On the interplay between Lorentzian Causality and Finsler metrics of Randers type

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Universidad de Granada

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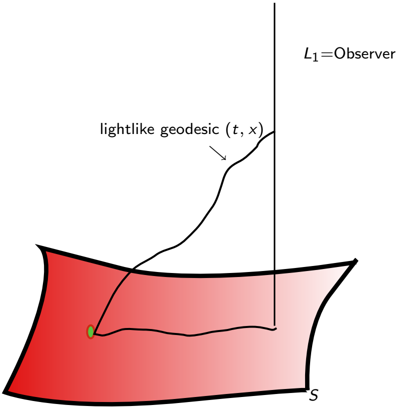
Interplay between Randers metrics and stationary spacetimes



$(\mathbb{R} \times S, l)$ is a standard stationary spacetime

$$l((\tau,y),(\tau,y)) = g_0(y,y) + 2g_0(\delta,y)\tau - \beta(x)\tau^2,$$

where $\beta(x) > 0$.

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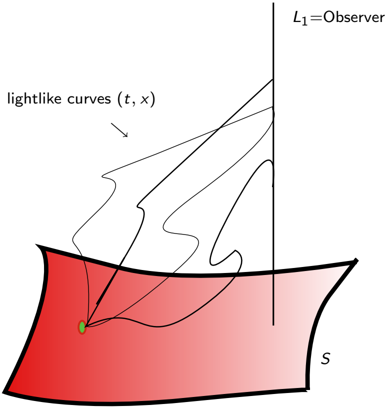

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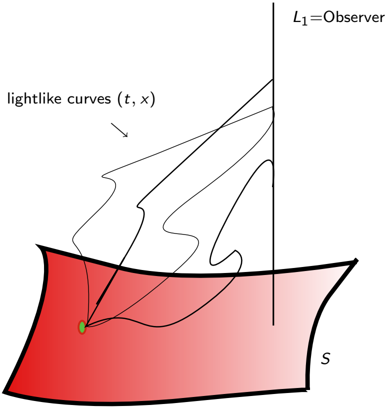

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$\Downarrow$ Fermat Principle

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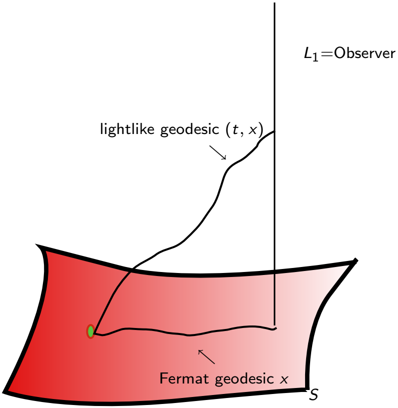

S is naturally endowed with a Randers metric F called the Fermat metric $F(x,v) = \frac{1}{\beta} g_0(v,\delta) + \sqrt{\frac{1}{\beta} g_0(v,v) + \frac{1}{\beta^2} g_0(v,\delta)^2}$

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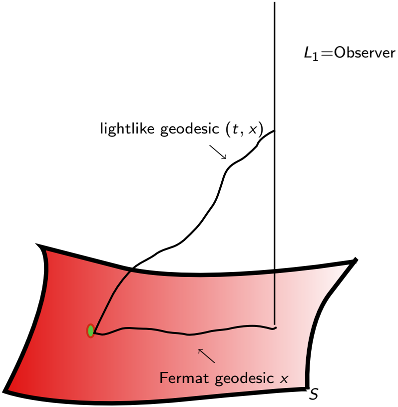

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where $\beta(x) > 0$.

This is because arrival time of lightlike curves is $AT(\gamma) =$
$\int_0^1 \left(\frac{1}{\beta} g_0(\dot{x},\delta) + \sqrt{\frac{1}{\beta} g_0(\dot{x},\dot{x}) + \frac{1}{\beta^2} g_0(\dot{x},\delta)^2} \right) \mathrm{d}s$

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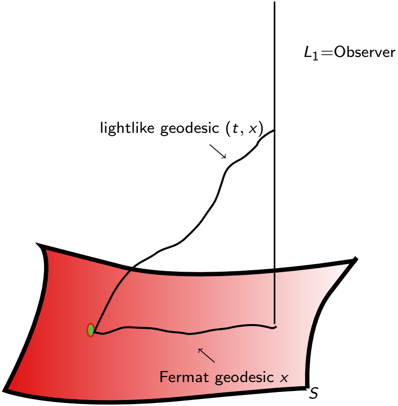

Causal properties of $(\mathbb{R} \times S, l)$

$\Updownarrow$

Hopf-Rinow properties of (S, F)

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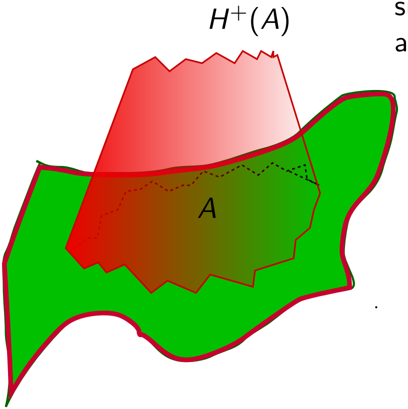

Cauchy horizons of a subset A contained in a slice $\{t_0\} \times S$

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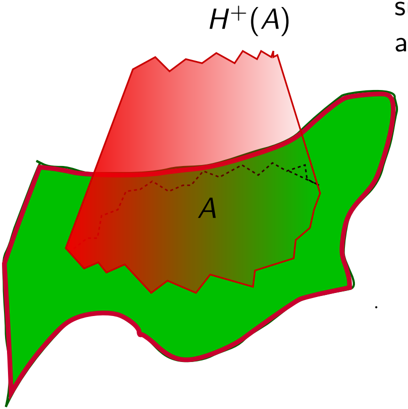

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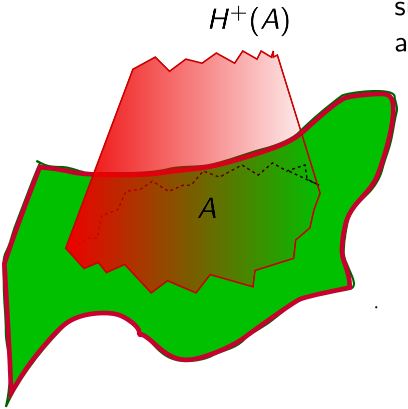

Cauchy horizons of a subset A contained in a slice $\{t_0\} \times S$

$\Downarrow$

are the graph of the distance function to the complementary A^c in (S, F)

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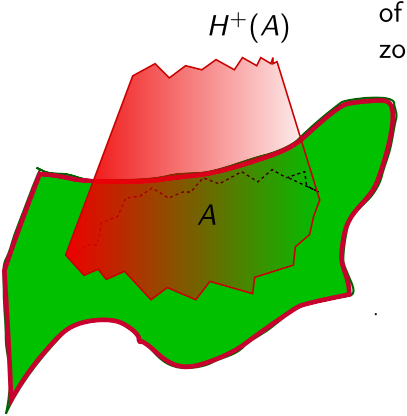

Differential properties of the Cauchy horizons in $(\mathbb{R} \times S, l)$

$\Updownarrow$

Differential properties of the distance function to a subset in (S, F)

Causal condition to have a standard splitting

Globally hyperbolic

$\Downarrow$

Causally simple

$\Downarrow$

Causally continuous

$\Downarrow$

Stably causal

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Strongly causal

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Distinguishing

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Causal condition to have a standard splitting

- A spacetime is Stationary if it admits a timelike Killing field.

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- How restrictive is to consider standard stationary spacetimes rather than stationary?

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Theorem (M. A. J.- M. Sánchez)

If a stationary spacetime L is distinguishing and the timelike Killing field is complete, then it is causally continuous and standard

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(c) *a slice $\{t_0\} \times S$, $t_0 \in \mathbb{R}$, is a Cauchy hypersurface if and only if the Fermat metric F on S is forward and backward complete.*

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$\bar{B}^+(p,r) = \{q : d(p,q) \leq r\}$ and
$\bar{B}^-(p,r) = \{q : d(q,p) \leq r\}$ $d(p,q) \neq d(q,p)$!!!!

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Cauchy developments and distance function to a subset

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Let $(\mathbb{R} \times S, g)$ be a standard stationary spacetime such that $\{t_0\} \times S$ is Cauchy, and $A_{t_0} = \{t_0\} \times A$. Then

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Cauchy horizons can be seen as the graph of the distance function to a subset!!!!

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Photograph courtesy of Yan Yan Li.

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$(\mathbb{R} \times S, g)$ $(n+1)$-standard stationary, with S Cauchy an $\Omega \subset S$, open connected with $C^{2,1}_{\mathrm{loc}}$-boundary $\partial\Omega$. If $A_{t_0} = \{t_0\} \times A$ and B is bounded then

$$\mathfrak{h}^{n-1}((\mathbb{R} \times B) \cap H^+_{\mathrm{mul}}(A)) < +\infty$$

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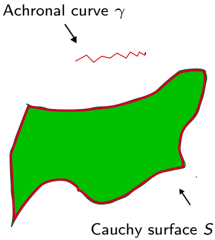

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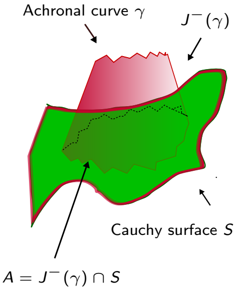

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Open problems

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(2) In the paper

G. W. Gibbons, C. A. R. Herdeiro, C. M. Warnick, M. C. Werner, *Stationary Metrics and Optical Zermelo-Randers-Finsler Geometry.*, Phys.Rev.D79: 044022,2009

the authors show that Fermat metrics with constant flag curvature correspond with locally conformally flat stationary spacetimes, but the converse is not true.

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(3) Which is the condition in the Fermat metric that characterizes conformally flatness for the stationary spacetime?

Bibliography

Bibliography

More information in:

- E. Caponio, M. A. J. and M. Sánchez, *The interplay between Lorentzian causality and Finsler metrics of Randers type.*, arxiv: 0903.3501, preprint 2009.
- E. Caponio, M. A. J. and A. Masiello, *On the energy functional on Finsler manifolds and applications to stationary spacetimes,* arxiv: 0702323, preprint 2007.