Singly-periodic Improper Affine Spheres

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Abstract

In this paper we give a characterization of complete embedded ends of singly-periodic improper affine spheres in terms of their conformal representation.

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1 Introduction

The study of locally strongly convex improper affine spheres is locally equivalent (see [2], [3]) to the study of convex solutions of the Monge-Ampère equation

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = 1 \tag{1}$$

on a planar domain. Since the underlying almost-complex structure of (1) is integrable, a conformal representation in terms of two meromorphic functions can be given for improper affine spheres (see Sect. 2). Indeed, a pair of meromorphic functions on a Riemann surface produce an immersion of an improper affine sphere in $\mathbb{R}^3$, provided certain compatibility conditions are satisfied, one of which refers to the multi-valuation of the immersion. This paper is motivated by the existence of examples of improper affine spheres given by multi-valued immersions. These examples are all invariant under a translation and a family of screw motions (see Sect. 3).

The purpose of this paper is to study a class of improper affine spheres that contains the aforementioned examples. We shall call an improper affine sphere *singly-periodic* if it is connected and invariant under an infinite cyclic group of equiaffine transformations of $\mathbb{R}^3$ that acts freely on $\mathbb{R}^3$. Our last objective is to describe the complete embedded ends of *singly-periodic* improper affine spheres. In this context we obtain the following classification result:

> *A complete embedded end of a singly-periodic improper affine sphere is asymptotic to a half elliptic paraboloid, a surface $E_{\mathfrak{a}}$ (see Sect. 3) or to a surface $M_{a_0,t}$ (see Sect. 4).*

The paper is organized as follows. In Sect. 2 we recall some basic facts about improper affine spheres, emphasizing the conformal representation.

In Sect. 3 we introduce complete ends of singly-periodic improper affine spheres and give a first family of examples.

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Finally, Sect. 4, 5, 6 and 7 are devoted to give a characterization of the different types of those complete embedded ends.

2 Preliminaries

We refer the reader to [3], [4] and [10] for more details about locally strongly convex affine surfaces.

Let $\boldsymbol{\chi} : M \longrightarrow \mathbb{R}^3$ be an oriented immersed locally strongly convex improper affine sphere with a differentiable boundary (possibly empty). We shall denote by (x_1, x_2, x_3) a rectangular coordinate system in $\mathbb{R}^3$. By using an equiaffine transformation if necessary, we can assume that the affine normal vector of M is $\xi = (0, 0, 1)$. Then the projection on $\Pi \equiv \{x_3 = 0\}$ parallel to ξ, $p_\xi : M \longrightarrow \Pi$, is an immersion and M is, locally, the graph of a convex solution $f : \mathcal{B} \longrightarrow \mathbb{R}$ of (1) on a bounded convex domain $\mathcal{B}$ in Π. That is, in a neighbourhood of each point, M is given by

$$\boldsymbol{\chi}(x_1, x_2) = (\boldsymbol{\chi}_1(x_1, x_2), \boldsymbol{\chi}_2(x_1, x_2), \boldsymbol{\chi}_3(x_1, x_2)) = (x_1, x_2, f(x_1, x_2)) \ , \tag{2}$$

where $(x_1, x_2) \in \mathcal{B}$. Moreover, it is easy to prove that the affine metric and the affine conormal map of the improper affine sphere are given on this neighbourhood by

$$ds^2 = \sum_{i,j=1}^{2} f_{ij} dx_i dx_j \ , \tag{3}$$

$$N = (N_1, N_2, N_3) = \left(-\frac{\partial f}{\partial x_1}, -\frac{\partial f}{\partial x_2}, 1 \right) \ , \tag{4}$$

where we denote by $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$, for $i, j = 1, 2$. Conversely, the graph of a convex solution of (1) is an improper affine sphere with affine normal vector field $\xi = (0, 0, 1)$ and affine metric given by (3).

Using standard notation of complex analysis (see [1]), one can define the functions $F, G : \mathcal{B} \longrightarrow \mathbb{C}$ given by

$$G(x_1, x_2) = \left(x_1 + \frac{\partial f}{\partial x_1} \right) + \mathrm{i} \left(x_2 + \frac{\partial f}{\partial x_2} \right) \ , \tag{5}$$

$$F(x_1, x_2) = \left(x_1 - \frac{\partial f}{\partial x_1} \right) + \mathrm{i} \left(-x_2 + \frac{\partial f}{\partial x_2} \right) \ , \tag{6}$$

with (x_1, x_2) in the former convex domain $\mathcal{B}$. Taking into account (2), (4), (5) and (6), the functions F and G can be written

$$G = (\boldsymbol{\chi}_1 - N_1) + \mathrm{i} \, (\boldsymbol{\chi}_2 - N_2) \ , \tag{7}$$

$$F = (\boldsymbol{\chi}_1 + N_1) - \mathrm{i} \, (\boldsymbol{\chi}_2 + N_2) \ , \tag{8}$$

and thus, F and G are globally defined on M. From the above considerations, it is possible to give the following conformal representation for improper affine spheres (see [5], [6]).

Theorem 1 (Conformal representation)

i) Let $\boldsymbol{\chi} : M \longrightarrow \mathbb{R}^3$ be an improper affine sphere with affine normal $\xi = (0, 0, 1)$ and consider on M the structure of Riemann surface induced by its affine metric. Then there exist holomorphic functions F and G on M such that dG does not vanish on M, $|dF| < |dG|$ and M can be represented, up a vertical translation, by the immersion

$$\boldsymbol{\chi} = \left(\frac{G + \overline{F}}{2}, \frac{1}{8}|G|^2 - \frac{1}{8}|F|^2 + \frac{1}{4}\mathrm{Re}(GF) - \frac{1}{2}\mathrm{Re} \int F dG \right) \ . \tag{9}$$

Moreover, the affine metric and the affine conormal map are given by

$$ds^2 = \frac{1}{4}\left(|dG|^2 - |dF|^2\right) , \tag{10}$$

$$N = \left(\frac{\overline{F} - G}{2}, 1\right) . \tag{11}$$

ii) Conversely, let M be a Riemann surface, F and G two holomorphic functions on M such that dG does not vanish on M and $|dF| < |dG|$. Then (9) *defines an improper affine sphere with affine normal* $(0, 0, 1)$ *and with affine metric and affine conormal map given by* (10) *and* (11)*, respectively. Moreover, $\mathbf{X}$ is singly-valued if and only if FdG does not have real periods.*

The pair (F, G) is called a *conformal representation* of the improper affine sphere M. This conformal representation, analogously to Weierstrass data in the theory of minimal surfaces, has become a powerful tool in the study of improper affine spheres (see [5], [6]). Furthermore, we can consider the holomorphic function $\Psi : M \longrightarrow D = \{z \in \mathbb{C} \mid |z| < 1\}$ given by $\Psi = \frac{dF}{dG}$ that is known as the *Gauss map* of M. Taking into account (10), it is easy to prove that the expression of the affine Gauss curvature as a function of F and G is

$$\kappa = \frac{8|G''F' - F''G'|^2}{\left(|G'|^2 - |F'|^2\right)^3} .$$

Thus, the affine Gauss curvature of an improper affine sphere is always non negative.

Now, let g be an equiaffine transformation on $\mathbb{R}^3$ such that dg preserves the vector $(0, 0, 1)$. Then, g can be written as

$$g\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ \alpha_1 & \alpha_2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} , \tag{12}$$

where $ad - bc = 1$.

Clearly, if $\mathbf{X} : M \longrightarrow \mathbb{R}^3$ is an improper affine sphere with affine normal vector $\xi = (0, 0, 1)$, then $g \circ \mathbf{X} : M \longrightarrow \mathbb{R}^3$ is also an improper affine sphere with affine normal vector $\xi = (0, 0, 1)$. Given (F, G) the conformal representation of $\mathbf{X} : M \longrightarrow \mathbb{R}^3$, we are interested in obtaining the conformal representation $(\widetilde{F}, \widetilde{G})$ of $g \circ \mathbf{X} : M \longrightarrow \mathbb{R}^3$.

In order to do this we consider a neighbourhood in M given as in (2). Then, its image under g is the graph of the function $\widetilde{f} : g_1(\mathcal{B}) \longrightarrow \mathbb{R}$ given by $\widetilde{f} = (f + \alpha_1 p_1 + \alpha_2 p_2 + \beta_3) \circ g_1^{-1}$, where g_1 is the equiaffine transformation in $\mathbb{R}^2$ given by $g_1(x_1, x_2) = (ax_1 + bx_2 + \beta_1, cx_1 + dx_2 + \beta_2)$ and $p_i : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are the standard projections. Therefore, taking into account (4), the affine conormal map $\widetilde{N}$ of $g(M)$ is given by

$$\widetilde{N}(g(p)) = \begin{pmatrix} d & -c & -d\alpha_1 + c\alpha_2 \\ -b & a & b\alpha_1 - a\alpha_2 \\ 0 & 0 & 1 \end{pmatrix} N(p) , \tag{13}$$

for all $p \in M$. Thus, from (7), (8), (12) and (13) we have the following expression for $\widetilde{G}$ and $\widetilde{F}$

$$\widetilde{G}(z) = \frac{1}{2}\left[(a + d - \mathrm{i}(b - c))G(z) + (a - d + \mathrm{i}(b + c))F(z)\right] + \beta_1 + d\alpha_1 - c\alpha_2 + \mathrm{i}(\beta_2 - b\alpha_1 + a\alpha_2) , \tag{14}$$

$$\widetilde{F}(z) = \frac{1}{2}\left[(a - d - \mathrm{i}(b + c))G(z) + (a + d + \mathrm{i}(b - c))F(z)\right] + \beta_1 - d\alpha_1 + c\alpha_2 - \mathrm{i}(\beta_2 + b\alpha_1 - a\alpha_2) . \tag{15}$$

Moreover, if we also assume that the equiaffine transformation g has no fixed points, then it is an elementary exercise to prove the following result.

Proposition 1 *Let $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be an equiaffine transformation without fixed points satisfying $dg(0,0,1) = (0,0,1)$, then there exists an equiaffine transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with $dT(0,0,1) = (0,0,1)$ such that $\widetilde{g} = T \circ g \circ T^{-1}$ can be written as follows:*

I) **I.1)**

$$\widetilde{g}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & \alpha_2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ 0 \\ \beta_3 \end{pmatrix},$$

with $t \neq 0$ and $\alpha_2, \beta_i \in \mathbb{R}$, for $i = 1, 3$ such that $\widetilde{g}$ has no fixed points.

I.2)

$$\widetilde{g}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ t & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \beta_3 \end{pmatrix},$$

with $t \neq 0$ and $\beta_3 \neq 0$.

II)

$$\widetilde{g}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha_1 & \alpha_2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta_2 \\ \beta_3 \end{pmatrix},$$

with β_2, β_3, α_1 and $\alpha_2 \in \mathbb{R}$ such that $\widetilde{g}$ has no fixed points.

III)

$$\widetilde{g}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \beta_3 \end{pmatrix},$$

with $t \in]0, 2\pi[$ and $\beta_3 \neq 0$.

IV)

$$\widetilde{g}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \beta_3 \end{pmatrix},$$

with $t \notin \{0, 1, -1\}$ and $\beta_3 \neq 0$.

We say that g is of type **I.1**, **I.2**, **II**, **III** or **IV** if there exists $\widetilde{g}$ of this type verifying the conditions of Proposition 1.

3 Complete ends of singly-periodic improper affine spheres

As we mentioned in Sect. 1, we say that an improper affine sphere is *singly-periodic* if it is connected and invariant under an infinite cyclic group of equiaffine transformations of $\mathbb{R}^3$ that acts freely on $\mathbb{R}^3$. The aim of this section is to introduce the study of complete ends of singly-periodic improper affine spheres.

Let $\boldsymbol{\chi} : M \longrightarrow \mathbb{R}^3$ be an improper affine sphere invariant under an equiaffine transformation $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that acts freely on $\mathbb{R}^3$. Notice that then g is as in Proposition 1. Moreover, there exists an isometry $\alpha : M \longrightarrow M$ of the affine metric such that $\boldsymbol{\chi} \circ \alpha = g \circ \boldsymbol{\chi}$. Thus, if as before we consider on M the structure of Riemann surface induced by the affine metric, we can consider the

Riemann surface $M/\langle\alpha\rangle$. Since the affine metric is invariant under $\langle\alpha\rangle$, it projects to a Riemannian metric on $M/\langle\alpha\rangle$. We are interested in understanding the singly-periodic improper affine spheres such that $M/\langle\alpha\rangle$ is homeomorphically $A = \{z \in \mathbb{C} \mid \frac{1}{r_1} \leq |z| < r_1\}$ with $1 < r_1 < \infty$ and the projected metric is complete.

Definition 1 *If $\boldsymbol{X} : M \longrightarrow \mathbb{R}^3$ is as before we say that M is a complete end of a singly-periodic improper affine sphere.*

Regarding the conformal structure of these improper affine spheres we can state the following result.

Lemma 1 *Let $\boldsymbol{X} : M \longrightarrow \mathbb{R}^3$ be a complete end of a singly-periodic improper affine sphere and $\alpha : M \longrightarrow M$ its associated isometry. Then $M/\langle\alpha\rangle$ is conformally the puncture disk $D^* = \{z \in \mathbb{C} \mid 0 < |z| \leq 1\}$.*

Proof: Suppose that $M/\langle\alpha\rangle$ is conformally A and the projected Riemannian metric on $M/\langle\alpha\rangle$ is written on A as

$$h(z) = \lambda(z)|dz| \; ,$$

with $\lambda > 0$. Hence we can introduce in $\widetilde{A} = \{z \in \mathbb{C} \mid \frac{1}{r_1} < |z| < r_1\}$ the metric

$$\widehat{h}(z) = \lambda(z)\lambda\left(1/z\right)|dz| \; .$$

Then one verifies easily that this metric is a complete Riemannian metric on $\widetilde{A}$. Moreover, taking into account that the affine Gauss curvature is non negative, it is not difficult to prove that the Gauss curvature of the metric $\widehat{h}$ is also non negative. Since $\widehat{h}$ is a complete Riemannian metric with non negative Gauss curvature on $\widetilde{A}$, a result by A. Huber (see Theorem 15 in [8]) says that $\widetilde{A}$ must be parabolic, but this is a contradiction. □

Throughout we always suppose that $M/\langle\alpha\rangle = D^*$ and the complete end is given by $\boldsymbol{X} : \mathbb{C}^- \longrightarrow \mathbb{R}^3$ where $\mathbb{C}^- = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq 0\}$. Moreover, since $\alpha : \mathbb{C}^- \longrightarrow \mathbb{C}^-$ is an isometry without fixed points, α has to be a translation and so we can assume that $\alpha(z) = z + 2\pi\mathrm{i}$. Finally, the projection $p : \mathbb{C}^- \longrightarrow D^*$ is given by $p(z) = \exp(z)$. Furthermore, denoting the conformal representation of the complete end by (F, G) and using (14) and (15) we have the following conditions on F and G

$$\begin{aligned} G(\alpha(z)) &= \frac{1}{2}\left[(a + d - \mathrm{i}(b - c))G(z) + (a - d + \mathrm{i}(b + c))F(z)\right] + \\ &\quad \beta_1 + d\alpha_1 - c\alpha_2 + \mathrm{i}(\beta_2 - b\alpha_1 + a\alpha_2) \; , \end{aligned} \tag{16}$$

$$\begin{aligned} F(\alpha(z)) &= \frac{1}{2}\left[(a - d - \mathrm{i}(b + c))G(z) + (a + d + \mathrm{i}(b - c))F(z)\right] + \\ &\quad \beta_1 - d\alpha_1 + c\alpha_2 - \mathrm{i}(\beta_2 + b\alpha_1 - a\alpha_2) \; . \end{aligned} \tag{17}$$

EXAMPLES OF SINGLY-PERIODIC IMPROPER AFFINE SPHERES

1. The elliptic paraboloid, given by the graph of the function $f(x_1, x_2) = \frac{x_1^2 + x_2^2}{2}$, is invariant under the following family of transformations of type **II**

$$g\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha_1 & \alpha_2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \frac{\alpha_1^2 + \alpha_2^2}{2} \end{pmatrix} ,$$

with $\alpha_i \in \mathbb{R}$, $i = 1, 2$.

2. Consider the following functions

$$G(\tau) = \frac{1}{\tau}\,, \quad F_{\mathfrak{a}}(\tau) = \mathfrak{a}\tau\ , \tag{18}$$

with $\mathfrak{a} \in \mathbb{C}$, that are holomorphic functions in the domain $\Omega = \{\tau \in \mathbb{C} \ \mid 0 < |\tau| < |\mathfrak{a}|^{-\frac{1}{2}}\}$ satisfying the conditions in Theorem 1. From (9) the improper affine spheres with conformal representation $(F_{\mathfrak{a}}, G)$, that we denote $E_{\mathfrak{a}}$, are given by the parametrization

$$\begin{aligned} \boldsymbol{\chi}_{\mathfrak{a}}(r,\theta) \ = \ & \left(\frac{1}{2}\left(\frac{1}{r}\cos\theta + \mathfrak{a}_1 r\cos\theta - \mathfrak{a}_2 r\sin\theta\right), \frac{-1}{2}\left(\frac{1}{r}\sin\theta + \mathfrak{a}_1 r\sin\theta + \mathfrak{a}_2 r\cos\theta\right),\right. \\ & \left.\frac{1}{8r^2} - \frac{|\mathfrak{a}|^2 r^2}{8} + \frac{\mathfrak{a}_1}{4} + \frac{\mathfrak{a}_1}{2}\log r - \frac{\mathfrak{a}_2}{2}\theta\right)\ , \end{aligned}$$

where $\mathfrak{a} = \mathfrak{a}_1 + \mathrm{i}\mathfrak{a}_2$ and $\tau = re^{\mathrm{i}\theta} \in \Omega$.

From (10) and (18), $E_{\mathfrak{a}}$ is clearly a complete end of an improper affine sphere. Moreover, $E_{\mathfrak{a}}$ is embedded since the intersection of $E_{\mathfrak{a}}$ with a vertical cylinder is a helix over the cylinder. We also observe that $E_{\mathfrak{a}}$ is invariant under the vertical translation (type **II**) of vector $(0,0,\mathfrak{a}_2 m\pi)$, for all $m \in \mathbb{Z}$. Finally, a direct computation proves that $E_{\mathfrak{a}}$ is also invariant under all the screw motions (type **III**) $g = t_w \circ \sigma$, where σ is a rotation of angle ϕ around the axis $\{x_1 = x_2 = 0\}$ and t_w is a translation of vector $w = (0,0,\frac{\mathfrak{a}_2}{2}\phi)$, for any $\phi \in \mathbb{R}$.

Observe that the immersion $\boldsymbol{\chi}_{\mathfrak{a}}$ is singly-valued if and only if $\mathfrak{a}_2 = 0$. In this case the immersion describes an improper affine sphere of revolution with logarithmic growth rate $\mathfrak{a}_1$ (see [5]). On the other hand, if $\mathfrak{a}_2 \neq 0$ the immersion $\boldsymbol{\chi}_{\mathfrak{a}}$ is multi-valued and we obtain a family of examples of complete ends of singly-periodic improper affine spheres (see Fig. 1 and Fig. 2). As before, $\mathfrak{a}_1$ will be called the *logarithmic growth rate* of the end $E_{\mathfrak{a}}$.

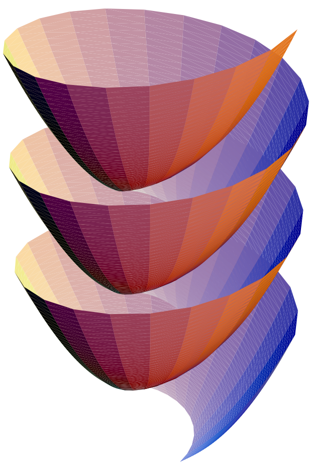

Figure 1: Surface $E_{\mathfrak{a}}$ for $\mathfrak{a} = \mathrm{i}$.

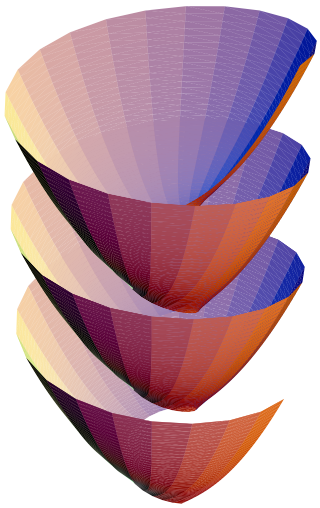

Figure 2: Surface $E_{\mathfrak{a}}$ for $\mathfrak{a} = -\mathrm{i}$.

4 Complete embedded ends of type I

Let $\boldsymbol{\chi} : \mathbb{C}^- \longrightarrow \mathbb{R}^3$ be a complete embedded end invariant under an equiaffine transformation $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ as in **I.1** and let (F, G) be its conformal representation. Then using (16) and (17) we have

$$\begin{aligned} G(\alpha(z)) &= \left(1 + \mathrm{i}\frac{t}{2}\right) G(z) + \mathrm{i}\frac{t}{2} F(z) + \beta_1 - t\alpha_2 + \mathrm{i}\alpha_2 \,, & (19) \\ F(\alpha(z)) &= -\mathrm{i}\frac{t}{2} G(z) + \left(1 - \mathrm{i}\frac{t}{2}\right) F(z) + \beta_1 + t\alpha_2 + \mathrm{i}\alpha_2 \,. & (20) \end{aligned}$$

Thus, the holomorphic one-form $\eta = dG + dF$ passes to the quotient D^*, namely, there exists $\widetilde{\eta}$ a holomorphic one-form on D^* verifying $\exp^* \widetilde{\eta} = \eta$. Since η never vanishes on $\mathbb{C}^-$ we can also consider the holomorphic function $H : \mathbb{C}^- \longrightarrow \mathbb{C}$ given by

$$H(z) = \frac{dG - dF}{\eta} - \frac{t}{2\pi} z \tag{21}$$

and the holomorphic one-form

$$\omega = \frac{2(G'' dF - F'' dG)}{(G' + F')^2} \,.$$

From (19) and (20) it is easy to check that H and ω also pass to the quotient D^* and then there exist a holomorphic function $\widetilde{H}$ and a holomorphic one-form $\widetilde{\omega}$ on D^* verifying $\widetilde{H} \circ \exp = H$ and $\exp^* \widetilde{\omega} = \omega$. From the definition of ω we also have

$$\omega = d\left(\frac{dG - dF}{\eta}\right) \,. \tag{22}$$

Hence we obtain that

$$dG - dF = f\, \eta \,,$$

where $f = \int \omega$. Hence, taking into account the definition of η, we can write

$$dG = \frac{1}{2}\,(1 + f\,)\,\eta \,, \quad dF = \frac{1}{2}\,(1 - f\,)\,\eta \,. \tag{23}$$

Then, using (9) and (23), the immersion can be expressed as follows

$$\boldsymbol{\chi} = (\boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \boldsymbol{\chi}_3) = \left(\frac{1}{2}\,\mathrm{Re}\int \eta, \frac{1}{2}\,\mathrm{Im}\int f\,\eta, \frac{1}{4}\,\mathrm{Re}\int \eta\,\mathrm{Re}\int f\,\eta - \frac{1}{4}\,\mathrm{Re}\int\left(\int \eta\right) f\,\eta\right) \,. \tag{24}$$

Hence a multi-valued parametrization of the end is given by

$$\widetilde{\boldsymbol{\chi}} = (\widetilde{\boldsymbol{\chi}}_1, \widetilde{\boldsymbol{\chi}}_2, \widetilde{\boldsymbol{\chi}}_3) = \left(\frac{1}{2}\,\mathrm{Re}\int \widetilde{\eta}, \frac{1}{2}\,\mathrm{Im}\int \widetilde{f}\,\widetilde{\eta}, \frac{1}{4}\,\mathrm{Re}\int \widetilde{\eta}\,\mathrm{Re}\int \widetilde{f}\,\widetilde{\eta} - \frac{1}{4}\,\mathrm{Re}\int\left(\int \widetilde{\eta}\right) \widetilde{f}\,\widetilde{\eta}\right) \,, \tag{25}$$

where $\widetilde{f} = \int \widetilde{\omega}$.

Definition 2 *The pair $(\widetilde{\eta}, \widetilde{\omega})$ will be called the conformal representation of this type of ends.*

The results of this section can be summarized in the following theorem.

Theorem 2 *Let* $\mathbf{X}:\mathbb{C}^-\longrightarrow\mathbb{R}^3$ *be a complete embedded end of an improper affine sphere invariant under an equiaffine transformation as in* **I.1** *and let* $(\widetilde{\eta},\widetilde{\omega})$ *be its conformal representation. Then* $t<0$, $\beta_1\neq 0$, *and* $\widetilde{\eta}$ *and* $\widetilde{\omega}$ *are holomorphic one-forms on* D^* *with a pole at zero of order one,* $\mathrm{Res}_0(\widetilde{\eta})=\frac{1}{\pi}(\alpha_2-\mathrm{i}\beta_1)$ *and* $\mathrm{Res}_0(\widetilde{\omega})=\frac{t}{2\pi}$.

Conversely, if $\widetilde{\eta}$ *and* $\widetilde{\omega}$ *are holomorphic one-forms on* D^* *with a pole at zero of order one,* $\mathrm{Im}(\mathrm{Res}_0(\widetilde{\eta}))\neq 0$ *and* $\mathrm{Res}_0(\widetilde{\omega})$ *a negative real number, then the immersion given in* (25) *contains a complete embedded subend invariant under an equiaffine transformation* g *of type* **I.1** *with* $\alpha_2=\pi\mathrm{Re}(\mathrm{Res}_0(\widetilde{\eta}))$, $\beta_1=-\pi\mathrm{Im}(\mathrm{Res}_0(\widetilde{\eta}))$ *and* $t=2\pi\mathrm{Res}_0(\widetilde{\omega})$.

Furthermore, there exist no complete embedded ends invariant under an equiaffine transformation as in **I.2**.

Proof of Theorem 2: For the sake of clarity, we develop the proof in different steps.

Step 1: *We assert:*

a) If $t>0$ *then* ds *cannot be a Riemannian metric.*

b) If $t<0$ *and the end is complete then* $\widetilde{\eta}$ *has a pole at* 0 *of order* $k+1\geq 1$ *and* $\mathrm{Res}_0(\widetilde{\eta})=\frac{1}{\pi}(\alpha_2-\mathrm{i}\beta_1)$ *and* $\widetilde{\omega}$ *has a pole at* 0 *of order one and* $\mathrm{Res}_0(\widetilde{\omega})=\frac{t}{2\pi}$.

From (21) and (22) we have

$$\widetilde{\omega}=\frac{t}{2\pi\tau}d\tau+d\widetilde{H}\ , \tag{26}$$

where $\tau=\exp(z)$. Then, taking into account (23) and (26), we have the following expression for the projected metric on D^*:

$$ds^2=\frac{1}{4}\mathrm{Re}(\widetilde{f})\,|\widetilde{\eta}|^2=\frac{1}{4}\left(\frac{t}{2\pi}\log r+\mathrm{Re}(\widetilde{H})\right)|\widetilde{\eta}|^2\ , \tag{27}$$

with $\tau=re^{\mathrm{i}\theta}$ and $\tau\in D^*$. In order that ds can be a Riemannian metric, it is necessary that the harmonic function $\frac{t}{2\pi}\log r+\mathrm{Re}(\widetilde{H})>0$ on D^*. Then it is not difficult to prove that $\widetilde{H}$ has a removable singularity at $\tau=0$ (see [1]) and so, using (26), we conclude that $\widetilde{\omega}$ is as in *b)*. Furthermore, if $t>0$ then $\frac{t}{2\pi}\log r+\mathrm{Re}(\widetilde{H})$ tends to $-\infty$ when $|\tau|$ tends to 0, and so there are points where ds is not a Riemannian metric.

Henceforth we suppose $t<0$. Thus, for r sufficiently small we have the following inequalities

$$\frac{t}{2\pi}\log r+\mathrm{Re}(\widetilde{H})\leq\frac{t}{2\pi}\log r+C_0\leq\frac{t-1}{2\pi}\log r\leq\frac{1-t}{2\pi}|\tau|^{-2}\ , \tag{28}$$

where C_0 is a constant. Finally, from (27) we obtain

$$ds^2\leq\frac{1-t}{8\pi}|\tau^{-1}\widetilde{\eta}|^2\ ,$$

Since $\mathbf{X}:\mathbb{C}^-\longrightarrow\mathbb{R}^3$ is a complete end and $\tau^{-1}\widetilde{\eta}$ is a holomorphic one-form without zeros in D^*, a well-known result (see Lemma 9.6 in [11]) says us that $\tau^{-1}\widetilde{\eta}$ has a pole at 0 of order $k+2\geq 1$. In fact, one can prove that $k>-1$ and then $\widetilde{\eta}$ has a pole at 0 of order $k+1\geq 1$. We proceed by contradiction. Suppose $k=-1$ and consider on D^* the divergent curve $\gamma(r)=r$ with $r\in]0,r_0]$ and r_0 sufficiently small so that inequalities in (28) are satisfied for $r\leq r_0$. Then from (27) and (28) the length of the curve γ respect to the affine metric satisfies

$$\int_\gamma ds\leq\frac{1}{2}\int_\gamma\left(\frac{t-1}{2\pi}\log r\right)^{\frac{1}{2}}|\widetilde{\eta}|\leq\frac{C_1}{2}\left(\frac{1-t}{2\pi}\right)^{\frac{1}{2}}\int_0^{r_0}(-\log r)^{\frac{1}{2}}dr\ ,$$

where C_1 is a positive constant. Since the integral on the righthand side converges, we conclude that γ has finite length, which is a contradiction. Finally, from (19), (20) and the definition of $\widetilde{\eta}$ we have $\mathrm{Res}_0(\widetilde{\eta}) = \frac{1}{\pi}(\alpha_2 - \mathrm{i}\beta_1)$.

Step 2: *We claim that if $t < 0$ and $k > 0$, then the end cannot be embedded. Then $\widetilde{\eta}$ has a pole at zero of order one and* $\mathrm{Im}(\mathrm{Res}_0(\widetilde{\eta})) \neq 0$.

For the sake of clarity, throughout this step we consider the parametrization given by (24). We shall prove our claim by finding two disjoint curves on $\mathbb{C}^-$ whose image by $\boldsymbol{x}$ intersect each other. The ideas of this proof and the proof of Step 3 in Sect. 5 are inspired in the works [7] and [9].

Assume that $\widetilde{\eta}$ has a pole at 0 of order $k+1 > 1$ and $\widetilde{\omega}$ is as in statement *b)* of Step 1. We distinguish two cases, the case $\beta_1 \neq 0$ and the case $\beta_1 = 0$.

CASE $\beta_1 \neq 0$.

In this case, it is easy to prove that there exists an equiaffine transformation $\widetilde{g}$ as in **I.1** with $t < 0$, $\beta_1 \neq 0$, $\beta_3 = 0$ and such that g and $\widetilde{g}$ are conjugate. From the above conditions, $\widetilde{\eta}$ and $\widetilde{\omega}$ can be written on D^* as follows

$$\widetilde{\eta} = \left(\sum_{\lambda=-k-1}^{\infty} \widehat{a}_\lambda \tau^\lambda \right) d\tau \ , \quad \widetilde{\omega} = \left(\frac{t}{2\pi\tau} + \sum_{\lambda=0}^{\infty} \widehat{b}_\lambda \tau^\lambda \right) d\tau \ , \tag{29}$$

with $\widehat{a}_\lambda, \widehat{b}_\lambda \in \mathbb{C}$, $\widehat{a}_{-1} = \frac{1}{\pi}(\alpha_2 - \mathrm{i}\beta_1)$ and $\widehat{a}_{-k-1} \neq 0$. Therefore we get the following expression for the holomorphic one-form η

$$\eta = \left(\sum_{\lambda=-k-1}^{\infty} \widehat{a}_\lambda \mathrm{e}^{\lambda z} \right) \mathrm{e}^z dz = \left(\sum_{\lambda=-k}^{\infty} a_\lambda \mathrm{e}^{\lambda z} \right) dz \ , \tag{30}$$

where $a_\lambda = \widehat{a}_{\lambda-1}$, $a_0 = \frac{1}{\pi}(\alpha_2 - \mathrm{i}\beta_1)$ and we recall $a_{-k} \neq 0$. Hence

$$\int \eta = a_0 z + \sum_{\lambda=-k}^{\infty} a'_\lambda \mathrm{e}^{\lambda z} \ , \tag{31}$$

where $a'_\lambda = \frac{a_\lambda}{\lambda}$ for $\lambda \neq 0$ and $a'_0 \in \mathbb{C}$. Moreover, from (24) and (31), the first component of the immersion $\boldsymbol{x}$ is given by

$$\boldsymbol{x}_1(z) = \frac{1}{2} \left\{ \frac{\alpha_2}{\pi} x + \frac{\beta_1}{\pi} y + \sum_{\lambda=-k}^{\infty} |a'_\lambda| \cos(\theta_\lambda + \lambda y) \mathrm{e}^{\lambda x} \right\} \ , \tag{32}$$

where $z = x + \mathrm{i}y$ and $\theta_\lambda = \arg(a'_\lambda)$. According to (29) and (30) we can write

$$f\,\eta = \left(\frac{t}{2\pi} z \sum_{\lambda=-k}^{\infty} a_\lambda \mathrm{e}^{\lambda z} + \sum_{\lambda=-k}^{\infty} b_\lambda \mathrm{e}^{\lambda z} \right) dz \ , \tag{33}$$

with $b_\lambda \in \mathbb{C}$. Using (24) and (33), we have the following expression for the second component of $\boldsymbol{x}$

$$\begin{aligned}
\boldsymbol{x}_2(z) &= \frac{1}{2}\mathrm{Im}\Big(\frac{t}{4\pi} a_0 z^2 + b_0 z + \frac{t}{2\pi} z \sum_{\substack{\lambda=-k \\ \lambda \neq 0}}^{\infty} a'_\lambda \mathrm{e}^{\lambda z} - \frac{t}{2\pi} \sum_{\lambda=-k}^{\infty} a''_\lambda \mathrm{e}^{\lambda z} + \sum_{\lambda=-k}^{\infty} b'_\lambda \mathrm{e}^{\lambda z}\Big) \qquad (34) \\
&= \frac{1}{2}\Big\{ \frac{t}{2\pi} \sum_{\substack{\lambda=-k \\ \lambda \neq 0}}^{\infty} |a'_\lambda| (x \sin(\theta_\lambda + \lambda y) + y \cos(\theta_\lambda + \lambda y)) \mathrm{e}^{\lambda x} + \sum_{\lambda=-k}^{\infty} |b'_\lambda| \sin(\gamma_\lambda + \lambda y) \mathrm{e}^{\lambda x} \\
&- \frac{t}{2\pi} \sum_{\lambda=-k}^{\infty} |a''_\lambda| \sin(\theta_\lambda + \lambda y) \mathrm{e}^{\lambda x} + \frac{t}{4\pi^2} \big(\beta_1 (y^2 - x^2) + 2\alpha_2 xy\big) + \mathrm{Re}(b_0) y + \mathrm{Im}(b_0) x \Big\} \ ,
\end{aligned}$$

where $a''_\lambda = \frac{a'_\lambda}{\lambda}$, $b'_\lambda = \frac{b_\lambda}{\lambda}$, $a''_0, b'_0 \in \mathbb{C}$ and $\gamma_\lambda = \arg(b'_\lambda)$. In a similar way, we deduce from (24), (31) and (33) that the asymptotic behaviour of $\mathbf{X}_3$ in this case is given by

$$\mathbf{X}_3(z) = \frac{t}{16k^2\pi}|a_{-k}|^2 x \mathrm{e}^{-2kx}(1 + h(x,y)) \;, \tag{35}$$

where $h(x,y)$ denotes a function such that $\forall y \in \mathbb{R}$, $\lim_{x\to-\infty} h(x,y) = 0$.

Now, we shall study the intersection of the end with the plane $x_2 = 0$. Denote by I_l the interval $I_l =]a_l^1, a_l^2[= \left]\frac{\theta_{-k}+\pi l}{k}, \frac{\theta_{-k}+\pi(l+1)}{k}\right[$, for $l \in \mathbb{Z}$. From (34) the second component of the immersion on the points $z = x_0 + iy$ for a fixed $x_0 < 0$ verifies

$$\begin{aligned} 2\mathbf{X}_2(x_0,y) &\le \frac{t}{2\pi}(x_0 - |y|) \sum_{\substack{\lambda = -k \\ \lambda \neq 0}}^{\infty} |a'_\lambda|\mathrm{e}^{\lambda x_0} - \frac{t}{2\pi}\sum_{\lambda=-k}^{\infty} |a''_\lambda|\mathrm{e}^{\lambda x_0} + \sum_{\lambda=-k}^{\infty} |b'_\lambda|\mathrm{e}^{\lambda x_0} \\ &+ \frac{t}{4\pi^2}\left(\beta_1(y^2 - x_0^2) + 2\alpha_2 x_0 y\right) + \mathrm{Re}(b_0)y + \mathrm{Im}(b_0)x_0 \;. \end{aligned}$$

Since the series $\sum_{\lambda=-k}^{\infty} a'_\lambda \tau^\lambda$, $\sum_{\lambda=-k}^{\infty} a''_\lambda \tau^\lambda$ and $\sum_{\lambda=-k}^{\infty} b'_\lambda \tau^\lambda$ represent the Laurent series of holomorphic functions on D^*, we have that they converge absolutely at each point of D^* and so the series in the above inequality converge. Hence we have for $y > 0$

$$2\mathbf{X}_2(x_0,y) \le \frac{t}{4\pi^2}\beta_1 y^2 + C_2 y + C_3 \;, \tag{36}$$

where C_i are constants for $i = 2,3$. Assume $\beta_1 > 0$. From (36), there exists $y_0 > 0$ such that $\mathbf{X}_2(x_0,y) < 0$ for $y \ge y_0$.

Moreover, if we fixed $y \in I_l$ for an odd l we obtain from (34) $\lim_{x\to-\infty}\mathbf{X}_2(x,y) = +\infty$. Therefore if $y \ge y_0$ and $y \in I_l$ for and odd l, there exists x_1 such that $\mathbf{X}_2(x_1,y) = 0$. We consider and odd $l_0 \in \mathbb{N}$ so that $a_{l_0}^1 \ge y_0$. Thus, if l is an odd integer satisfying $l \ge l_0$, we have $A_l =]-\infty, 0[\times I_l \cap \mathbf{X}_2^{-1}(0) \neq \emptyset$ and then A_l is a union of analytic curves. Furthermore, from (34) we have

$$\begin{aligned} \sin(\theta_{-k} - ky) &= -\frac{2\pi}{t|a'_{-k}|}x^{-1}\mathrm{e}^{kx}\{\frac{t}{2\pi}x \sum_{\substack{\lambda = -k+1 \\ \lambda \neq 0}}^{\infty} |a'_\lambda|\sin(\theta_\lambda + \lambda y)\mathrm{e}^{\lambda x} \\ &+ \frac{t}{2\pi}y \sum_{\substack{\lambda = -k \\ \lambda \neq 0}}^{\infty} |a'_\lambda|\cos(\theta_\lambda + \lambda y)\mathrm{e}^{\lambda x} - \frac{t}{2\pi}\sum_{\lambda=-k}^{\infty}|a''_\lambda|\sin(\theta_\lambda + \lambda y)\mathrm{e}^{\lambda x} \\ &+ \sum_{\lambda=-k}^{\infty}|b'_\lambda|\sin(\gamma_\lambda + \lambda y)\mathrm{e}^{\lambda x} + \frac{t}{4\pi^2}\left(\beta_1(y^2 - x^2) + 2\alpha_2 xy\right) + \mathrm{Re}(b_0)y + \mathrm{Im}(b_0)x\} \;. \end{aligned} \tag{37}$$

As consequence, we deduce the following parametrization of two arcs in A_l

$$\begin{aligned} \Gamma_l^1(x) &= x + \mathrm{i}y_l^1(x) \;, \quad x \in]-\infty, a_l[\\ \Gamma_l^2(x) &= x + \mathrm{i}y_l^2(x) \;, \quad x \in]-\infty, a_l[\end{aligned}$$

where y_l^j are defined by equation (37) and $a_l < 0$. Moreover, deriving in (37) we obtain $y_l^j(x) = a_l^j + O(x^{-1})$ for $j = 1,2$ where by $O(x^m)$ we denote a function such that $x^{-m}O(x^m)$ is bounded as $x \to -\infty$ (see Fig. 3).

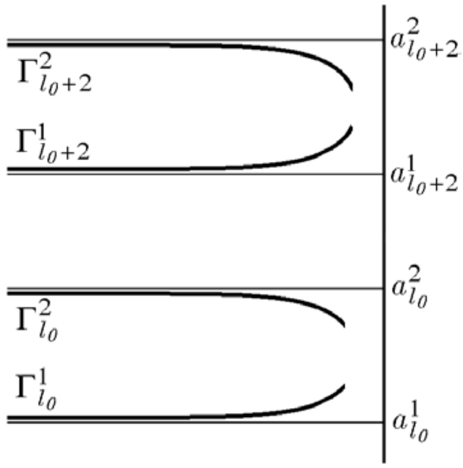

Figure 3:

Hence, taking into account (32), we deduce

$$\begin{aligned}\lim_{x\to-\infty} \boldsymbol{\chi}_1(\Gamma^1_l(x)) &= -\infty\ ,\\ \lim_{x\to-\infty} \boldsymbol{\chi}_1(\Gamma^2_l(x)) &= +\infty\ ,\\ \lim_{x\to-\infty} \boldsymbol{\chi}_1(\Gamma^1_{l_0+2kn}(x)) - \boldsymbol{\chi}_1(\Gamma^1_{l_0}(x)) &= n\beta_1\ ,\end{aligned}$$

for $n\in\mathbb{N}$ and l an odd integer satisfying $l\geq l_0$. According to (35), we also have

$$\lim_{x\to-\infty} \boldsymbol{\chi}_3(\Gamma^j_l(x)) = +\infty\ .$$

Moreover, since $\beta_3=0$ we have $\boldsymbol{\chi}_3(x+\mathrm{i}(y+2\pi n)) = \boldsymbol{\chi}_3(x+\mathrm{i}y)$, for all $n\in\mathbb{N}$ and then

$$\boldsymbol{\chi}_3(\Gamma^1_{l_0+2kn}(x)) = \boldsymbol{\chi}_3(\Gamma^1_{l_0}(x)+\mathrm{i}(2\pi n+O(x^{-1}))) \sim \boldsymbol{\chi}_3(\Gamma^1_{l_0}(x)+2\pi n\mathrm{i})) = \boldsymbol{\chi}_3(\Gamma^1_{l_0}(x))\ .$$

Then, the curve $\boldsymbol{\chi}(\Gamma^1_{l_0+2kn}(x))$ is asymptotic to the curve $\boldsymbol{\chi}(\Gamma^1_{l_0}(x))+(n\beta_1,0,0)$ and so there exists n sufficiently large so that $\boldsymbol{\chi}(\Gamma^1_{l_0+2kn})$ cuts the curve $\boldsymbol{\chi}(\Gamma^2_{l_0})$ (see Fig. 4) and consequently the end is not embedded.

If $\beta_1<0$ a similar argument with slight changes allows us to finish as before.

CASE $\beta_1=0$.

We observe from the expression of the equiaffine transformation g, that if $\beta_1=0$ then $\alpha_2=0$ in order that g has no fixed points. Thus, $\mathrm{Res}_0(\widetilde{\eta})=0$ and then, after a conformal reparametrization of a subend, we can assume the following expressions for η and ω

$$\eta = -k\mathrm{e}^{-kw}dw\ ,\quad \omega = \left(\frac{t}{2\pi}+\sum_{\lambda=1}^{\infty} b_\lambda \mathrm{e}^{\lambda w}\right)dw\,, \tag{38}$$

where $b_\lambda\in\mathbb{C}$ and $w\in\mathbb{C}^-_R=\{z\in\mathbb{C}\mid \mathrm{Re}(z)\leq R\}$ for some $R\leq 0$. According to (24) and (38) the expression of $\boldsymbol{\chi}_1$ is

$$\boldsymbol{\chi}_1(w) = \frac{1}{2}\mathrm{Re}(\mathrm{e}^{-kw}) = \frac{1}{2}\mathrm{e}^{-ku}\cos(kv)\ .$$

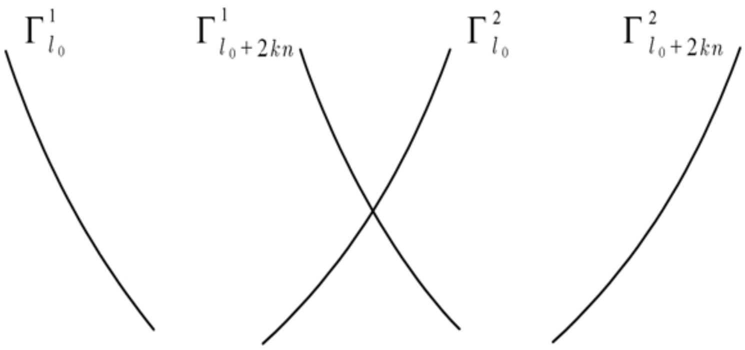

Figure 4:

where $w = u + \mathrm{i}v$. Thus, the intersection of the end with the plane $x_1 = 0$ can be parametrized by the rays, $\gamma_l(u) = u + \mathrm{i}\Theta_l$, where $u \leq R$ and $\Theta_l = l\frac{\pi}{k} - \frac{\pi}{2k}$, $l \in \mathbb{Z}$. We denote by $\Omega_l =]-\infty, R[\times]\Theta_{l-1}, \Theta_l[$. Given $\delta > 0$ sufficiently large, the level set $\boldsymbol{\chi}_1^{-1}(\delta) \cap \Omega_l$ for odd $|l|$ consists of a connected arc $\gamma_{\delta,l}$ that can be parametrized as

$$\gamma_{\delta,l}(v) = u(v) + \mathrm{i}v = -\frac{1}{k}\log\left(\frac{2\delta}{\cos(kv)}\right) + \mathrm{i}v\ , \quad v \in]\Theta_{l-1}, \Theta_l[\ .$$

Notice that the curve $\gamma_{\delta,l}$ is asymptotic to the rays γ_{l-1} and γ_l (see Fig. 5). Taking into account

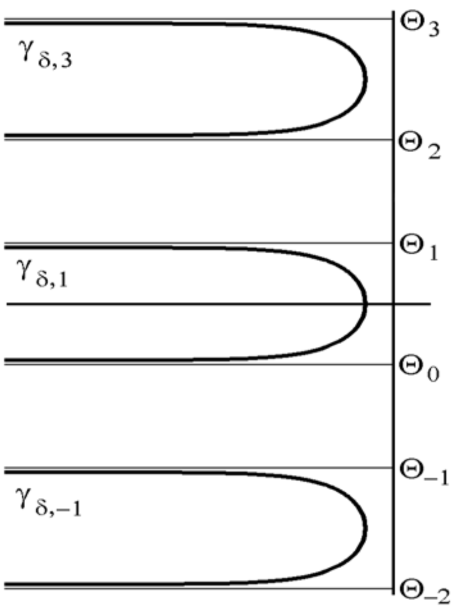

Figure 5:

(24) and (38), it is not hard to see that the asymptotic behaviour of $\boldsymbol{\chi}_2$ is as follows

$$\boldsymbol{\chi}_2(\gamma_{\delta,l}(v)) = \frac{t}{4\pi}u(v)\mathrm{e}^{-ku(v)}(1 + h_1(v))\ ,$$

when $v \to \Theta_{l-1}$ and h_1 is a function such that $\lim_{v\to\Theta_{l-1}} h_1(v) = 0$. Furthermore, if $v \to \Theta_l$ we get

$$\boldsymbol{\chi}_2(\gamma_{\delta,l}(v)) = -\frac{t}{4\pi}u(v)\mathrm{e}^{-ku(v)}(1 + h_2(v)) \,,$$

where h_2 is a function such that $\lim_{v\to\Theta_l} h_2(v) = 0$. Similarly, from (35) and (38) we have

$$\boldsymbol{\chi}_3(\gamma_{\delta,l}(v)) = \frac{t}{16\pi}u(v)\mathrm{e}^{-2ku(v)}(1 + h_3(v)) \,,$$

when v tends to the extremes of the interval $]\Theta_{l-1}, \Theta_l[$ and h_3 is a function such that $\lim_{v\to\Theta_{l-1}} h_3(v) = \lim_{v\to\Theta_l} h_3(v) = 0$.

Moreover we have

$$g(\boldsymbol{\chi}(\gamma_{\delta,l})) = g(\delta, \boldsymbol{\chi}_2(\gamma_{\delta,l}), \boldsymbol{\chi}_3(\gamma_{\delta,l})) = (\delta, \boldsymbol{\chi}_2(\gamma_{\delta,l}) + t\delta, \boldsymbol{\chi}_3(\gamma_{\delta,l}) + \beta_3) \,.$$

Thus, g is a translation of vector $(0, t\delta, \beta_3)$ on the curve $\boldsymbol{\chi}(\gamma_{\delta,l})$. From the properties of the curves $\boldsymbol{\chi}(\gamma_{\delta,l})$ it is clear that the curve $g(\boldsymbol{\chi}(\gamma_{\delta,l}))$ cuts the curve $\boldsymbol{\chi}(\gamma_{\delta,l})$ and then the end cannot be embedded (see Fig. 6).

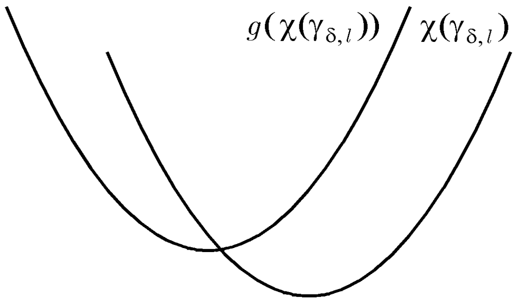

Figure 6:

Consequently, $k = 0$ and therefore $\widetilde{\eta}$ has a pole at zero of order one. Moreover, $\mathrm{Im}(\mathrm{Res}_0(\widetilde{\eta})) = -\frac{\beta_1}{\pi} \neq 0$. If not, since g has not fixed points $\alpha_2 = 0$ and $\mathrm{Res}_0(\widetilde{\eta}) = 0$.

Step 3: *We claim that if $\widetilde{\eta}$ and $\widetilde{\omega}$ are as in Theorem 2, then the multi-valued parametrization $\widetilde{X}$ given by (25) contains a complete embedded subend invariant under an equiaffine transformation g of type* **I.1**.

Assume that $\widetilde{\eta}$ and $\widetilde{\omega}$ are as in Theorem 2. Since $\widetilde{\eta}$ has a pole at 0 of order one and $\mathrm{Res}_0(\widetilde{\eta}) = \frac{1}{\pi}(\alpha_2 - \mathrm{i}\beta_1)$ we can assume, after a conformal reparametrization of a subend, that the expression of $\widetilde{\eta}$ is as follows

$$\widetilde{\eta} = \frac{a_0}{\zeta}d\zeta \,, \tag{39}$$

where $a_0 = \frac{1}{\pi}(\alpha_2 - \mathrm{i}\beta_1) \neq 0$ and $\zeta \in D^*_\varepsilon = \{z \in \mathbb{C} \mid 0 < |z| < \varepsilon\}$ for some $0 < \varepsilon \leq 1$. We recall that $\beta_1 \neq 0$. From (39) we obtain

$$\int \widetilde{\eta} = a_0 \log(\zeta) \,. \tag{40}$$

Furthermore, since $\widetilde{\omega}$ has a pole at 0 of order one and $\mathrm{Res}_0(\widetilde{\omega}) = \frac{t}{2\pi}$ we can write

$$\widetilde{\omega} = \left(\frac{t}{2\pi\zeta} + \sum_{\lambda=0}^{\infty} b_{\lambda+1}\zeta^\lambda\right) d\zeta \,, \tag{41}$$

with $b_\lambda \in \mathbb{C}$. Hence we obtain

$$\widetilde{f}\,\widetilde{\eta} = \frac{ta_0}{2\pi}\frac{\log(\zeta)}{\zeta} + \frac{b_0}{\zeta} + \sum_{\lambda=0}^{\infty} b'_\lambda \zeta^\lambda \ , \tag{42}$$

with $b'_\lambda = \frac{a_0 b_{\lambda+1}}{\lambda+1}$ for $\lambda > 0$ and $b_0 \in \mathbb{C}$. From (19), (23), (40) and (42) it is easy to see that $b_0 = \frac{it}{2}\overline{a_0}$. Clearly, from (39) and (41), the affine metric given in (27) is a complete Riemannian metric on $D^*_{\varepsilon_0}$, for some $0 < \varepsilon_0 \leq \varepsilon$.

Now, we shall show that for ε_0 sufficiently small the subend is also embedded. Indeed, we prove that this subend is a graph over the plane $x_3 = 0$.

From (25), (40) and (42) we get

$$\begin{aligned}
\widetilde{\boldsymbol{\chi}}_1(\zeta) &= \frac{1}{2}\mathrm{Re}(a_0\log(\zeta)) = \frac{1}{2\pi}\left(\alpha_2 \log r + \beta_1\theta\right) \ , && (43)\\
\widetilde{\boldsymbol{\chi}}_2(\zeta) &= \frac{1}{2}\mathrm{Im}\left(\frac{t}{4\pi}a_0\log(\zeta)^2 + b_0\log(\zeta) + \sum_{\lambda=0}^{\infty} b''_\lambda\zeta^\lambda\right) = \frac{1}{2}\left\{\mathrm{Re}(b_0)\theta + \mathrm{Im}(b_0)\log r\right. && (44)\\
&+ \left.\frac{t}{4\pi^2}\left(\beta_1(\theta^2 - (\log r)^2) + 2\alpha_2\theta\log r\right) + \sum_{\lambda=0}^{\infty}|b''_\lambda|\sin(\theta_\lambda + \lambda\theta)r^\lambda\right\} \ ,
\end{aligned}$$

where $b''_\lambda = \frac{b'_\lambda}{\lambda}$ for $\lambda > 0$, $b''_0 \in \mathbb{C}$ and $\theta_\lambda = \arg(b''_\lambda)$. Then the intersection of the end with the plane $x_1 = \delta$ for δ a real constant can be parametrized on $D^*_{\varepsilon_0}$ by the curve

$$\gamma_\delta(r) = r\exp\left(-\frac{\alpha_2}{\beta_1}\log r + \frac{2\pi\delta}{\beta_1}\right) \ ,$$

with $r \leq \varepsilon_0$. Substituting in (44) we obtain

$$\begin{aligned}
\widetilde{\boldsymbol{\chi}}_2(\gamma_\delta(r)) &= \frac{1}{2}\left\{-\frac{|a_0|^2 t}{4\beta_1}(\log r)^2 + \left(\mathrm{Im}(b_0) - \frac{\alpha_2}{\beta_1}\mathrm{Re}(b_0)\right)\log r + \frac{\delta^2 t}{\beta_1} + \frac{2\pi\delta}{\beta_1}\mathrm{Re}(b_0)\right.\\
&+ \left.\sum_{\lambda=0}^{\infty}|b''_\lambda| \ \sin\left(\theta_\lambda + \lambda\left(-\frac{\alpha_2}{\beta_1}\log r + \frac{2\pi\delta}{\beta_1}\right)\right)r^\lambda\right\} \ ,
\end{aligned}$$

Hence we deduce

$$\frac{d\,\widetilde{\boldsymbol{\chi}}_2}{dr}(\gamma_\delta(r)) = -\frac{|a_0|^2 t}{4\beta_1}r^{-1}\log r + O(r^{-1}) \ ,$$

where $O(r^m)$ denotes a function such that $r^{-m}O(r^m)$ is bounded (independently of δ) as $r \to 0$. Therefore there exists $0 < \varepsilon_1 \leq \varepsilon_0$ such that $\widetilde{\boldsymbol{\chi}}_2 \circ \gamma_\delta :]0, \varepsilon_1] \longrightarrow \mathbb{R}$ is a one-to-one function for all $\delta \in \mathbb{R}$. Consequently, $\widetilde{\boldsymbol{\chi}}(\gamma_\delta(]0, \varepsilon_1]))$ is a graph over the line $\{x_1 = \delta, x_3 = 0\}$ and so the subend $\widetilde{\boldsymbol{\chi}} : D^*_{\varepsilon_1} \longrightarrow \mathbb{R}^3$ is also a graph over the plane $x_3 = 0$.

Step 4: We claim that there exist no complete embedded ends invariant under an equiaffine transformation g as in **I.2**.

We shall deduce this fact using the former steps. Observe that if $\boldsymbol{\chi} : \mathbb{C}^- \longrightarrow \mathbb{R}^3$ is an end invariant under g, then the end is also invariant under the equiaffine transformation g^2 given by

$$g^2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ t' & 1 & 0 \\ 0 & \alpha'_2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \beta'_1 \\ 0 \\ \beta'_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2\beta_3 \end{pmatrix} .$$

Thus, g^2 is an equiaffine transformation of type **I.1** with $\alpha'_2 = \beta'_1 = 0$. But, according to the above steps there exist no complete embedded ends invariant under such a transformation and therefore neither exist complete embedded ends invariant under g.

Remark 1 *Obviously, when we use the parametrization* (25) *to give an immersion* $\widetilde{\mathbf{x}}$ *invariant under an equiaffine transformation* g *of type* **I.1** *the integration constants must be chosen according to* g.

Remark 2 *We denote by* $M_{a_0,t}$ *the complete embedded end given by Theorem 2 and the meromorphic one-forms* $\widetilde{\eta} = \frac{a_0}{\zeta}$, $\widetilde{\omega} = \frac{t}{2\pi\zeta}$ *with* $a_0 \in \mathbb{C}$ *such that* $\mathrm{Im}(a_0) \neq 0$, $t < 0$ *and* $\zeta \in D^*$ *(see Fig. 7). Notice that a generic end as in Theorem 2 is asymptotic to an end* $M_{a_0,t}$.

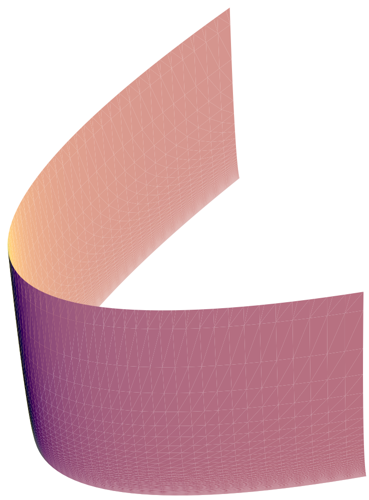

Figure 7:

5 Complete embedded ends of type II

Let $\mathbf{x} : \mathbb{C}^- \longrightarrow \mathbb{R}^3$ be a complete end invariant under an equiaffine transformation $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ as in **II**. Then using (16) and (17) we have

$$G(\alpha(z)) = G(z) + \alpha_1 + \mathrm{i}(\beta_2 + \alpha_2)\ , \quad F(\alpha(z)) = F(z) - \alpha_1 - \mathrm{i}(\beta_2 - \alpha_2)\ . \tag{45}$$

Thus, we have that dG and dF pass to the quotient D^*, namely, there exist $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ holomorphic one-forms on D^* verifying $\exp^* \widetilde{\omega}_1 = dG$ and $\exp^* \widetilde{\omega}_2 = dF$. Taking into account (10) we have the following expression for the projected metric on D^*

$$ds^2 = \frac{1}{4}\left(|\widetilde{\omega}_1(\tau)|^2 - |\widetilde{\omega}_2(\tau)|^2\right) \leq \frac{1}{4}|\widetilde{\omega}_1(\tau)|^2\ , \tag{46}$$

with $\tau = \exp(z)$ and $\tau \in D^*$. Since $\mathbf{X} : \mathbb{C}^- \longrightarrow \mathbb{R}^3$ is a complete end and $\widetilde{\omega}_1$ is a holomorphic one-form without zeros in D^*, we have (see Lemma 9.6 in [11]) that $\widetilde{\omega}_1$ has a pole at 0 of order $k+1 \geq 1$. Therefore as $|\widetilde{\omega}_2| < |\widetilde{\omega}_1|$, we deduce that $\widetilde{\omega}_2$ has at most a pole at 0 of order less or equal that $k+1$.

Denote $G_0 = \int \widetilde{\omega}_1$ and $F_0 = \int \widetilde{\omega}_2$ that are not singly-valued functions in general. Notice that G_0 and F_0 are also defined by the equalities $G_0 \circ \exp = G$ and $F_0 \circ \exp = F$. From these definitions we can give a multi-valued parametrization of the end on D^* as follows

$$\widetilde{\mathbf{X}} = (\widetilde{\mathbf{X}}_1, \widetilde{\mathbf{X}}_2, \widetilde{\mathbf{X}}_3) = \left(\frac{1}{2}(G_0 + \overline{F_0}), \frac{1}{8}(|G_0|^2 - |F_0|^2) + \frac{1}{4}\mathrm{Re}(G_0 F_0) - \frac{1}{2}\mathrm{Re} \int F_0 \, \widetilde{\omega}_1 \right) . \qquad (47)$$

Definition 3 *The pair $(\widetilde{\omega}_1, \widetilde{\omega}_2)$ will be called the conformal representation of this type of ends.*

The following result is a characterization, in terms of $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$, of the complete embedded ends invariant under an equiaffine transformation of type **II**.

Theorem 3 *Let $\mathbf{X} : \mathbb{C}^- \longrightarrow \mathbb{R}^3$ be a complete embedded end of an improper affine sphere invariant under an equiaffine transformation as in* **II** *and let $(\widetilde{\omega}_1, \widetilde{\omega}_2)$ be its conformal representation and F_0 as before. Then we have one of the following situations:*

i) *If $k > 0$, then up an equiaffine transformation, the equiaffine transformation is a vertical translation of vector $(0, 0, \beta_3)$, $\widetilde{\omega}_1$ has a pole at zero of order $k+1$ and $\mathrm{Res}_0(\widetilde{\omega}_1) = 0$, $\widetilde{\omega}_2$ has a zero at zero of order $k-1$ and $\mathrm{Im}\,(\mathrm{Res}_0(F_0\widetilde{\omega}_1)) = \frac{\beta_3}{\pi}$.*

Conversely, if $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ are as above the multi-valued parametrization $\widetilde{\mathbf{X}} : D^* \longrightarrow \mathbb{R}^3$ given by (47) contains a complete embedded subend invariant under a vertical translation of vector $(0, 0, \pi\,\mathrm{Im}\,(\mathrm{Res}_0(F_0\widetilde{\omega}_1))$.

ii) *If $k = 0$, then up an equiaffine transformation, the equiaffine transformation writes as*

$$\widetilde{g} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha_2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha_2 \\ 0 \end{pmatrix} , \qquad (48)$$

with $\alpha_2 \neq 0$, $\widetilde{\omega}_1$ has a pole at zero of order one and $\mathrm{Res}_0(\widetilde{\omega}_1) = \frac{\alpha_2}{\pi}$, $\widetilde{\omega}_2$ is a holomorphic one-form on D and $\mathrm{Im}\,(\mathrm{Res}(F_0\widetilde{\omega}_1)) = -\frac{\alpha_2^2}{2\pi}$.

Conversely, if $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ are as above the multi-valued parametrization $\widetilde{\mathbf{X}} : D^* \longrightarrow \mathbb{R}^3$ given by (47) contains a complete embedded subend invariant under an equiaffine transformation as in (48) with $\alpha_2 = \pi\,\mathrm{Res}_0(\widetilde{\omega}_1)$.

Remark 3 *Observe that in both cases of Theorem 3, F_0 is a well defined holomorphic function on D and then it makes sense to consider $\mathrm{Res}_0(F_0\widetilde{\omega}_1)$.*

Proof of Theorem 3: Let G_0 and F_0 be defined as before. From the conditions on $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ we have the following expressions for G_0 and F_0

$$G_0(\tau) = \sum_{\lambda=-k}^{\infty} a'_\lambda \tau^\lambda + A\log(\tau) \quad , \quad F_0(\tau) = \sum_{\lambda=p}^{\infty} b'_\lambda \tau^\lambda + B\log(\tau) \, , \qquad (49)$$

where $k \geq 0$, $A \neq 0$ if $k = 0$ and p is an integer number such that $-p \leq k$. Moreover, from (45) it is easy to see that $A = \frac{1}{2\pi}(\alpha_2 + \beta_2 - \mathrm{i}\alpha_1)$ and $B = \frac{1}{2\pi}(\alpha_2 - \beta_2 + \mathrm{i}\alpha_1)$.

We recall that if we consider an equiaffine transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ as in (12) the functions G_0 and F_0 of the new singly-periodic improper affine sphere $T \circ \widetilde{\mathbf{X}} : D^* \longrightarrow \mathbb{R}^3$ can be obtained from the equations (14) and (15). Hence taking into account (49) it is possible to find a transformation T of the form

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a' & 0 & 0 \\ c' & d' & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} ,$$

with $a'd' = 1$, so that the Gauss map satisfies $\Psi(0) = \frac{\widetilde{\omega}_2}{\widetilde{\omega}_1}(0) = 0$ for the improper affine sphere $T \circ \widetilde{\mathbf{x}} : D^* \longrightarrow \mathbb{R}^3$. It is easy to check that the condition on the Gauss map is equivalent to $-p < k$ if $k > 0$ and $B = 0$ if $k = 0$ in (49). For the sake of simplicity, we assume this when necessary. Observe that the new improper affine sphere is invariant under the equiaffine transformation $T \circ g \circ T^{-1}$ that is of the same type as g.

We distinguish two cases, the case $k > 0$ and the case $k = 0$.

CASE $k > 0$.

Now, we consider a new coordinate ζ given by

$$\zeta = \tau H_0(\tau)^{-\frac{1}{k}} = \tau \left(\sum_{\lambda=0}^{\infty} a'_{\lambda-k} \tau^{\lambda} \right)^{-\frac{1}{k}} ,$$

with $\zeta \in D^*_\varepsilon$, $0 < \varepsilon \leq 1$. Since $a'_{-k} \neq 0$, we can insure that H_0 has a k-th holomorphic root on D^*_ε for ε sufficiently small. From (49), we can write G_0 and F_0 in the new coordinate as

$$G_0(\zeta) = \frac{1}{\zeta^k} + A\log(\zeta) + AH_1(\zeta) \quad , \quad F_0(\zeta) = \sum_{\lambda=p}^{\infty} b_\lambda \zeta^\lambda + B\log(\zeta) + BH_1(\zeta) \ , \tag{50}$$

where $b_\lambda \in \mathbb{C}$ and H_1 denotes a holomorphic function on $D_\varepsilon = \{z \in \mathbb{C} \mid |z| < \varepsilon\}$.

As before, taking into account (50) we can consider an equiaffine transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ of the form

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha'_1 & \alpha'_2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \beta'_1 \\ \beta'_2 \\ 0 \end{pmatrix} ,$$

so that $b_0 = 0$ in the expression of the corresponding F_0 of the improper affine sphere $T \circ \widetilde{\mathbf{x}} : D^* \longrightarrow \mathbb{R}^3$. Observe that the new improper affine sphere is invariant under the equiaffine transformation $T \circ g \circ T^{-1}$ that is of the same type as g. Then we can assume $b_0 = 0$ in the expression of F_0 given in (50).

Our objective in this case is to prove that the end can not be embedded either $p < k$ or $(A, B) \neq (0, 0)$. Some steps in the proof of this fact are analogous to the proof by E. Toubiana for singly-periodic minimal surfaces invariant under a translation (see Lemma 2 in [13]). Thereby, we shall insist in the points in which our proof is different and refer the reader to [13] for more details.

The argument of Toubiana's proof is very geometric. We consider a cylinder and represent by (θ, h) a point on it, where θ and h are its argument and its hight, respectively. Our aim is to prove that if certain conditions on p, A and B are satisfied, then it is possible to construct a curve $\alpha(t)$ in the intersection of the end and the cylinder such that $\theta(t)$ is a monotone function. If there exist three points on the curve (θ, h_1), $(\theta + 2\epsilon\pi n, h_2)$ and $(\theta + 2\epsilon\pi m, h_3)$ with $0 < n < m$ natural numbers, $\epsilon = \pm 1$ and $h_1 < h_2$ and $h_3 < h_2$ the curve could not be embedded and neither is the end.

As the study of this case is rather long, we shall divide it in several steps.

Step 1: $p < 0$.

Denote by $\zeta = re^{\mathrm{i}\theta}$. Taking into account (50), we have the following expression for the two first components of the multi-valued parametrization given by (47):

$$\widetilde{\mathbf{X}}_1 + \mathrm{i}\widetilde{\mathbf{X}}_2(r,\theta) = \frac{1}{2}\zeta^{-k}(1 + f(r,\theta)) \ , \tag{51}$$

where $f(r,\theta)$ is a function such that $\forall\theta \in \mathbb{R}$, $\lim_{r\to 0} f(r,\theta) = 0$. Similarly, from (47) and (50) we deduce

$$\widetilde{\mathbf{X}}_3(r,\theta) = \frac{1}{8}r^{-2k} + r^{(-k+p)}\left(-\frac{k+p}{4(k-p)}|b_p|\sin(\theta_p + (k-p)\theta) + t(r,\theta)\right) \ , \tag{52}$$

where $\theta_p = \arg\left(\mathrm{i}\overline{b_p}\right)$ and $t(r,\theta)$ is a real function such that $\forall\theta \in \mathbb{R}$, $\lim_{r\to 0} t(r,\theta) = 0$.

Since $-k < p < 0$ we have $0 < -\frac{p}{k} < 1$ and then it is possible to find $\sigma_0 \in \left]\frac{-\theta_p}{k-p}, \frac{-\theta_p+\pi}{k-p}\right[$ and $n_0, n \in \mathbb{N}$ such that $\sigma_0 - \frac{2\pi p}{k(k-p)} \in \left]\frac{-\theta_p+\pi}{k-p}, \frac{-\theta_p+2\pi}{k-p}\right[$ and $\sigma_0 - \frac{2\pi p n_0}{k(k-p)} \in \left]\frac{-\theta_p+2\pi n}{k-p}, \frac{-\theta_p+\pi(2n+1)}{k-p}\right[$. Thus, using (52) we have for $\zeta \in D_\varepsilon^*$, $0 < \varepsilon < 1$, that

$$\begin{aligned} \widetilde{\mathbf{X}}_3\left(r, \sigma_0 + \frac{2\pi}{k}\right) &- \widetilde{\mathbf{X}}_3\left(r, \sigma_0\right) > 0 \ , \\ \widetilde{\mathbf{X}}_3\left(r, \sigma_0 + \frac{2\pi}{k}\right) &- \widetilde{\mathbf{X}}_3\left(r, \sigma_0 + \frac{2\pi n_0}{k}\right) > 0 \ . \end{aligned}$$

In order to conclude that the end is not embedded it is sufficient to prove that there exist three points $\zeta_1, \zeta_2, \zeta_3 \in D_\varepsilon^*$ such that $(\widetilde{\mathbf{X}}_1 + \mathrm{i}\widetilde{\mathbf{X}}_2)(\zeta_1) = (\widetilde{\mathbf{X}}_1 + \mathrm{i}\widetilde{\mathbf{X}}_2)(\zeta_2) = (\widetilde{\mathbf{X}}_1 + \mathrm{i}\widetilde{\mathbf{X}}_2)(\zeta_3)$ and whose third coordinate fulfill the inequalities $\widetilde{\mathbf{X}}_3(\zeta_2) - \widetilde{\mathbf{X}}_3(\zeta_1) > 0$ and $\widetilde{\mathbf{X}}_3(\zeta_2) - \widetilde{\mathbf{X}}_3(\zeta_3) > 0$. Taking into account (51), this part of the argument follows from Toubiana's paper.

Step 2: $0 < p < k$, $|A| \neq |B|$.

Suppose $|A| < |B|$. As before, the first components of the immersion are given by (51). However, from (47) and (50) the third coordinate of the immersion (47) is given by

$$\begin{aligned} \widetilde{\mathbf{X}}_3(r,\theta) &= \frac{r^{-2k}}{8} + \frac{|\overline{A} - B|}{4}\cos(\gamma_1 - k\theta)r^{-k}\log r + \frac{|A|^2 - |B|^2}{8}\theta^2 + \theta(C + \widetilde{t}_1(r,\theta)) \\ &+ r^{-k}\left(\frac{|\overline{A} + B|}{4}\theta\sin(\gamma_2 - k\theta) + C' + \widetilde{t}_2(r,\theta)\right) \ , \end{aligned} \tag{53}$$

where $\gamma_1 = \arg\left(\overline{A} - B\right)$, $\gamma_2 = \arg\left(\overline{A} + B\right)$, $C, C' \in \mathbb{R}$, $\widetilde{t}_i(r,\theta)$ are real functions such that $\widetilde{t}_i(r,\theta) = \widetilde{t}_i(r,\theta + 2\pi)$ and $\forall\theta \in \mathbb{R}$, $\lim_{r\to 0}\widetilde{t}_i(r,\theta) = 0$ for $i = 1, 2$. Consider $\sigma_1 \in \mathbb{R}$ such that $\sin(\gamma_2 - k\sigma_1) = 0$. From (53), we have for $m \in \mathbb{N}$ sufficiently large and r sufficiently small

$$\widetilde{\mathbf{X}}_3\left(r, \sigma_1\right) - \widetilde{\mathbf{X}}_3\left(r, \sigma_1 - 2\pi m\right) = \pi m\left(\frac{|A|^2 - |B|^2}{2}(-\pi m + \sigma_1) + 2C + 2\widetilde{t}_1(r,\sigma_1)\right) > 0 \ ,$$

$$\widetilde{\mathbf{X}}_3\left(r, \sigma_1\right) - \widetilde{\mathbf{X}}_3\left(r, \sigma_1 + 2\pi m\right) = \pi m\left(\frac{|A|^2 - |B|^2}{2}(-\pi m - \sigma_1) - 2C - 2\widetilde{t}_1(r,\sigma_1)\right) > 0 \ ,$$

Since we have $\sigma_1 - 2\pi m < \sigma_1 < \sigma_1 + 2\pi m$, we can use again Toubiana's argument to conclude that in our assumptions, $0 < p < k$ and $|A| < |B|$, the improper affine sphere is not embedded. Reasoning in a similar way we obtain the same conclusion when $|A| > |B|$.

Step 3: $0 < p < k$, $|A| = |B| \neq 0$.

Since $|A| = |B|$, from the expressions of A and B we have that either $\alpha_2 = 0$ or $\beta_2 = 0$. If $\beta_2 = 0$, taking into account that the equiaffine transformation g has no fixed points, we deduce

$\alpha_1 = \alpha_2 = 0$. But, then $A = B = 0$, contrary to our assumption. Therefore $\alpha_2 = 0$, $\beta_2 \neq 0$ and $A = \frac{1}{2\pi}(\beta_2 - \mathrm{i}\alpha_1) = -B$. Furthermore, it is not hard to see that, up an equiaffine transformation, we can assume $\beta_3 = 0$. Our next purpose is to prove that if $A \neq 0$ the immersion is neither an embedding. Assume $A \neq 0$.

From (50) we observe that $G_0 + F_0$ can be written as

$$(G_0 + F_0)(\zeta) = \zeta^{-k}\left(1 + \widehat{H}(\zeta)\right) ,$$

where $\widehat{H}$ is a holomorphic function in D_ε. Thus, we can consider

$$\widetilde{\zeta} = \zeta\left(1 + \widehat{H}(\zeta)\right)^{-\frac{1}{k}} ,$$

with $\widetilde{\zeta} \in D^*_{\varepsilon_0}$, $0 < \varepsilon_0 < \varepsilon$. Observe that for ε_0 sufficiently small $\widetilde{\zeta}$ is a new holomorphic coordinate. Hence we can write G_0 and F_0 as

$$G_0(\widetilde{\zeta}) = \frac{1}{\widetilde{\zeta}^k} + A\log(\widetilde{\zeta}) + \widehat{H}_1(\widetilde{\zeta}) \quad , \quad F_0(\widetilde{\zeta}) = -A\log(\widetilde{\zeta}) - \widehat{H}_1(\widetilde{\zeta}) , \tag{54}$$

where $\widehat{H}_1$ is a holomorphic function on D_{ε_0}. Therefore if we denote $\widetilde{\zeta} = s\mathrm{e}^{\mathrm{i}\Theta}$, from (54) we have the following expressions for the components of the immersion (47)

$$\begin{aligned}
\widetilde{\mathbf{x}}_1(s,\Theta) &= \frac{1}{2}s^{-k}\cos(k\Theta) , &(55)\\
\widetilde{\mathbf{x}}_2(s,\Theta) &= \frac{1}{2}s^{-k}\left(-\sin(k\Theta) + t_1(s,\Theta)\right) , &(56)\\
\widetilde{\mathbf{x}}_3(s,\Theta) &= \frac{1}{8}s^{-2k}\left(1 + t_2(s,\Theta)\right) , &(57)
\end{aligned}$$

where by $t_i(s,\Theta)$ we denotes a function such that $\forall\Theta \in \mathbb{R}$, $\lim\limits_{s\to 0} t_i(s,\Theta) = 0$ for $i = 1,2$. From (55) we have that the intersection curves of the end with the plane $x_1 = 0$ are a family of curves parametrized on $D^*_{\varepsilon_0}$ by the $2k$ rays $\{\widetilde{\zeta} = s\mathrm{e}^{\mathrm{i}\Theta_l} \mid 0 < s < \varepsilon_0\}$ where $\Theta_l = l\frac{\pi}{k} - \frac{\pi}{2k}$, $l = 1, ..., 2k$. Denote by

$$\Gamma_i(s) = \widetilde{\mathbf{x}}\left(s\mathrm{e}^{\mathrm{i}\Theta_i}\right) ,$$

with $0 < s < \varepsilon_0$ and $i = 1,2$. From (56) and (57) these two planar curves have the following expansions

$$\begin{aligned}
\Gamma_1(s) &= \left(0, \frac{1}{2}s^{-k}\left(-1 + O(s)\right), \frac{1}{8}s^{-2k}\left(1 + O(s)\right)\right) , &(58)\\
\Gamma_2(s) &= \left(0, \frac{1}{2}s^{-k}\left(1 + O(s)\right), \frac{1}{8}s^{-2k}\left(1 + O(s)\right)\right) , &(59)
\end{aligned}$$

where the expression $O(s^n)$ will be used to indicate a term which is bounded in absolute value by a constant times s^n for s small. As the end is invariant under an equiaffine transformation g of the type **II** we have that the image under g^q for any $q \in \mathbb{Z}$ of the curves Γ_i is also on the end. Moreover, taking into account that $\alpha_2 = \beta_3 = 0$ in the expression of g in **II**, it is easy to see that

$$g^q\left(\Gamma_i(s)\right) = \Gamma_i(s) + (0, q\beta_2, 0) ,$$

it is to say, ${g^q}_{|\Gamma_i}$ is a translation in the direction of $(0,1,0)$. From (58) and (59) it is clear that the curves Γ_i are asymptotic to a parabola. Thus, we can find a suitable q such that $g^q\left(\Gamma_1(s)\right)$

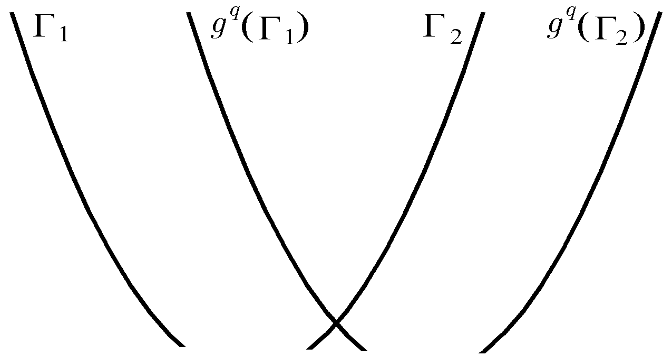

Figure 8:

intersects the curve Γ_2 and then the end is not embedded (see Fig. 8).

Step 4: $0 < p < k$**,** $A = B = 0$**.**

From (50) the expressions of G_0 and F_0 become

$$G_0(\zeta) = \frac{1}{\zeta^k} \quad , \quad F_0(\zeta) = \sum_{\lambda=p}^{\infty} b_\lambda \zeta^\lambda \ .$$

As in the Step 1 we have the immersion given by (51) and (52). Therefore the same argument presented there, with slight changes, proves that the end is not embedded.

Consequently, if the end is embedded is necessary that $p \geq k$ and $A = B = 0$. Then, g must be a vertical translation of vector $(0, 0, \beta_3)$ and from (50) the functions G_0 and F_0 write as follows

$$G_0(\zeta) = \frac{1}{\zeta^k} \ , \quad F_0(\zeta) = \sum_{\lambda=k}^{\infty} b_\lambda \zeta^\lambda \ . \tag{60}$$

Hence $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ are as in *i)*. Finally, the expression for the third component of (47) is given by

$$\widetilde{\mathbf{x}}_3(r, \theta) = \frac{r^{-2k}}{8} + \frac{k}{2} \left(\mathrm{Re}(b_k) \log r - \mathrm{Im}(b_k)\theta\right) + C + \widetilde{t}(r, \theta) \ ,$$

where $C \in \mathbb{R}$ and $\widetilde{t}(r, \theta)$ is as in Step 2. Hence $\mathrm{Im}\left(\mathrm{Res}_0(F_0 \widetilde{\omega}_1)\right) = -k \, \mathrm{Im}(b_k) = \frac{\beta_3}{\pi}$.

Step 5: The purpose of this step is to prove that if $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ are as in *i)*, the end given by the multi-valued parametrization (47) has a complete embedded subend invariant under the vertical translation of vector $(0, 0, \pi \, \mathrm{Im}\left(\mathrm{Res}_0(F_0 \widetilde{\omega}_1)\right)$. Suppose that $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ are as in *i)*. Then up an equiaffine transformation and a conformal reparametrization of a subend, we can assume that G_0 and F_0 are as in (60). Clearly, from (46) and (60) the metric given in (3) is a complete Riemannian metric on D^*_ε.

Now, taking into account (60), the multi-valued immersion (47) is given by

$$\begin{aligned}
\widetilde{\mathbf{x}}_1(r, \theta) &= \frac{1}{2} r^{-k} \cos(k\theta)(1 + \widetilde{t}_1(r, \theta)) \ , \\
\widetilde{\mathbf{x}}_2(r, \theta) &= \frac{1}{2} r^{-k} \sin(k\theta)(-1 + \widetilde{t}_2(r, \theta)) \ , \\
\widetilde{\mathbf{x}}_3(r, \theta) &= \frac{1}{8} r^{-2k} + \frac{k}{2} \left(\mathrm{Re}(b_k) \log r - \mathrm{Im}(b_k)\theta\right) + C + \widetilde{t}_3(r, \theta) \ ,
\end{aligned} \tag{61}$$

where $C \in \mathbb{R}$ and $\widetilde{t}_i(r,\theta)$ are as in Step 2. Clearly, $\widetilde{\mathbf{X}}(D^*)$ is invariant under the vertical translation of vector $(0,0,\beta_3) = (0,0,-k\pi \mathrm{Im}(b_k))$. Thus, we have

$$\begin{aligned}
\tfrac{\partial \widetilde{\mathbf{X}}_1}{\partial r}(r,\theta) &= -\tfrac{k}{2} r^{-k-1}\cos(k\theta)(1+\widetilde{t}_4(r,\theta)) \ , \\
\tfrac{\partial \widetilde{\mathbf{X}}_2}{\partial r}(r,\theta) &= \tfrac{k}{2} r^{-k-1}\sin(k\theta)(1+\widetilde{t}_5(r,\theta)) \ , \\
\tfrac{\partial \widetilde{\mathbf{X}}_1}{\partial \theta}(r,\theta) &= -\tfrac{k}{2} r^{-k}\sin(k\theta)(1+\widetilde{t}_6(r,\theta)) \ , \\
\tfrac{\partial \widetilde{\mathbf{X}}_2}{\partial \theta}(r,\theta) &= -\tfrac{k}{2} r^{-k}\cos(k\theta)(1+\widetilde{t}_7(r,\theta)) \ ,
\end{aligned}$$

where the functions $\widetilde{t}_i(r,\theta)$ are as before. Hence it is easy to see that $(\widetilde{\mathbf{X}}_1, \widetilde{\mathbf{X}}_2)$ is a one-to-one function on $\Omega_{l,\varepsilon_l} =]0,\varepsilon_l[\times]\Theta_{l-1},\Theta_l[$, where $\Theta_l = l\frac{2\pi}{k}$ and $l \in \mathbb{Z}$ and so we obtain that on Ω_{l,ε_l} the immersion is a graph over the plane $\{x_3 = 0\}$. Furthermore, if $l = 1,...,k+1$ from (61) we deduce that $\widetilde{\mathbf{X}}(\Omega_{l,\varepsilon_l})$ is asymptotic to $t_{v_l}(\widetilde{\mathbf{X}}(\Omega_{1,\varepsilon_1}))$, where t_{v_l} is the translation of vector $v_l = (0,0,-\pi(l-1)\mathrm{Im}(b_k))$. Then, there is ε' such that $\widetilde{\mathbf{X}}(\Omega_{l_1,\varepsilon'}) \cap \widetilde{\mathbf{X}}(\Omega_{l_2,\varepsilon'}) = \emptyset$ for all $l_1, l_2 \in \{1,...,k+1\}$. Therefore $\widetilde{\mathbf{X}}$ is an embedding on $]0,\varepsilon'[\times[0,\frac{2\pi}{k}+2\pi[$. Since $\widetilde{\mathbf{X}}(r,\theta+2\pi m) = \widetilde{\mathbf{X}}(r,\theta) + (0,0,m\beta_3)$, we conclude that $\widetilde{\mathbf{X}}$ is an embedding on $]0,\varepsilon'[\times\mathbb{R}$.

CASE $k = 0$.

From the reasoning at the beginning of the proof we can assume $B = 0$ in (49). Hence, taking into account that $A \neq 0$ we have $\alpha_2 = \beta_2 \neq 0$ and $\alpha_1 = 0$. Furthermore, up an equiaffine transformation, we can also assume $\beta_3 = 0$. Therefore, up an equiaffine transformation, g writes as in (48) and the expressions of G_0 and F_0 given in (49) are as follows

$$G_0(\tau) = A\log(\tau) + H_2(\tau) \quad , \quad F_0(\tau) = H_3(\tau) \ ,$$

where $A = \frac{\alpha_2}{\pi}$ and H_i are holomorphic functions in D_ε for $i = 2,3$. Thus, we deduce that $\widetilde{\omega}_1$ has a pole at zero of order one and $\mathrm{Res}_0(\widetilde{\omega}_1) = \frac{\alpha_2}{\pi}$ and $\widetilde{\omega}_2$ is a holomorphic one-form on D_ε.

We consider now a new coordinate given by

$$\zeta = \tau \exp\left(\frac{1}{A}(H_2 - H_3)\right) \ .$$

Clearly, ζ is a holomorphic coordinate defined in $D^*_{\varepsilon_0}$ for some $\varepsilon_0 > 0$. In the new coordinate we have

$$G_0(\zeta) = A\log(\zeta) + \sum_{\lambda=0}^{\infty} b_\lambda \zeta^\lambda \ , \quad F_0(\zeta) = \sum_{\lambda=0}^{\infty} b_\lambda \zeta^\lambda \ , \tag{62}$$

with $b_\lambda \in \mathbb{C}$. Thus, we have the following expression for the multi-valued parametrization (47) in the new coordinate

$$\widetilde{\mathbf{X}}_1(\zeta) = \frac{A}{2}\log r + \sum_{\lambda=0}^{\infty} |b_\lambda|\cos(\theta_\lambda + \lambda\theta) r^\lambda \ , \tag{63}$$

$$\widetilde{\mathbf{X}}_2(\zeta) = \frac{A}{2}\theta \ , \tag{64}$$

$$\widetilde{\mathbf{X}}_3(\zeta) = \frac{A^2}{8}((\log r)^2 + \theta^2) + \frac{A}{2}\mathrm{Im}(b_0)\theta + C + \widetilde{t}(r,\theta) \ , \tag{65}$$

where $\zeta = re^{i\theta}$, $\theta_\lambda = \arg(b_\lambda)$, $C \in \mathbb{R}$ and $\widetilde{t}(r,\theta)$ is a real function such that $\widetilde{t}(r,\theta) = \widetilde{t}(r,\theta+2\pi)$ and $\forall\theta \in \mathbb{R}$, $\lim_{r\to 0} \widetilde{t}(r,\theta) = 0$. Hence, using (48), (64) and (65) it is easy to see that $\mathrm{Im}\left(\mathrm{Res}_0(F_0\widetilde{\omega}_1)\right) = \frac{\alpha_2}{\pi}\mathrm{Im}(b_0) = -\frac{\alpha_2^2}{2\pi}$.

Next, we shall prove that if $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ are as in *ii)* then the end contains a complete embedded subend. Suppose that $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ are as in *ii)*. Then up a conformal reparametrization of a subend, we can assume that G_0 and F_0 are as in (62) and hence there exists $\varepsilon_1 > 0$ such that the metric given in (10) is a complete Riemannian metric on $D^*_{\varepsilon_1}$. Moreover, the immersion is given by (63), (64) and (65) and then the intersection of the end with the plane $x_2 = \delta$ for δ a real constant can be parametrized on $D^*_{\varepsilon_1}$ by the curve

$$\gamma_\delta(r) = r e^{\mathrm{i}\frac{2\delta}{A}} ,$$

with $r < \varepsilon_1$. Substituting in the expression of $\widetilde{\boldsymbol{\chi}}_1$ we obtain

$$\widetilde{\boldsymbol{\chi}}_1(r) = \frac{A}{2}\log r + \sum_{\lambda=0}^{\infty} |b_\lambda| \cos\left(\theta_\lambda + \lambda\frac{2\delta}{A}\right) r^\lambda .$$

Hence we deduce

$$\frac{d\widetilde{\boldsymbol{\chi}}_1}{dr}(r) = \frac{A}{2r}(1 + O(r)) ,$$

where $O(r^m)$ denotes a function such that $r^{-m}O(r^m)$ is bounded (independently of δ) as $r \to 0$. Therefore there exists $0 < \varepsilon_2 < \varepsilon_1$ such that $\widetilde{\boldsymbol{\chi}}_{1|\gamma_\delta} :]0, \varepsilon_2] \longrightarrow \mathbb{R}$ is a one-to-one function for all $\delta \in \mathbb{R}$. Consequently, $\widetilde{\boldsymbol{\chi}}(\gamma_\delta(]0, \varepsilon_2]))$ is a graph over the line $\{x_2 = \delta, x_3 = 0\}$ and so the end is also a graph over the plane $x_3 = 0$ (see Fig. 9).

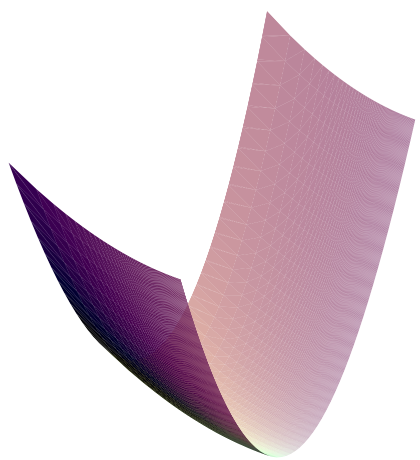

Figure 9:

Remark 4 *Notice that the complete embedded end given by* (61) *is asymptotic to the end* $E_{\mathfrak{a}}$ *with* $\mathfrak{a} = b_k = -\frac{1}{k}\mathrm{Res}_0(F_0\widetilde{\omega}_1)$. *As before the real number* $\mathfrak{a}_1 = \mathrm{Re}(b_k)$ *will be called the logarithmic growth rate of the end.*

On the other hand, it is easy to see that the complete embedded end given by (63), (64) *and* (65) *is asymptotic to a half elliptic paraboloid (see Fig. 9).*

6 Complete embedded ends of type III

Let $\mathbf{X} : \mathbb{C}^- \longrightarrow \mathbb{R}^3$ be a complete end invariant under an equiaffine transformation $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ as in **III**. Then using (16) and (17) we have

$$G(\alpha(z)) = \mathrm{e}^{\mathrm{i}t} G(z) \ , \qquad F(\alpha(z)) = \mathrm{e}^{-\mathrm{i}t} F(z) \ . \tag{66}$$

Consider $a \in]0,1[$ such that $\mathrm{e}^{\mathrm{i}t} = \mathrm{e}^{2\pi \mathrm{i} a}$. Then, from (66) we have

$$G(\alpha(z)) = \mathrm{e}^{2\pi \mathrm{i} a} G(z) \ , \quad F(\alpha(z)) = \mathrm{e}^{-2\pi \mathrm{i} a} F(z) \ . \tag{67}$$

Hence it is easy to check that the holomorphic one-forms given by

$$\omega_1 = \mathrm{e}^{-az} dG \ , \quad \omega_2 = \mathrm{e}^{az} dF \ , \tag{68}$$

pass to the quotient, namely, there exist $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ holomorphic one-forms on D^* satisfying $\exp^* \widetilde{\omega}_1 = \omega_1$ and $\exp^* \widetilde{\omega}_2 = \omega_2$. Thus, from (68) we have the following inequalities for the affine metric given in (10)

$$ds^2 \leq \frac{1}{4}|dG|^2 = \frac{1}{4}|\mathrm{e}^{az}\omega_1|^2 = \frac{1}{4}|\tau|^{2a}|\widetilde{\omega}_1|^2 \leq \frac{1}{4}|\widetilde{\omega}_1|^2 \ , \tag{69}$$

where $\tau = r\mathrm{e}^{\mathrm{i}\theta} = \exp(z)$, $\tau \in D^*$. Observe that $\widetilde{\omega}_1$ is always different from zero. Therefore the same argument presented in Sect. 5 proves that $\widetilde{\omega}_1$ has a pole at 0 of order $k+1 \geq 1$. Moreover, as $|dF| < |dG|$, from (68) we obtain the following inequality

$$|\widetilde{\omega}_2| < |\tau|^{2a}|\widetilde{\omega}_1| \ .$$

Consequently, if $a \in]0, \frac{1}{2}]$ then $\widetilde{\omega}_2$ has at most a pole at 0 of order k and if $a \in]\frac{1}{2}, 1[$ then $\widetilde{\omega}_2$ has at most a pole at 0 of order $k-1$. Hence $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ write as

$$\widetilde{\omega}_1 = \left(\sum_{\lambda=-k-1}^{\infty} \widehat{a}_\lambda \tau^\lambda \right) d\tau \ , \quad \widetilde{\omega}_2 = \left(\sum_{\lambda=p-1}^{\infty} \widehat{b}_\lambda \tau^\lambda \right) d\tau \ , \tag{70}$$

with $\widehat{a}_\lambda, \widehat{b}_\lambda \in \mathbb{C}$, $\widehat{a}_{-k-1} \neq 0$ and $p \in \mathbb{Z}$ such that $-p \leq k-1$ if $a \in]0, \frac{1}{2}]$ and $-p \leq k-2$ if $a \in]\frac{1}{2}, 1[$. Now we can prove that $k+1 \geq 2$. We proceed by contradiction. Suppose $k = 0$ and consider on D^* the divergent curve $\gamma(r) = r$ with $r \in]0, r_0[$ and $r_0 \in]0,1[$. Then from (69) and (70) the length of the curve γ respect to the affine metric satisfies

$$\int_\gamma ds \leq \frac{1}{2}\int_\gamma |\tau|^a |\widetilde{\omega}_1| \leq \frac{1}{2}\int_0^{r_0} \sum_{\lambda=-1}^{\infty} |\widehat{a}_\lambda| r^{\lambda+a} dr = \frac{1}{2}\sum_{\lambda=-1}^{\infty} \frac{|\widehat{a}_\lambda|}{\lambda+1+a} r_0^{\lambda+1+a} \ ,$$

Hence we conclude that γ has finite length, which is a contradiction.

Denote by $G_0 = G \circ \exp^{-1}$ and $F_0 = F \circ \exp^{-1}$ that are not singly-valued on D^* in general. Then, taking into account (68) and (70), we have the following expressions for G_0 and F_0

$$G_0 = \sum_{\lambda=-k}^{\infty} a'_\lambda \tau^{\lambda+a} + a_0 \ , \quad F_0 = \sum_{\lambda=p}^{\infty} b'_\lambda \tau^{\lambda-a} + b_0 \ , \tag{71}$$

where $a'_\lambda = \frac{\widehat{a}_{\lambda-1}}{\lambda+a}$, $b'_\lambda = \frac{\widehat{b}_{\lambda-1}}{\lambda-a}$, $a_0, b_0 \in \mathbb{C}$ and $\tau \in D^*$. From (67) we deduce that $\tau^{-a}G_0$ and $\tau^a F_0$ are singly-valued holomorphic functions on D^* and then $a_0 = b_0 = 0$ in (71). Clearly $\widetilde{\eta}_1 = \frac{dG_0}{G_0}$ is a well defined holomorphic one-form on D^* with a pole of order one at 0 and $\widetilde{\eta}_2 = F_0 dG_0$ is a well defined holomorphic one-form on D^* with at most a pole at 0 and the functions G_0 and F_0 can be obtained from these forms by $G_0 = \exp\left(\int \widetilde{\eta}_1\right)$ and $F_0 = \exp\left(-\int \widetilde{\eta}_1\right)\frac{\widetilde{\eta}_2}{\widetilde{\eta}_1}$. Hence, taking into account (9), we can give the following multi-valued parametrization of the end

$$\widetilde{\boldsymbol{\chi}} = (\widetilde{\boldsymbol{\chi}}_1, \widetilde{\boldsymbol{\chi}}_2, \widetilde{\boldsymbol{\chi}}_3) = \left(\frac{1}{2}(G_0 + \overline{F_0}), \frac{1}{8}(|G_0|^2 - |F_0|^2) + \frac{1}{4}\mathrm{Re}\left(\frac{\widetilde{\eta}_2}{\widetilde{\eta}_1}\right) - \frac{1}{2}\mathrm{Re}\int \widetilde{\eta}_2\right) . \tag{72}$$

Definition 4 *The pair $(\widetilde{\eta}_1, \widetilde{\eta}_2)$ will be called the conformal representation of this type of ends.*

Then, we can prove the following result that is a characterization, in terms of $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$, of complete embedded ends invariant under a screw motion as in **III**.

Theorem 4 *Let $\boldsymbol{\chi} : \mathbb{C}^- \longrightarrow \mathbb{R}^3$ be a complete embedded end of an improper affine sphere invariant under an equiaffine transformation of type* **III** *and let $(\widetilde{\eta}_1, \widetilde{\eta}_2)$ be its conformal representation. Then, up an equiaffine transformation, the holomorphic one-forms $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$ have a pole at zero of order one, $\mathrm{Res}_0(\widetilde{\eta}_1) = -k+a$, where k is a positive integer number and $a = \frac{t}{2\pi}$, and $\mathrm{Im}\,(\mathrm{Res}_0(\widetilde{\eta}_2)) = \frac{\beta_3}{\pi}$.*

Conversely, if $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$ are as above the multi-valued parametrization $\widetilde{\boldsymbol{\chi}} : D^* \longrightarrow \mathbb{R}^3$ given by (72) contains a complete embedded subend invariant under a screw motion as in **III** *of angle $t = 2\pi a$ and vector $(0, 0, \pi\,\mathrm{Im}\,(\mathrm{Res}_0(\widetilde{\eta}_2)))$.*

Proof of Theorem 4: From the above reasoning we have the following expressions for G_0 and F_0

$$G_0 = \sum_{\lambda=-k}^{\infty} a'_\lambda \tau^{\lambda+a} , \quad F_0 = \sum_{\lambda=p}^{\infty} b'_\lambda \tau^{\lambda-a} . \tag{73}$$

We recall that if $a \neq \frac{1}{2}$ then $-p + a < k - a$. On the other hand, if $a = \frac{1}{2}$ and $p = -k+1$ we can consider, as in Sect. 5, an equiaffine transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $\Psi(0) = \frac{dF_0}{dG_0}(0) = 0$, or equivalently the coefficient of the term $\tau^{-k+\frac{1}{2}}$ in the expression of the function F_0 associated to the end $T \circ \boldsymbol{\chi} : \mathbb{C}^- \longrightarrow \mathbb{R}^3$ is zero. Observe that $T \circ g \circ T^{-1} = g$. Then, we can assume that G_0 and F_0 can be written as in (73) with $-p + a < k - a$.

In order to study whether the end is an embedding, we consider a new holomorphic coordinate

$$\zeta = \tau H_4(\tau)^{\frac{1}{-k+a}} = \tau \left(\sum_{\lambda=0}^{\infty} a'_{\lambda-k} \tau^\lambda\right)^{\frac{1}{-k+a}} ,$$

with $\zeta = r e^{i\theta} \in D^*_\varepsilon$, $0 < \varepsilon < 1$. We can insure that $H_4^{\frac{1}{-k+a}}$ is a holomorphic function on D^*_ε for ε sufficiently small. In the new coordinate G_0 and F_0 write

$$G_0 = \zeta^{-k+a} , \quad F_0 = \sum_{\lambda=p}^{\infty} b_\lambda \zeta^{\lambda-a} .$$

where as before $-p + a < k - a$. Then, substituting these expressions in (72) we obtain

$$(\widetilde{\boldsymbol{\chi}}_1 + i\widetilde{\boldsymbol{\chi}}_2)(r, \theta) = \frac{1}{2}\zeta^{-k+a}(1 + f(r, \theta)) ,$$

$$\widetilde{\mathbf{x}}_3(r,\theta) = \frac{r^{2(-k+a)}}{8} + r^{(-k+p)}\left(-\frac{k+p-2a}{4(k-p)}|b_p|\sin(\theta_p + (k-p)\theta) + \widetilde{t}(r,\theta)\right) \tag{74}$$

$$+ \frac{k-a}{2}\left(\mathrm{Re}(b_k)\log r - \mathrm{Im}(b_k)\theta\right) \ , \tag{75}$$

where $f(r,\theta)$, θ_p and $\widetilde{t}(r,\theta)$ are as in Sect. 5. Now, our objective is to prove that the end can not be embedded if $p < k$. Assume $p < k$, then taking into account that $0 < \frac{|p-a|}{k-a} < 1$ and reasoning as in Steps 1 and 4 of Sect. 5 we deduce that if the end is embedded then $p \geq k$ and so, the functions G_0 and F_0 write as follows

$$G_0(\zeta) = \zeta^{-k+a} \ , \quad F_0(\zeta) = \sum_{\lambda=k}^{\infty} b_\lambda \zeta^{\lambda-a} \ . \tag{76}$$

Hence $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$ have a pole at zero of order one and $\mathrm{Res}(\widetilde{\eta}_1) = -k+a$. Finally, from (74) we obtain $\mathrm{Im}\left(\mathrm{Res}_0(\widetilde{\eta}_2)\right) = (-k+a)\,\mathrm{Im}(b_k) = \frac{\beta_3}{\pi}$.

Suppose now that $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$ are as in Theorem 4. Then after a conformal reparametrization of a subend, G_0 and F_0 are as in (76) and the immersion (72) can be expressed as follows

$$\begin{aligned}
\widetilde{\mathbf{x}}_1(r,\theta) &= \frac{1}{2} r^{-k+a}\cos((k-a)\theta)(1+\widetilde{t}_1(r,\theta)) \ , \\
\widetilde{\mathbf{x}}_2(r,\theta) &= \frac{1}{2} r^{-k+a}\sin((k-a)\theta)(-1+\widetilde{t}_2(r,\theta)) \ , \\
\widetilde{\mathbf{x}}_3(r,\theta) &= \frac{1}{8} r^{-2k} + \frac{k-a}{2}\left(\mathrm{Re}(b_k)\log(r) - \mathrm{Im}(b_k)\theta\right) + C + \widetilde{t}_3(r,\theta) \ ,
\end{aligned}$$

where $C \in \mathbb{R}$ and $\widetilde{t}_i(r,\theta))$ are as before. Therefore, using the reasoning presented in Step 5 of Section 5, we obtain that on $\Omega_{l,\varepsilon_l} =]0,\varepsilon_l[\times]\Theta_{l-1},\Theta_l[$ the immersion is a graph over the plane $\{x_3 = 0\}$, where now $\Theta_l = l\frac{2\pi}{k-a}$ and $l \in \mathbb{Z}$. Consider a natural number l_0 such that $l_0 > k+1-a$. Observe that if $l = 1,...,l_0$ from (61) we deduce that $\widetilde{\mathbf{x}}(\Omega_{l,\varepsilon_l})$ is asymptotic to $t_{v_l}(\widetilde{\mathbf{x}}(\Omega_{1,\varepsilon_1}))$, where t_{v_l} is the translation of vector $v_l = (0,0,-\pi(l-1)\mathrm{Im}(b_k))$. Then, there is ε' such that $\widetilde{\mathbf{x}}(\Omega_{l_1,\varepsilon'}) \cap \widetilde{\mathbf{x}}(\Omega_{l_2,\varepsilon'}) = \emptyset$ for all $l_1, l_2 \in \{1,...,l_0\}$. Therefore $\widetilde{\mathbf{x}}$ is an embedding on $]0,\varepsilon'[\times[0, l_0\frac{\pi}{2(k-a)}[$. Since $\widetilde{\mathbf{x}}(r,\theta+2\pi m) = g(\widetilde{\mathbf{x}}(r,\theta))$, we conclude that $\widetilde{\mathbf{x}}$ is an embedding on $]0,\varepsilon'[\times\mathbb{R}$.

Remark 5 *Observe that the complete embedded end described above is also asymptotic to the surface* $E_{\mathfrak{a}}$ *with* $\mathfrak{a} = \frac{1}{-k+a}\mathrm{Res}_0(\widetilde{\eta}_2)$.

7 Nonexistence of ends of type IV

The purpose of this section is to prove that there exist no ends invariant under an equiaffine transformation of this type. We proceed by contradiction. Suppose that there exists $\mathbf{x} : \mathbb{C}^- \longrightarrow \mathbb{R}^3$ an improper affine sphere invariant under an equiaffine transformation $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ as in **IV** with $t > 0$. Then using (16) and (17) we have

$$G(\alpha(z)) = \frac{t+t^{-1}}{2}G(z) + \frac{t-t^{-1}}{2}F(z) \ , \quad F(\alpha(z)) = \frac{t-t^{-1}}{2}G(z) + \frac{t+t^{-1}}{2}F(z) \ .$$

Hence we obtain

$$(dG+dF)(\alpha(z)) = t(dG+dF)(z) \ , \quad (dG-dF)(\alpha(z)) = t^{-1}(dG-dF)(z) \ . \tag{77}$$

Consider $a \in \mathbb{R}$, $a \neq 0$ such that $t = \mathrm{e}^{2a\pi}$. Then, from (77) it is easy to check that the following holomorphic one-forms

$$\omega_1 = \mathrm{e}^{\mathrm{i}az}(dG + dF) , \quad \omega_2 = \mathrm{e}^{-\mathrm{i}az}(dG - dF) , \tag{78}$$

pass to the quotient, namely, there exist $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ holomorphic one-forms on D^* satisfying $\exp^* \widetilde{\omega}_1 = \omega_1$ and $\exp^* \widetilde{\omega}_2 = \omega_2$.

Taking into account $|dF| < |dG|$ and (78) we deduce that the holomorphic one-forms $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ have no zeros or poles on D^*. Then, $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ can be written as

$$\widetilde{\omega}_1 = \tau^m \mathrm{e}^{P_1(\tau)} d\tau , \quad \widetilde{\omega}_2 = \tau^n \mathrm{e}^{P_2(\tau)} d\tau , \tag{79}$$

where $\tau = \exp(z)$, $m, n \in \mathbb{Z}$ and P_i are holomorphic functions on D^* with at most a pole at zero (a proof of this fact can be found in Theorem 1.1 of [12]). According to (79) we have

$$\omega_1 = \exp\left((m+1)z + P_1(\mathrm{e}^z)\right) dz , \quad \omega_2 = \exp\left((n+1)z + P_2(\mathrm{e}^z)\right) dz . \tag{80}$$

Thus, taking into account (78) and (80) we can give the following expression for the affine metric

$$\begin{aligned} ds^2 &= \frac{1}{4}\left(|dG|^2 - |dF|^2\right) = \frac{1}{4}\mathrm{Re}\left(\overline{\exp(-\mathrm{i}az)}\exp(\mathrm{i}az)\overline{\omega}_1\omega_2\right) \\ &= \frac{1}{4}\mathrm{Re}\left(\exp((m+1+\mathrm{i}a)\overline{z} + (n+1+\mathrm{i}a)z + (\overline{P_1} + P_2)(\mathrm{e}^z))\right)|dz|^2 \\ &= \frac{1}{4}\exp((m+n+2)x + \mathrm{Re}(P_1+P_2)(e^z))\cos(h(x,y))|dz|^2 , \end{aligned}$$

where $z = x + \mathrm{i}y$ and $h(x,y) = 2ax + y(-m+n) + \mathrm{Im}(-P_1+P_2)(e^z)$. Clearly, the function $h(x,y)$ is not bounded for any $a \in \mathbb{R}$, $a \neq 0$. Therefore the affine metric cannot be a Riemannian metric on D^*_ε for all $0 < \varepsilon < 1$ and this leads to a contradiction.

Observe that if $\mathbf{X} : \mathbb{C}^- \longrightarrow \mathbb{R}^3$ is an improper affine sphere invariant under an equiaffine transformation g of type **IV** with $t < 0$, then it is also invariant under the equiaffine transformation g^2 given by

$$g^2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t^2 & 0 & 0 \\ 0 & t^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2\beta_3 \end{pmatrix} .$$

But, according to the above reasoning there exist no locally strongly convex ends invariant under such a transformation and therefore neither exist ends invariant under g.

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