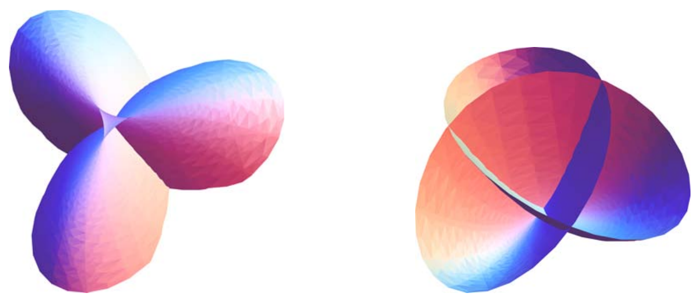

Figure 8: Improper affine map with three swallowtails

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