The space of Parabolic Affine Spheres with fixed compact boundary

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Abstract

The aim of this paper is to study the moduli space of solutions of the Dirichlet problem associated to the equation of Monge-Ampère type, $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = 1$, on an exterior planar domain. We prove that this moduli space is either empty or a differentiable manifold of dimension five.

1 Introduction

Some recent progresses in differential geometry and partial differential equations are based on the theory of moduli spaces. In this context we remark the equation associated to the minimal surfaces and the works of R. Böhme, F. Tomi, J. Tromba and B. White, ([BT], [TT], [W]), for compact minimal surfaces; and the study of J. Pérez and A. Ros ([PR]) for the non-compact case.

The aim of this paper is to study, in a similar way, the following unimodular Hessian equation,

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = 1 \quad in \quad \Omega\,, \tag{1}$$

where Ω is a planar domain and f is in the usual Hölder space $\mathcal{C}^{2,\alpha}(\overline{\Omega})$. Without loss of generality we shall consider only locally convex solutions of (1).

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This equation arises in the context of an affine differential geometric problem as the equation of a parabolic affine sphere in the unimodular affine real 3-space (see [C], [CY] and [LSZ]). Contrary to the case of smooth bounded domains, little is known about solutions of (1) when the domain is unbounded. Here, we recall the famous result of K. Jörgens which asserts that all solutions of (1) on $\Omega = \mathbb{R}^2$ are quadratic polynomials (see [J]).

Recently, the authors proved in [FMM2] that the behaviour at infinity of a solution of (1) depends on five real numbers which give the shape and the logarithmic growth of the solution at infinity. This behaviour is presented in a brief way in Sect. 2.

In Sect. 3 we introduce the space of solutions of

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = 1 \quad in \quad \Omega\,, \quad f_{|\partial\Omega} = \varphi\,, \tag{2}$$

when Ω is the exterior of a plane Jordan curve of class $\mathcal{C}^\infty$ and $\varphi \in C^\infty(\partial\Omega)$. By using a transformation of Ω onto a bounded domain we show that a solution of (2) has a singularity at the origin which is described by the five aforementioned real numbers, then classical results for linear elliptic operators and the Implicit Function Theorem are used in order to prove that the moduli space of solutions of (2) is either empty or a 5-dimensional differentiable manifold (consequently, the infinitesimal deformations of the solutions are described by the variation of the numbers describing the singularity at the origin).

2 Preliminaries

Throughout we use standard notation of complex analysis (see [A]). In particular we use $\Re$ and $\Im$ to denote real and imaginary part, respectively. As we mentioned in the introduction when Ω is the exterior of a plane Jordan curve γ, it is possible to give a good description of the solutions of (1) at infinity. The purpose of this section is to present this description. We refer the reader to [FMM1] and [FMM2] for more details.

Let $f : \Omega \longrightarrow \mathbb{R}$ be a solution of (1) on Ω and let $M_f = \{(x_1, x_2, f(x_1, x_2)) \mid (x_1, x_2) \in \Omega\}$ be its graph. Then it is known (see [C], [CY] and [LSZ]) that M_f is a parabolic affine sphere with affine normal vector $\xi = (0, 0, 1)$ and affine metric given by

$$ds^2 = \sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j\,. \tag{3}$$

Moreover, the transformation $L_f : \Omega \longrightarrow \mathbb{C}$ given by

$$z = L_f(w) = w + 2\,\frac{\partial f}{\partial \overline{w}}\,, \tag{4}$$

with $w = x_1 + \mathrm{i}x_2$, is a global diffeomorphism from $\widehat{\Omega} \subset \Omega$ onto Ω_R (the exterior of a disk of radius R). This transformation allows to define a holomorphic function $F : L_f(\widehat{\Omega}) \longrightarrow \mathbb{C}$ as

$$F(z) = \overline{w} - 2\,\frac{\partial f}{\partial w}\ , \tag{5}$$

where $z = L_f(w)$. Furthermore, this function can be written on Ω_R as

$$F(z) = \mu z + \nu + \sum_{n=1}^{\infty} \frac{a_n}{z^n}\ , \tag{6}$$

where $\mu, \nu, a_n \in \mathbb{C}$ for $n \geq 2$, $a_1 \in \mathbb{R}$ and $|\mu| < 1$. From (4) and (5), w and the derivative of f are

$$w = \frac{1}{2}\left(z + \overline{F(z)}\right)\ , \quad \frac{\partial f}{\partial w} = \frac{1}{4}\left(\overline{z} - F(z)\right)\ . \tag{7}$$

Thus, the original function f can be recovered as

$$f(w) = \frac{1}{8}|z|^2 - \frac{1}{8}|F(z)|^2 + \frac{1}{4}\Re(zF(z)) - \frac{1}{2}\Re\int_{z_0}^{z} F(\zeta)d\zeta\ . \tag{8}$$

By using (4), (5), (6) and (8) we obtain

$$\begin{aligned} f(w) \ &= \ \frac{1}{2}\left(x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2}\right) - \frac{\nu_1}{4}\left(x_1 + \frac{\partial f}{\partial x_1}\right) + \frac{\nu_2}{4}\left(x_2 + \frac{\partial f}{\partial x_2}\right) - \\ &- \ \frac{a_1}{4}\log(|z|^2) + O(1)\ , \end{aligned} \tag{9}$$

where $\nu = \nu_1 + \mathrm{i}\nu_2$. The expression $O(|w|^n)$ will be used to indicate a term which is bounded in absolute value by a constant times $|w|^n$ for $|w|$ large. Since $\lim_{|z|\to\infty}(F(z) - \mu z) = \nu$, from (4) and (5) one has,

$$\frac{\partial f}{\partial x_1} = \frac{1}{1-|\mu|^2}\{(1 + |\mu|^2 - 2\mu_1)x_1 + 2\mu_2 x_2 + a - (1-\mu_1)R_1 + \mu_2 R_2\}, \tag{10}$$

$$\frac{\partial f}{\partial x_2} = \frac{1}{1-|\mu|^2}\{(1 + |\mu|^2 + 2\mu_1)x_2 + 2\mu_2 x_1 + b + (1+\mu_1)R_2 - \mu_2 R_1\}, \tag{11}$$

where $\mu = \mu_1 + \mathrm{i}\mu_2$, $a = -\nu_1 + \mu_1\nu_1 + \mu_2\nu_2$, $b = \nu_2 - \mu_2\nu_1 + \mu_1\nu_2$ and $R_1 = \Re(F(z) - \mu z - \nu)$ and $R_2 = \Im(F(z) - \mu z - \nu)$. Moreover, since F is a holomorphic function of z, from (7) one has,

$$f_{11} = \frac{(1-r_1)^2 + r_2^2}{1 - r_1^2 - r_2^2}\ , \quad f_{22} = \frac{(1+r_1)^2 + r_2^2}{1 - r_1^2 - r_2^2}\ , \quad f_{12} = \frac{2r_2}{1 - r_1^2 - r_2^2}\ , \tag{12}$$

where $r_1 = \Re\left(\frac{\partial F}{\partial z}\right)$, $r_2 = \Im\left(\frac{\partial F}{\partial z}\right)$ and $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$.

Finally, with the above notations and using (9), (10) and (11) we obtain that f is given, on the exterior of some plane Jordan curve, by the expression

$$f(w) = \mathcal{E}(f)(w) - \frac{a_1}{4}\log(|z|^2) + O(1)\ , \tag{13}$$

where

$$\begin{aligned}\mathcal{E}(f)(w) &= \frac{1}{2(1-|\mu|^2)}\left\{\left(1+|\mu|^2-2\mu_1\right)x_1^2 + \left(1+|\mu|^2+2\mu_1\right)x_2^2\right\} + \\ &+ \frac{1}{1-|\mu|^2}\left\{2\mu_2 x_1 x_2 + a x_1 + b x_2 + \frac{\nu_2 b - \nu_1 a}{4}\right\}\ .\end{aligned} \tag{14}$$

Definition 1 *When k is a large positive number, from* (13) *and* (14)*, the ellipse $\mathcal{E}_k \equiv \mathcal{E}(f)(w) = k$, gives the shape of M_f at infinity. The ellipse $\mathcal{E}_k$ will be called* **the ellipse at infinity** *associated to f.*

The real number a_1 that appears in the expression (13) *is known as* **the logarithmic growth rate** *of the function f and it will be denoted by* $\mathbf{log}(f)$*. One can observe that* $\mathbf{log}(f)$ *indicates how much the graph of f moves away from the elliptic paraboloid.*

As L_f is known as the transformation of Lewy (*see* [S]*, pp.* 167) *, the function F given by* (5) *will be called* **the Lewy function** *of f.*

3 The space of solutions

In this section we shall study the moduli space of solutions of (2) in $\mathcal{C}^{2,\alpha}(\overline{\Omega})$, when Ω is the exterior of a plane Jordan curve γ of class $\mathcal{C}^\infty$ and $\varphi \in \mathcal{C}^\infty(\partial\Omega)$. Clearly, we can suppose, up a translation, that the origin is not in $\Omega \cup \gamma$.

Firstly, by using regularity results about Monge-Ampère equation (see [Au], [GT]) we can observe that if f is a $\mathcal{C}^{2,\alpha}(\overline{\Omega})$ solution of (2), then $f \in \mathcal{C}^\infty(\overline{\Omega})$. Therefore, throughout we denote by $\mathcal{M}$ the set of solutions of (2) of class $\mathcal{C}^\infty$. Moreover, on $\mathcal{M}$ we consider the $\mathcal{C}^\infty$-compact topology. The sequential convergence in this topology is the uniform convergence of the function and all of its derivatives on each compact.

Consider $f \in \mathcal{M}$ and let L_f and F be the Lewy transformation and the Lewy function associated to f. From (6), F can be written as

$$F(z) = \mu z + \nu + \sum_{n=1}^{\infty} \frac{a_n}{z^n}\ ,$$

on the exterior of a disk, with the same notations as before. Consider now the unimodular affine transformation $A_{\mu,\nu} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$A_{\mu,\nu}(x_1,x_2) = (1-|\mu|^2)^{-\frac{1}{2}}((1-\mu_1)x_1 + \mu_2 x_2 - \nu_1, \mu_2 x_1 + (1+\mu_1)x_2 + \nu_2)\ , \tag{15}$$

where $(x_1, x_2) \in \mathbb{R}^2$, $\mu = \mu_1 + i\mu_2$ and $\nu = \nu_1 + i\nu_2$. Then, there exist positive real numbers R_1 and R_2 (depending on μ and ν) with $R_1 < R_2$ and an embedding $L_{\mu,\nu} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that

$$L_{\mu,\nu} = \begin{cases} A_{\mu,\nu} & \text{on} \quad \Omega_{R_2} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 > R_2^2\} \, , \\ Id & \text{on} \quad B_{R_1}(0) \, , \end{cases}$$

where we denote by Id the identity transformation on $\mathbb{R}^2$ and by $B_{R_1}(0)$ the disk of radius R_1 and center 0 in $\mathbb{R}^2$. We take R_1 and R_2 large enough in order to get $\partial\Omega \subseteq B_{R_1}(0)$ and $B_{R_1}(0) \subset Int(A_{\mu,\nu}(\partial B_{R_2}(0))$. Thus, we have $L_{\mu,\nu}(\overline{\Omega}) = \overline{\Omega}$. Let T be the inversion, namely, $T(w) = \frac{1}{w}$. We label $\zeta = (T \circ L_{\mu,\nu})(w)$ the new coordinate and $\mathcal{B}$ the simply connected bounded domain in $\mathbb{C}$ given by $\mathcal{B} = T(\Omega) \cup \{0\}$. From (4), (10) and (11), we have $|z|^2 = r^{-2} t(\zeta)$ for large $|z|$, where $r = |\zeta|$ and t is a non vanishing regular function on $\mathcal{B}$. Then from (13) and (14), $\widetilde{f} = f \circ L_{\mu,\nu}{}^{-1} \circ T$ can be written as

$$\widetilde{f}(\zeta) = \frac{1}{2r^2} + \frac{a_1}{2} \log(r) + h(\zeta) \, , \tag{16}$$

for some function $h \in \mathcal{C}^\infty(\overline{\mathcal{B}})$.

For any planar domain D and any function $\beta \in \mathcal{C}^\infty(\partial D)$ we are going to label

$$\begin{aligned} \mathcal{C}^{k,\alpha}_\beta(\overline{D}) &= \{f \in \mathcal{C}^{k,\alpha}(\overline{D}) |\, f = \beta \quad \text{in} \quad \partial D\} \, , \\ \mathcal{C}^{k,\alpha}_\beta(\overline{D} \setminus \{0\}) &= \{f \in \mathcal{C}^{k,\alpha}(\overline{D} \setminus \{0\}) |\, f = \beta \quad \text{in} \quad \partial D\} \, , \\ \mathcal{C}^{\infty}_\beta(\overline{D}) &= \{f \in \mathcal{C}^{\infty}(\overline{D}) |\, f = \beta \quad \text{in} \quad \partial D\} \, , \\ \mathcal{C}^{\infty}_\beta(\overline{D} \setminus \{0\}) &= \{f \in \mathcal{C}^{\infty}(\overline{D} \setminus \{0\}) |\, f = \beta \quad \text{in} \quad \partial D\}. \end{aligned}$$

For the former domain $\mathcal{B}$, there exist a positive real number ε and a radial function $\delta \in \mathcal{C}^\infty(\overline{\mathcal{B}})$, such that $B_\varepsilon(0) \subset \mathcal{B}$, $\delta(r) = 0$, for $0 \le r \le \frac{\varepsilon}{2}$, and $\delta(r) = 1$, for $r \ge \varepsilon$.

Moreover, we shall denote by U_a the function $U_a \in \mathcal{C}^{2,\alpha}_0(\overline{\mathcal{B}} \setminus \{0\})$ given by

$$U_a(\zeta) = a \frac{(1 - \delta(r))}{2} \log(r) \, , \tag{17}$$

for $a \in \mathbb{R}$. Taking (16) and the above definitions into account, $\widetilde{f}$ has the following expression on $\mathcal{B}$

$$\widetilde{f}(\zeta) = \frac{1 - \delta(r)}{2r^2} + U_{a_1}(\zeta) + \widetilde{h}(\zeta) \, , \tag{18}$$

where $\tilde{h} \in \mathcal{C}^\infty_\varphi(\overline{\mathcal{B}})$. If for $a \in \mathbb{R}$ we denote

$$\Phi_a(\zeta) = \frac{1}{2r^2} + \frac{a}{2} \log(r) \, , \tag{19}$$

then $\widetilde{f}$ can be also expressed as

$$\widetilde{f}(\zeta) = (1-\delta(r))\Phi_{a_1}(\zeta) + \widetilde{h}(\zeta)\ . \tag{20}$$

Notice that in this expression $\delta\cdot\Phi_{a_1}\in\mathcal{C}^\infty(\overline{\mathcal{B}})$ by the properties of δ.

By taking g_f^*, the pull-back of the affine metric ds^2 given in (3) by the transformation ${L_{\mu,\nu}}^{-1}\circ T$, we obtain a Riemannian metric on $\mathcal{B}\setminus\{0\}$. Furthermore, when $|w|$ is large, from (6), (7) and (12), we have

$$f_{11} = \frac{1+|\mu|^2-2\mu_1}{1-|\mu|^2} + O(|w|^{-1})\ , f_{22} = \frac{1+|\mu|^2+2\mu_1}{1-|\mu|^2} + O(|w|^{-1}),$$

$$f_{12} = \frac{2\mu_2}{1-|\mu|^2} + O(|w|^{-1}).$$

From here it is easy to see that $\overline{g}_f = r^4 g_f^*$ is a well-defined Riemannian metric on $\mathcal{B}$.

Now, we consider for functions $f\in\mathcal{C}^{2,\alpha}(\overline{\Omega})$, the Hessian operator given by $\mathrm{Hess}(f) = \det(\nabla^2 f) - 1$, where $\nabla^2 f$ denotes the Hessian matrix associated to the metric $|dw|^2$. Clearly, this is an elliptic operator on the set $\mathcal{M}$. This allows us to define an elliptic operator acting on functions $\widetilde{f}\in\mathcal{C}^{2,\alpha}(\overline{\mathcal{B}}\setminus\{0\})$ given by

$$\begin{aligned} H(\widetilde{f}\,) &= \mathrm{Hess}(\widetilde{f}\circ T)\circ T = r^6\left\{r^2\det\left(\nabla_0^2\widetilde{f}\right) - 2\left\langle\overline{\zeta}\,,\nabla_0\widetilde{f}\right\rangle\left(D_{11}\widetilde{f}-D_{22}\widetilde{f}\right)\right. \\ &- \left. 4\left(uD_2\widetilde{f}+vD_1\widetilde{f}\right)D_{12}\widetilde{f} - 4\left|\left|\nabla_0\widetilde{f}\right|\right|^2\right\} - 1\ , \end{aligned} \tag{21}$$

where $\nabla_0^2\widetilde{f}$ and $\nabla_0\widetilde{f}$ denote the Hessian matrix and the gradient associated to the metric $|d\zeta|^2$, respectively; $\zeta = u + \mathrm{i}v$ and $D_1\widetilde{f} = \frac{\partial\tilde{f}}{\partial u}$, $D_2\widetilde{f} = \frac{\partial\tilde{f}}{\partial v}$, $D_{11}\widetilde{f} = \frac{\partial^2\tilde{f}}{\partial u\partial u}$, $D_{12}\widetilde{f} = \frac{\partial^2\tilde{f}}{\partial u\partial v}$ and $D_{22}\widetilde{f} = \frac{\partial^2\tilde{f}}{\partial v\partial v}$ are the corresponding derivatives of $\widetilde{f}$.

A straight calculation using (21) gives the following result,

Lemma 1 *Let $a\in\mathbb{R}$, and for $k\geq 2$ let $\widetilde{f}\in\mathcal{C}^{k,\alpha}(\overline{\mathcal{B}}\setminus\{0\})$ be a function of the type $\widetilde{f} = \Phi_a + \Psi + \widetilde{h}$ where $\widetilde{h}\in\mathcal{C}^{k,\alpha}_\varphi(\overline{\mathcal{B}})$ and $\Psi\in\mathcal{C}^\infty(\overline{\mathcal{B}})$ depend analytically on a. Then $\frac{1}{r^4}H$ is a $\mathcal{C}^{k-2,\alpha}(\overline{\mathcal{B}})$-valued operator of the type*

$$\frac{1}{r^4}H\left(\widetilde{f}\right) = r^4\det\left(\nabla_0^2\widetilde{h}\right) + \sum_{i,j=1}^{2} A_{ij}\left(\zeta, a, \nabla_0\widetilde{h}\right)D_{ij}\widetilde{h} + B\left(\zeta, a, \nabla_0\widetilde{h}\right)\ ,$$

which depends analytically on $a\in\mathbb{R}, \nabla_0\widetilde{h}$ and $\nabla_0^2\widetilde{h}$.

As we announced we have the following result,

Theorem 1 *Let $\mathcal{M}$ be the set of solutions of* (2)*. If $\mathcal{M}$ is not empty, then it is a 5-dimensional differentiable manifold.*

Proof: We can consider the map $\mathfrak{t} : B_1(0) \times \mathbb{C} \times \mathbb{R} \times \mathcal{C}^{2,\alpha}_{\varphi}(\overline{\mathcal{B}}) \longrightarrow \mathcal{C}^{2,\alpha}_{\varphi}(\overline{\Omega})$, given by $\mathfrak{t}(\mu, \nu, a, \widetilde{h}) = \widetilde{f}_{a,\widetilde{h}} \circ T \circ L_{\mu,\nu}$ where $\widetilde{f}_{a,\widetilde{h}}(\zeta) = (1 - \delta(r))\Phi_a(\zeta) + \widetilde{h}(\zeta)$. With this notation we define, for each $k \geq 5$, the map

$$\mathcal{H} : B_1(0) \times \mathbb{C} \times \mathbb{R} \times \mathcal{C}^{k,\alpha}_{\varphi}(\overline{\mathcal{B}}) \longrightarrow \mathcal{C}^{k-2,\alpha}(\overline{\mathcal{B}}) ,$$

given by

$$\mathcal{H}(\mu, \nu, a, \widetilde{h})(\zeta) = \frac{1}{r^4}\mathrm{Hess}(f)(w) ,$$

where $f = \mathfrak{t}(\mu, \nu, a, \widetilde{h})$ and $w = ({L_{\mu,\nu}}^{-1} \circ T)(\zeta)$. It is easy to prove that $\mathcal{H}$ is a differentiable map of $\{\mu, \nu, a, \widetilde{h}\}$. Furthermore, $\mathcal{H}(\mu, \nu, a, \widetilde{h}) \in \mathcal{C}^{k-2,\alpha}(\overline{\mathcal{B}} \setminus \{0\})$. Then it suffices to prove $\mathcal{H}$ is well-defined in a neighbourhood of $\zeta = 0$. But near $\zeta = 0$ we have ${L_{\mu,\nu}}^{-1} \circ T = A^{-1}_{\mu,\nu} \circ T$ and since $A_{\mu,\nu}$ is an unimodular affine transformation we have

$$\mathcal{H}(\mu, \nu, a, \widetilde{h})(\zeta) = \frac{1}{r^4}\left(\det\left(\nabla^2(\widetilde{f}_{a,\widetilde{h}} \circ T)\right)(T(\zeta)) - 1\right) = \frac{1}{r^4} H\left(\widetilde{f}_{a,\widetilde{h}}\right)(\zeta) .$$

Thus, using the Lemma 1, we obtain that $\mathcal{H}$ is well-defined.

We shall denote by $\mathcal{N}$ the set $\mathcal{N} = \mathcal{H}^{-1}(0)$ (observe that this set is the same for all k). In order to prove that $\mathcal{N}$ is a submanifold of $B_1(0) \times \mathbb{C} \times \mathbb{R} \times \mathcal{C}^{k,\alpha}_{\varphi}(\overline{\mathcal{B}})$ we shall compute the differential of $\mathcal{H}$ at a point $(\mu, \nu, a, \widetilde{h}) \in \mathcal{N}$. Given $(\vec{\mu}, \vec{\nu}, \vec{a}, \vec{h}) \in \mathbb{C}^2 \times \mathbb{R} \times \mathcal{C}^{k,\alpha}_{0}(\overline{\mathcal{B}})$, let β be the curve $\beta(t) = (\mu, \nu, a, \widetilde{h}) + t(\vec{\mu}, \vec{\nu}, \vec{a}, \vec{h})$, with $t \in [-t_0, t_0]$ and t_0 a positive real number, then we have

$$\left.\frac{d}{dt}\right|_{t=0} (\mathcal{H} \circ \beta)(t)(\zeta) = \frac{1}{r^4} \left.\frac{d}{dt}\right|_{t=0} \det\left(\nabla^2 f(t)\right)(w(t)) ,$$

where we denote by $w(t) = (L^{-1}_{\mu(t),\nu(t)} \circ T)(\zeta)$, $f(t) = \mathfrak{t}(\beta(t)) = \widetilde{f}(t) \circ T \circ L_{\mu(t),\nu(t)}$, $\widetilde{f}(t) = (1 - \delta)\Phi_{a+t\vec{a}} + \widetilde{h} + t\vec{h}$, $\mu(t) = \mu + t\vec{\mu}$ and $\nu(t) = \nu + t\vec{\nu}$. We observe that $f(0) = f$ and $\widetilde{f}(0) = \widetilde{f}_{a,\widetilde{h}}$. Since the function $f(t)$ is a differentiable function of t we have the following expression for it

$$f(t) = f + t\widehat{f} + t^2\overline{f}_t , \tag{22}$$

where $\widehat{f} = \left.\frac{d}{dt}\right|_{t=0} f(t)$ and $\overline{f}_t \in \mathcal{C}^{k-2,\alpha}(\overline{\mathcal{B}})$. Then, using (17), (19), (20) and (22), the differential of $\mathcal{H}$ is

$$d\mathcal{H}_{(\mu,\nu,a,\widetilde{h})}(\vec{\mu}, \vec{\nu}, \vec{a}, \vec{h})(\zeta) \;=\; \frac{1}{r^4} \left.\frac{d}{dt}\right|_{t=0} \det\left(\nabla^2\left(f + t\widehat{f} + t^2\overline{f}_t\right)\right)(w(t)) =$$

$$= \frac{1}{r^4}\frac{d}{dt}\Big|_{t=0}\left(\det(\nabla^2 f) + t\Delta_f \widehat{f} + t^2 l_t\right)(w(t)) =$$
$$= \frac{1}{r^4}\left(\Delta_f \widehat{f}\right)(w) = \overline{\Delta}_f(\widehat{f}\circ {L_{\mu,\nu}}^{-1}\circ T)(\zeta)\ ,$$

where Δ_f and $\overline{\Delta}_f$ denote the Laplace-Beltrami operator associated to the affine metric ds^2 of M_f given by (3) and the Riemannian metric $\overline{g}_f$, respectively, and l_t is a differentiable function of t.

Furthermore, $\widetilde{f}(t)$ can be written as

$$\widetilde{f}(t) = \widetilde{f}_{a,\widetilde{h}} + t\vec{f}\ , \tag{23}$$

where $\vec{f} = U_{\vec{a}} + \vec{h}$. Hence we obtain the following expression for the differential of $\mathcal{H}$

$$d\mathcal{H}_{(\mu,\nu,a,\widetilde{h})}(\vec{\mu},\vec{\nu},\vec{a},\vec{h}) = \overline{\Delta}_f(\vec{f} + \Gamma^{(\vec{\mu},\vec{\nu})}_{(\mu,\nu,a,\widetilde{h})})\ , \tag{24}$$

where $\Gamma^{(\vec{\mu},\vec{\nu})}_{(\mu,\nu,a,\widetilde{h})} = \frac{d}{dt}\big|_{t=0}(\widetilde{f}_{a,\widetilde{h}}\circ T\circ L_{\mu(t),\nu(t)})\circ {L_{\mu,\nu}}^{-1}\circ T$.

From (24) we have that $\overline{\Delta}_f(U_{\vec{a}} + \Gamma^{(\vec{\mu},\vec{\nu})}_{(\mu,\nu,a,\widetilde{h})}) \in \mathcal{C}^{k-2,\alpha}(\overline{\mathcal{B}})$. Then, given $\phi\in\mathcal{C}^{k-2,\alpha}(\overline{\mathcal{B}})$, we know, (see [GT]), that there exists a unique function $\vec{h}\in\mathcal{C}^{k,\alpha}(\overline{\mathcal{B}})$ such that

$$\begin{cases} \overline{\Delta}_f\vec{h} &= -\overline{\Delta}_f(U_{\vec{a}} + \Gamma^{(\vec{\mu},\vec{\nu})}_{(\mu,\nu,a,\widetilde{h})}) + \phi \qquad on\ \ \mathcal{B}\ , \\ \vec{h} &= 0 \qquad\qquad\qquad\qquad\qquad\quad on\ \ \partial\mathcal{B}\ , \end{cases}$$

and $d\mathcal{H}_{(\mu,\nu,a,\widetilde{h})}$ is a surjective map. Moreover, if we denote by $\vec{h}^{(\vec{\mu},\vec{\nu},\vec{a})}_{(\mu,\nu,a,\widetilde{h})}$ the unique solution of the above problem for $\phi = 0$, then

$$\text{Ker}\left(d\mathcal{H}_{(\mu,\nu,a,\widetilde{h})}\right) = \left\{\left(\vec{\mu},\vec{\nu},\vec{a},\vec{h}^{(\vec{\mu},\vec{\nu},\vec{a})}_{(\mu,\nu,a,\widetilde{h})}\right)\ |\ (\vec{\mu},\vec{\nu},\vec{a})\in\mathbb{C}^2\times\mathbb{R}\right\} \tag{25}$$

and thereby Ker $\left(d\mathcal{H}_{(\mu,\nu,a,\widetilde{h})}\right)$ splits and so $\mathcal{H}$ is a submersion on $\mathcal{N}$.

If $\mathcal{N}$ is not empty, from the Implicit Function Theorem, we have that $\mathcal{N}$ is a 5-dimensional differentiable submanifold of $B_1(0)\times\mathbb{C}\times\mathbb{R}\times\mathcal{C}^{k,\alpha}_\varphi(\overline{\mathcal{B}})$, for each $k\geq 5$, whose tangent space at a point $(\mu,\nu,a,\widetilde{h})\in\mathcal{N}$ is given by (25). Furthermore, by using the Maximum Principle at infinity (see [FMM2]) and (25), it is not difficult to prove that the map $\Psi:\mathcal{N}\longrightarrow\mathbb{R}^5$ given by $\Psi(\mu,\nu,a,\widetilde{h}) = (\mu,\nu,a)$ is an embedding for each of the former manifold structures on $\mathcal{N}$. Then any two of them are diffeomorphic and the topology associated to these structures is the topology that $\mathcal{N}$ has as a subset of $B_1(0)\times\mathbb{C}\times\mathbb{R}\times\mathcal{C}^\infty_\varphi(\overline{\mathcal{B}})$, where on $\mathcal{C}^\infty_\varphi(\overline{\mathcal{B}})$ we consider the $\mathcal{C}^\infty$ topology, that is, the topology of the uniform convergence of the function and all of its derivatives. Thereby, if we consider on $\mathcal{M} = \mathfrak{t}(\mathcal{N})$ the topology $\mathcal{T}$ such as the map $\mathfrak{t}$ is an homeomorphism, we have that $(\mathcal{M}, \mathcal{T})$ is a 5-dimensional differentiable manifold. In the following result we shall prove that the topology $\mathcal{T}$ and the $\mathcal{C}^\infty$-compact topology are equivalent on $\mathcal{M}$ and thus the proof of the Theorem 1 will be concluded.

Proposition 1 *Let $\{f_n\} \subset \mathcal{M}$ and $f_0 \in \mathcal{M}$, then the following assertions are equivalent:*
a) $\{f_n\}$ converges to f_0 in the topology $\mathcal{T}$.
b) $\{f_n\}$ converges to f_0 in the $\mathcal{C}^\infty$-compact topology.

Proof: It is clear that *a)* implies *b)*.

Conversely, we assume the assertion *b)* holds. The functions f_n can be written as $f_n = \mathfrak{t}(\mu_n, \nu_n, a_n, \widetilde{h}_n)$, for $n \geq 0$. We denote by $z_n = L_{f_n}(w)$ and by F_n the Lewy function of f_n given in (4) and (5) for $n \geq 0$. From expression (6) we can compute the numbers $\{\mu_n, \nu_n, a_n\}$ by means of integrals of the function F_n along a suitable curve, for $n \geq 0$. Using (4) and (5) again, these integrals can be given in terms of $\frac{\partial f_n}{\partial x_i}$ and $\frac{\partial^2 f_n}{\partial x_i \partial x_j}$, for $i, j = 1, 2$. Since $\{f_n\}$ converges to f_0 in $\mathcal{C}^k$ on each compact of Ω for all $k \geq 0$, we have that $\left\{\frac{\partial f_n}{\partial x_i}\right\}$ and $\left\{\frac{\partial^2 f_n}{\partial x_i \partial x_j}\right\}$ converge to $\frac{\partial f_0}{\partial x_i}$ and $\frac{\partial^2 f_0}{\partial x_i \partial x_j}$, respectively in $\mathcal{C}^k$ on each compact of Ω for all $k \geq 0$ and $i, j = 1, 2$. Hence $\{\mu_n\} \longrightarrow \mu_0$, $\{\nu_n\} \longrightarrow \nu_0$ and $\{a_n\} \longrightarrow a_0$.

Now we define, for $k \geq 5$, the map

$$\mathcal{G} : B_1(0) \times \mathbb{C} \times \mathbb{R} \times \mathcal{C}^{k,\alpha}_{\varphi}(\overline{\mathcal{B}}) \longrightarrow B_1(0) \times \mathbb{C} \times \mathbb{R} \times \mathcal{C}^{k-2,\alpha}(\overline{\mathcal{B}}) ,$$

given by

$$\mathcal{G}(\mu, \nu, a, \widetilde{h}) = \left(\mu, \nu, a, \mathcal{H}(\mu, \nu, a, \widetilde{h})\right) .$$

As before $\mathcal{G}$ is a differentiable map on its variables $\{\mu, \nu, a, \widetilde{h}\}$ and taking (24) into account, its differential is given by

$$\begin{aligned} d\mathcal{G}_{(\mu,\nu,a,\widetilde{h})}(\vec{\mu}, \vec{\nu}, \vec{a}, \vec{h}) &= \left(\vec{\mu}, \vec{\nu}, \vec{a}, d\mathcal{H}_{(\mu,\nu,a,\widetilde{h})}(\vec{\mu}, \vec{\nu}, \vec{a}, \vec{h})\right) = \\ &= \left(\vec{\mu}, \vec{\nu}, \vec{a}, \overline{\Delta}_f \left(\vec{h} + U_{\vec{a}} + \Gamma^{(\vec{\mu},\vec{\nu})}_{(\mu,\nu,a,\widetilde{h})}\right)\right) , \end{aligned}$$

with the former notations. Hence, it is easy to check that $\mathcal{G}$ is a local diffeomorphism at each point $(\mu, \nu, a, \widetilde{h}) \in \mathcal{N}$. From the convergence of the sequences $\{\mu_n\}, \{\nu_n\}$ and $\{a_n\}$, we have that $\mathcal{G}(\mu_n, \nu_n, a_n, \widetilde{h}_n) = (\mu_n, \nu_n, a_n, 0)$ lies in a neighbourhood W of $(\mu_0, \nu_0, a_0, 0)$ in $B_1(0) \times \mathbb{C} \times \mathbb{R} \times \mathcal{C}^{k-2,\alpha}(\overline{\mathcal{B}})$ where the map $\mathcal{G}$ is a diffeomorphism. Moreover, from the Maximum Principle at infinity for solutions of (2) (see [FMM2]), there exists only one pre-image of $(\mu_n, \nu_n, a_n, 0)$. Therefore $(\mu_n, \nu_n, a_n, \widetilde{h}_n) \in \mathcal{G}^{-1}(W)$, and then $\{\widetilde{h}_n\}$ converges to $\widetilde{h}_0$ in $\mathcal{C}^k$ for all $k \geq 0$. Thus, one obtains *a)*. □

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