NON EXISTENCE RESULTS AND CONVEX HULL PROPERTY FOR MAXIMAL SURFACES IN $\mathbb{L}^3$

ROSA M.B. CHAVES AND LEONOR FERRER*

ABSTRACT. In this paper we deal with properly immersed maximal surfaces with non empty boundary and singularities in the 3-dimensional Minkowski space. We use maximum principle and scaling arguments to obtain non existence results for these surfaces when the boundary is planar. We also give sufficient conditions in order that such surfaces satisfy the convex hull property.

1. INTRODUCTION

In the last years, maximal hypersurfaces in a Lorentzian manifold have been a focus of considerable interest. These are spacelike submanifolds of codimension one with zero mean curvature. Such hypersurfaces, and in general those having constant mean curvature, have a special significance in classical Relativity (see [17]).

When the ambient space is the flat Minkowski space $\mathbb{L}^{n+1}$, Calabi [2] ($n \leq 3$) and Cheng and Yau [3] (for arbitrary dimension) proved that a complete maximal hypersurface is necessarily a spacelike hyperplane. The preceding result is valid if we substitute the hypothesis of completeness for the one of properness (see [7]).

Therefore, it does not make any sense to consider global problems on regular maximal hypersurfaces in $\mathbb{L}^{n+1}$. Interesting problems are then, those that deal with hypersurfaces with non empty boundary or having certain type of singularities. In this line, Bartnik and Simon [1] obtained results on the existence and regularity of spacelike solutions to the boundary value problem for the mean curvature operator in $\mathbb{L}^{n+1}$ and Kobayashi [14] investigated such surfaces with conelike singularities. In [6] Estudillo and Romero defined a class of maximal surfaces with singularities of other type and studied criteria for such surfaces to be a plane. On the other hand, Klyachin and Mikyukov [12] tackle the problem of existence of solutions to the maximal hypersurface equation in $\mathbb{L}^{n+1}$ with prescribed boundary conditions and a finite number of singularities. It should be remarked that Fernández, López and Souam [9] proved that a complete embedded maximal surface with a finite set of singularities is an entire graph over any spacelike plane and that this family of maximal graphs has a structure of moduli space. Finally, we would like to mention the paper [22] where topological obstructions to the existence of this type of surfaces are given by Umehara and Yamada.

Maximal surfaces in $\mathbb{L}^3$ and minimal surfaces in the Euclidean space are closely related. Firstly, both families are solutions of variational problems: local maxima (minima) for the area functional and both admit a Weierstrass representation (see [13] for maximal surfaces). Moreover, the maximal surface equation as well as the minimal surface equation are quasilinear elliptic equations and therefore we have a maximum principle for them.

* Research partially supported by MCYT-FEDER grant number MTM2004-00160.
2000 Mathematics Subject Classification: primary 53C50; secondary 53C42, 53C80.
Keywords and phrases: maximal surfaces.

Otherwise, contrarily to the minimal case, solutions to the maximal surface equation can have isolated singularities, that is to say, points where the solution is not differentiable. These points correspond to possible degeneracy of the ellipticity of the maximal surface equation. Geometrically at these singular points the Gauss curvature blows up, the Gauss map has no well-defined limit and the surface is asymptotic to the light cone.

Moreover, in the minimal case, the maximum principle has been used by Schoen [21], Hoffman-Meeks [11], Meeks-Rosenberg [18], López-Martín [16], among others, to obtain remarkable results. In this paper we apply maximum principle and scaling arguments to properly immersed maximal surfaces with non empty boundary and isolated singularities in $\mathbb{L}^3$. We get two types of results: non existence results for properly immersed maximal surfaces with singularities and planar boundary contained in a timelike or lightlike plane and results that generalize the convex hull property for such surfaces. Recall that a surface satisfies the convex hull property if and only if it lies in the convex hull of its boundary. Although it is well-known that compact maximal surfaces in $\mathbb{L}^3$ verify the convex hull property, just for having non positive Euclidean Gauss curvature (see [20]), this is not the general case. We give sufficient conditions so that a properly immersed maximal surface (not necessarily compact and with singularities) verifies the convex hull property.

The present paper is laid out as follows. Section 2 contains the notations and definitions we need in the paper. In this section we also describe the behavior of maximal surfaces around an isolated singularity and present the maximal surfaces we use as barriers: Lorentzian catenoids, Riemann type maximal surfaces, Scherk's type maximal surfaces as well as spacelike planes. We finish the section giving a first generalization of the convex hull property to compact maximal surfaces with singularities.

In Section 3 we obtain non existence results for properly immersed maximal surfaces with singularities and boundary contained in a timelike plane. In particular, if $\mathsf{C}^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 \leq 0, x_3 \geq 0\}$ we show that:

Theorem A. *There does not exist a connected properly immersed maximal surface M such that $M \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 \geq 0, -\tan(\theta)x_2 + x_3 \geq 0\}$ and $\partial(M) \subset \mathsf{C}^+ \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 = 0\}$, for $\theta \in]-\frac{\pi}{4}, \frac{\pi}{4}[$.*

The preceding theorem holds even if we allow certain singularities (see Theorem 3.5).

Section 4 is devoted to study properly immersed maximal surfaces whose boundary is contained in a spacelike plane. Let us consider $V(\theta, \delta, \delta') = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \geq 0, -\tan(\theta)x_2 + x_3 \leq 0, x_1 - \cot(\delta)x_2 + \cot(\theta)(\cot(\delta) - \cot(\delta')) \geq 0\}$, for $\theta \in]0, \frac{\pi}{4}[$ and $\delta, \delta' \in]0, \pi[$ (see Fig. 6). Our main result of this section is:

Theorem B. *Let M be a connected properly immersed maximal surface such that $M \subset V(\theta, \delta, \delta')$ and $\partial(M)$ is contained in a spacelike plane. Then M is a planar region.*

As in the previous theorem, Theorem B holds even if we allow certain singularities (see Theorem 4.2 and Corollary 4.3). We would like to emphasize that in the proof of the above theorem a barrier surface is constructed ad hoc using the aforementioned Bartnik and Simon existence result. Moreover, the previous theorem is valid if we substitute the region $V(\theta, \delta, \delta')$ for C^+ (see Proposition 4.4).

Finally, in Sect. 5 we exploit the results obtained in the preceding sections to give non existence results for properly immersed maximal surfaces with the boundary on a lightlike plane and to state the following convex hull property:

Proposition C. *Any connected properly immersed maximal surface with singularities contained either in $V(\theta, \delta, \delta')$ or C^+ lies in the convex hull of its boundary and some of its singularities.*

We refer the reader to Propositions 5.3 and 5.4 for a precise formulation of this result.

Acknowledgments: We are indebted to F. Martín, I. Fernández and specially to F.J. López for helpful conversations during the preparation of this work. This paper was carried out during the second author's visit to the IME, Universidade de São Paulo (Brasil). The second author is grateful to the people at the institute for their hospitality. The authors would also like to thank the referee for valuable suggestions.

2. Preliminaries

We denote by $\mathbb{L}^3$ the three dimensional Lorentz-Minkowski space $(\mathbb{R}^3, \langle\ ,\ \rangle)$, where $\langle\ ,\ \rangle = dx_1^2 + dx_2^2 - dx_3^2$. A vector $v \in \mathbb{R}^3 - \{(0,0,0)\}$ is called *spacelike, timelike or lightlike* if $\langle v, v\rangle$ is positive, negative or zero, respectively. The vector $(0,0,0)$ is considered a spacelike vector. We say that a plane in $\mathbb{L}^3$ is *spacelike, timelike or lightlike* if the induced metric is Riemannian, non degenerate and indefinite or degenerate, respectively. We also say that an affine plane in $\mathbb{L}^3$ is *spacelike, timelike or lightlike* if it is parallel to a spacelike, timelike or lightlike vectorial plane.

The *light cone* at $y = (y_1, y_2, y_3) \in \mathbb{L}^3$ is defined as

$$C(y) = \{x \in \mathbb{L}^3 \mid \langle x - y, x - y\rangle = 0\}\ .$$

We also denote $C^+(y) = C(y) \cap \{x_3 \geq y_3\}$ and $C^-(y) = C(y) \cap \{x_3 \leq y_3\}$. Observe that lightlike vectors in $\mathbb{L}^3$ lie in the light cone $C((0,0,0))$.

Let us denote by $\mathbb{H}^2 = \mathbb{H}^2_+ \cup \mathbb{H}^2_-$, where $\mathbb{H}^2_+ = \{x \in \mathbb{L}^3 \mid \langle x, x\rangle = -1\} \cap \{x_3 \geq 0\}$ and $\mathbb{H}^2_- = \{x \in \mathbb{L}^3 \mid \langle x, x\rangle = -1\} \cap \{x_3 \leq 0\}$.

Consider $\sigma : \overline{\mathbb{C}} - \{|z| = 1\} \to \mathbb{H}^2$ the stereographic projection for $\mathbb{H}^2$ given by

$$\sigma(z) = \left(\frac{2\Im(z)}{|z|^2 - 1}, \frac{2\Re(z)}{|z|^2 - 1}, \frac{|z|^2 + 1}{|z|^2 - 1}\right), \tag{2.1}$$

where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\sigma(\infty) = (0,0,1)$ and $\Im$ and $\Re$ stands for the imaginary and real part of the complex numbers.

An immersion $X : M \to \mathbb{L}^3$ is *spacelike* if the tangent plane at any point is spacelike. If X is spacelike M is orientable, that is to say, the Gauss Map N is globally well defined and $N(M)$ lies in one of the components of $\mathbb{H}^2$.

A *maximal immersion* is a spacelike immersion $X : M \to \mathbb{L}^3$ such that its mean curvature vanishes. In this case $X(M)$ is said to be a *maximal surface* in $\mathbb{L}^3$. Using isothermal parameters compatible with a fixed orientation $N : M \to \mathbb{H}^2$, M has in a natural way a conformal structure, and the map $g = \sigma^{-1} \circ N$ is meromorphic. Moreover, there exists a holomorphic 1-form Φ_3 on M such that the 1-forms

$$\Phi_1 = \frac{i}{2}\left(\frac{1}{g} - g\right)\Phi_3\ ,\ \Phi_2 = -\frac{1}{2}\left(\frac{1}{g} + g\right)\Phi_3 \tag{2.2}$$

are holomorphic, and together with Φ_3, have no real periods on M and no common zeros. Up to a translation, the immersion is given by

$$X = \Re\int(\Phi_1, \Phi_2, \Phi_3)\ . \tag{2.3}$$

The induced Riemannian metric ds^2 on M is given by $ds^2 = \lambda(du^2 + dv^2)$, where $z = u + iv$ is a conformal parameter and $\lambda = \frac{1}{2}(|\Phi_1|^2 + |\Phi_2|^2 - |\Phi_3|^2) = \left(\frac{|\Phi_3|}{2}\left(\frac{1}{|g|} - |g|\right)\right)^2$. Since M is spacelike, we have $|g| \neq 1$ on M and we can assume $|g| < 1$.

Conversely, let M, g and Φ_3 be a Riemann surface, a meromorphic map on M and a holomorphic 1-form on M. If $|g(p)| \neq 1, \forall p \in M$ and the 1-forms Φ_j, $j = 1, 2, 3$, defined as above are holomorphic, have no real periods and no common zeros then the conformal immersion X defined in (2.3) is maximal and its Gauss map is $\sigma \circ g$. We call (M, g, Φ_3) the Weierstrass representation of X. For more details see [13].

2.1. **The maximum principle for maximal surfaces.** A maximal surface in $\mathbb{L}^3$ can be represented locally as a graph $x_3 = u(x_1, x_2)$ of a smooth function u with $u_{x_1}^2 + u_{x_2}^2 < 1$, satisfying the equation

$$(1 - u_{x_1}^2)u_{x_2x_2} + 2u_{x_1}u_{x_2}u_{x_1x_2} + (1 - u_{x_2}^2)u_{x_1x_1} = 0 \ . \tag{2.4}$$

The maximum principle for elliptic quasilinear equations then gives the following maximum principle for maximal surfaces:

Maximum principle. *Let S_1 and S_2 be two maximal surfaces in $\mathbb{L}^3$ which intersect tangentially at a point $p \in S_1 \cap S_2$. Suppose that u_i, for $i = 1, 2$ denotes the function defining S_i around p and that $u_1 \geq u_2$ (we say S_1 is above S_2 or S_2 is below S_1). Then $S_1 = S_2$ locally around p.*

2.2. **Maximal surfaces with singularities.** If in a maximal immersion $X : M \to \mathbb{L}^3$ we allow points $q \in M$ where the induced metric is not Riemannian we say that X (respectively, $X(M)$) has singularities and q (respectively, $X(q)$) is called a *singular point*. The different kind of isolated singularities of maximal surfaces as well as the behavior of maximal surfaces around these points are well-known (see [14], [5], [19] and [9]). We need to recall some aspects of this behavior and go deeply into some of them.

Let D be an open disc and $X : D \to \mathbb{L}^3$ a maximal immersion with a singular point in $q \in D$. There are two possibilities: either N extends continuously to q (q is a *spacelike* singular point) or not (q is a *lightlike* singular point).

In the second case $D - \{q\}$ with the induced metric is conformally equivalent to $\{z \in \mathbb{C}, 0 < r < |z| < 1\}$ and X extends to a conformal map $X : A_r \to \mathbb{L}^3$ with $X(\mathbb{S}^1) = X(q) = p$, where $A_r = \{z \in \mathbb{C}, r < |z| \leq 1\}$ and $\mathbb{S}^1 = \{z \in \mathbb{C}, |z| = 1\}$. Denote by $J(z) = \frac{1}{\overline{z}}$ the inversion about $\mathbb{S}^1$. Then Schwarz reflection allows us to assert that X extends analytically to $B_r = \{z \in \mathbb{C}, r < |z| < \frac{1}{r}\}$ and satisfies $X \circ J = -X + 2p$. Therefore if (g, Φ_3) are the Weierstrass data of the extended immersion we have that $J^*(\Phi_k) = -\overline{\Phi_k}$ for $k = 1, 2, 3$, where by $J^*(\Phi_k)$ we denote the pullback of Φ_k under J, more precisely if $\Phi_k = f_k dz$ then $J^*(\Phi_k) = \frac{-1}{\overline{z}^2}(f_k \circ J)d\overline{z}$. Thus $g \circ J = \frac{1}{\overline{g}}$ and consequently $|g| = 1$ on $\mathbb{S}^1$. Let Π be a spacelike plane containing $p = X(\mathbb{S}^1)$ and label $\pi : \mathbb{L}^3 \to \Pi$ as the Lorentzian orthogonal projection. If n (always even) and m denote the number of zeros of Φ_3 on $\mathbb{S}^1$ and the degree of the map $g : \mathbb{S}^1 \to \mathbb{S}^1$, respectively, it can be proved

Lemma 2.1. ([9]) *There exists a small closed disc U in Π centered at p such that $(\pi \circ X)^{-1}(p) \cap V = \mathbb{S}^1$ and $(\pi \circ X) : V - \mathbb{S}^1 \to U - \{p\}$ is a covering of $m + \frac{n}{2}$ sheets, where V is the annular connected component of $(\pi \circ X)^{-1}(U)$ containing $\mathbb{S}^1$.*

As a consequence, X is an embedding around q if and only if $m = 1$ and $n = 0$. In this case the point $p = X(\mathbb{S}^1)$ is said to be a *conelike singularity* of the maximal surface $X(D)$.

Moreover, for r_0 close enough to 1 , $X(A_{r_0})$ is the graph of a function u over Π. Locally, conelike singularities are points where the function defining the graph is not differentiable and correspond to possible degeneracy of the equation (2.4). Moreover, the graph of u is either above Π and asymptotic to $C^+(p)$ or below Π and asymptotic to $C^-(p)$, and the point p is called a *downward* or *upward pointing conelike singularity*, respectively.

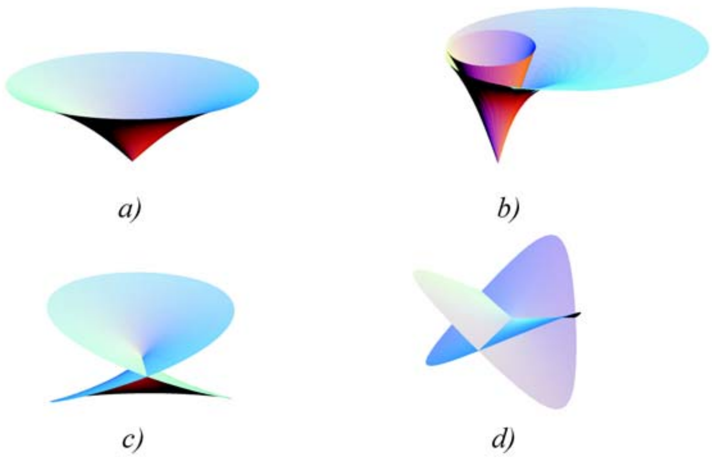

FIGURE 1. Different types of isolated singularities: Figures a), b) and c) correspond to lightlike singularities while d) is an example of a spacelike singularity. More precisely: a) A downward pointing conelike singularity ($m = 1$, $n = 0$), b) a downward pointing lightlike singularity with $m = 2$, $n = 0$, c) a lightlike singularity with $m = 1$, $n = 2$ and d) a spacelike singularity with $n = 2$.

Furthermore, we can also prove:

Lemma 2.2. *Let D be an open disc and $X : D \to \mathbb{L}^3$ a maximal immersion with a lightlike singular point in $q \in D$ and denote $p = X(q)$. Then, the neighborhoods U and V of Lemma 2.1 can be chosen to verify:*

i) *If p is a lightlike singularity with $n = 0$, then $X(V)$ is either over Π and asymptotic to $C^+(p)$ or below Π and asymptotic to $C^-(p)$ (see Fig. 1.a) and 1.b)).*
ii) *On the contrary, if p is a lightlike singularity with $n > 0$, there exist points of $X(V)$ in both sides of the plane Π. In particular there exist a pair of curves α, β in V starting at q such that $X(\alpha) - \{p\}$ is over Π and asymptotic to $C^+(p)$ and $X(\beta) - \{p\}$ is below Π and asymptotic to $C^-(p)$ (see Fig. 1.c)).*

Proof: Up to a Lorentzian isometry we can assume $\Pi = \{x_3 = 0\}$ and $p = (0, 0, 0)$. Let $X : A_r \to \mathbb{R}^3$ be a conformal reparametrization of the maximal immersion with $X(\mathbb{S}^1) = p$ and consider U, V as in Lemma 2.1. A thoughtful reading of the proof of Lemma 2.1 in [9] will convince the reader that the same arguments prove i).

For the proof of assertion ii) we use again the ideas of the Lemma 2.1 in [9]. Observe that the Weierstrass data can be written on a neighborhood of $\mathbb{S}^1$ as

$$g(z) = z^m \ , \quad \Phi_3(z) = i\frac{\prod_{j=1}^{n}(z - a_j)}{z^{\frac{n}{2}+1}} f(z)dz \ , \tag{2.5}$$

where $a_1, \ldots, a_n$ are the zeros of Φ_3 on $\mathbb{S}^1$ (each zero appears as many times as its multiplicity) and f is a non-vanishing holomorphic function. Recall that the multiplicity of the zero of Φ_3 at a_i coincides with the number of nodal curves of the harmonic function x_3 meeting at a_i minus one. By the maximum principle there are no domains bounded by nodal curves and x_3 changes sign when crossing a nodal curve. Note that since $n \geq 2$ there are points of $X(V)$ in both sides of Π and there exist at least a pair of domains $\Gamma, \Gamma' \subset V$ bounded by a pair of nodal curves of x_3, a piece of $\partial V - \mathbb{S}^1$ and a point or a piece of $\mathbb{S}^1$, such that $x_3(X(\Gamma)) > 0$ and $x_3(X(\Gamma')) < 0$.

In order to finish the assertion ii) we shall prove that the image of all the curves $\rho_\theta(s) = se^{i\theta}$ for $\theta \in K = [0, 2\pi] - \{\arg(a_1), \ldots, \arg(a_n)\}$ is asymptotic to the cone $C(p)$. Taking into account (2.5) we can write

$$X(\rho_\theta(s)) = \Re \int_1^s \frac{i \prod_{j=1}^n (te^{i\theta} - a_j)}{t^{\frac{n}{2}+1}(e^{i\theta})^{\frac{n}{2}}} f(te^{i\theta}) \left(\frac{i}{2}(\frac{e^{-im\theta}}{t^m} - t^m e^{im\theta}), -\frac{1}{2}(\frac{e^{-im\theta}}{t^m} + t^m e^{im\theta}), 1 \right) dt \, .$$

As $J^*(\Phi_3) = -\overline{\Phi_3}$, we deduce $\Im \left(\frac{i \prod_{j=1}^n (e^{i\theta} - a_j)}{(e^{i\theta})^{\frac{n}{2}}} f(e^{i\theta}) \right) = 0$. Using this it is straightforward to see

$$\lim_{s \to 1} \left\| \frac{X(\rho_\theta(s))}{x_3(X(\rho_\theta(s)))} - (\sin(m\theta), -\cos(m\theta), 1) \right\|_1 = 0 \, ,$$

where $\| \cdot \|_1$ is the $\mathcal{C}^1$ norm in $\mathcal{C}^1(K, \mathbb{R}^3)$. Therefore, we can consider a pair of curves $\alpha \in \Gamma$ and $\beta \in \Gamma'$ satisfying the requirements of statement ii). $\square$

Definition 2.3. If the point p is as in the statement i) of Lemma 2.2 we say that p is a *downward* or *upward pointing lightlike singularity*, respectively. We also name both types of singularities *general conelike singularities*.

If D is an open disc and $X : D \to \mathbb{L}^3$ a maximal immersion with a spacelike singular point in $q \in D$, the local behavior at the singularity is similar to the case of minimal surfaces in $\mathbb{R}^3$ (see [4], [6] and [9]): X is not a topological embedding, $D - \{q\}$ with the induced metric is conformally equivalent to $\{z \in \mathbb{C}, 0 < |z| < 1\}$, the Weierstrass data (g, Φ_3) extend analytically to q, $|g(q)| < 1$ and $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ has a zero at q. Furthermore, up to a Lorentzian isometry we can assume that the tangent plane of $X(D)$ at $p = X(q)$ is $\Pi = \{x_3 = 0\}$ and $p = (0,0,0)$. Observe that the Weierstrass data of the immersion can be written as

$$g(z) = z^m f(z) \, , \quad \Phi_3(z) = z^{m+n} dz \, ,$$

where $m > 0$, n is the zero order of Φ at q and f is a holomorphic function with $f(0) \neq 0$. Furthermore, up to a rotation around the axis x_3, we can assume $\Im(f(0)) = 0$. From here it is easy to obtain that the asymptotic behavior of the immersion around the singularity is in polar coordinates

$$X(se^{i\theta}) = \Big(\tfrac{-s^{n+1}}{2f(0)(n+1)} \sin((n+1)\theta) + O(s^{n+2}), \quad \tfrac{-s^{n+1}}{2f(0)(n+1)} \cos((n+1)\theta) + O(s^{n+2}), \tfrac{s^{m+n+1}}{m+n+1} \cos((m+n+1)\theta) \Big) \, , \tag{2.6}$$

where by $O(s^{n+2})$ we denote a function such that $s^{-n-2}O(s^{n+2})$ is bounded as $s \to 0$. Therefore, it is clear that X has a branch point at q of order n in the sense of Gulliver-Osserman-Royden ([10]). From Lemma 2.12 in [10] we have

Lemma 2.4. *Let $X : D \to \mathbb{L}^3$ be a maximal immersion with a spacelike singular point in $q \in D$, $p = X(q)$ and S an embedded surface in $\mathbb{L}^3$ with $p \in S$. Suppose that for a neighborhood V of q, $X(V)$ lies on one side of S. Then the tangent plane to S at p coincides with the tangent plane to $X(D)$ at p.*

Remark 2.5. *In the case of spacelike singularities, we always assume that the immersion $X : D \to \mathbb{L}^3$ is not a branched covering of an embedded surface, it is to say that q is not a false branch point (see* [10]*).*

Finally, we like to mention a well known property of maximal surfaces with singularities (see for example [8]).

Lemma 2.6. *Let $X : M \to \mathbb{L}^3$ be a maximal immersion with isolated singularities. Then for all $q \in M$ there exists a neighborhood V, such that $X(V) - \{X(q)\}$ is contained in the exterior of $C(X(q))$.*

Remark 2.7. *Let S be an embedded surface and $p \in S$. If the tangent plane of S at p is spacelike then S can be written in a neighborhood of p as the graph of a function h on a domain Ω of the plane $\{x_3 = 0\}$. Let M be another surface (possibly with singularities) and denote by π the orthogonal projection on $\{x_3 = 0\}$. In this context, we say that M is above (below) S in a neighborhood of p if $x_3(p') \geq h(x_1, x_2)$ ($x_3(p') \leq h(x_1, x_2)$) for all $p' \in M \cap \pi^{-1}(x_1, x_2)$, $(x_1, x_2) \in \Omega$.*

2.3. **Maximal surfaces with boundary.** Let S' be a maximal surface with possibly isolated singularities. Consider $S \subset S'$ such that the topological boundary of S in S' is not empty and at least piecewise $\mathcal{C}^1$. Then S will be called a *maximal surface with boundary*, the topological boundary of S in S' is said to be the *boundary* of S and it will be denoted as $\partial(S)$. We also denote $\mathrm{Int(S)} = \mathrm{S} - \partial(\mathrm{S})$ and we refer to $\mathrm{Int(S)}$ as the *interior* of S. Moreover, S is said a *properly immersed maximal surface with boundary* if S' is a maximal surface properly immersed in $\mathbb{L}^3$. Note that from our definition, some of the singularities could be on the boundary of S.

At this point, we would like to mention that if S is a maximal surface with boundary, since the components of a maximal immersion are harmonic functions, we deduce that the intersection of S with any plane Π containing $\partial(S)$ in one of the half spaces determinated by Π, is a union of piecewise analytic curves and therefore each connected component of $S - (S \cap \Pi)$ is a maximal surface with boundary.

2.4. **Parabolicity of maximal surfaces.** In [7] I. Fernández and F.J. López proved the following result

Theorem *(Fernández-López) Let M be a properly immersed maximal surface with boundary such that, except for a compact set, it is contained in the region $\{x \in \mathbb{L}^3 \mid \langle x, x\rangle \geq \varepsilon\}$, for $\varepsilon > 0$. Then M is relative parabolic, it is to say, bounded harmonic functions on M are determined uniquely by their values at the boundary and the interior isolated singularities.*

An immediate consequence of this theorem is

Corollary 2.8. *Let M be a connected properly immersed maximal surface with boundary such that $M \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq x_3 \leq k\}$ and the boundary and the singularities are contained in $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = k\}$, for $k > 0$. Then M is a planar region.*

We would like to mention that in [7] the definition of a maximal surface with boundary is more general than in this paper.

2.5. **Our barriers.** Let us introduce some notation. For any $v \in \mathbb{R}^3 - \{(0,0,0)\}$ and $y \in \mathbb{R}^3$ we define

$$H(y,v) = \{x \in \mathbb{R}^3 \mid \langle v, x-y\rangle_e = 0\}\,,$$

$$H^+(y,v) = \{x \in \mathbb{R}^3 \mid \langle v, x-y\rangle_e \geq 0\}\,,$$

$$H^-(y,v) = \{x \in \mathbb{R}^3 \mid \langle v, x-y\rangle_e \leq 0\}\,,$$

where $\langle\cdot,\cdot\rangle_e$ denotes the Euclidean metric of $\mathbb{R}^3$.

Furthermore, for $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ and $t \in \mathbb{R}$, we introduce the following terminology

$$\Pi_{\theta t} = H((0,0,t),(0,-\tan(\theta),1))\ .$$

$$\Pi^+_{\theta t} = H^+((0,0,t),(0,-\tan(\theta),1))\ .$$

$$\Pi^-_{\theta t} = H^-((0,0,t),(0,-\tan(\theta),1))\ .$$

In particular we denote by $\Pi_\theta = \Pi_{\theta 0}$, $\Pi^+_\theta = \Pi^+_{\theta 0}$ and $\Pi^-_\theta = \Pi^-_{\theta 0}$.

We also consider for $\alpha \in]-\frac{\pi}{4}, \frac{\pi}{4}[$

$$\Sigma_\alpha = H((0,0,0),(0,1,-\tan(\alpha)))\ .$$

$$\Sigma^+_\alpha = H^+((0,0,0),(0,1,-\tan(\alpha)))\ .$$

$$\Sigma^-_\alpha = H^-((0,0,0),(0,1,-\tan(\alpha)))\ .$$

Observe that $\Pi_{\frac{\pi}{4}t}$ and $\Pi_{-\frac{\pi}{4}t}$ are lightlike planes while $\Pi_{\theta t}$ are spacelike planes for $\theta \in]-\frac{\pi}{4}, \frac{\pi}{4}[$. Furthermore, for all $\theta \in]-\frac{\pi}{4}, \frac{\pi}{4}[$ it is always possible to consider an orthochronous hyperbolic rotation f_s of $\mathbb{L}^3$ of the form

$$f_s\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(s) & \sinh(s) \\ 0 & \sinh(s) & \cosh(s) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

that preserves the forward (backward) light cone $C^+((0,0,0))$ ($C^-((0,0,0))$) and verifies $f_s(\Pi_\theta) = \Pi_0$. Analogously, for $\theta \in]-\frac{\pi}{4}, \frac{\pi}{4}[$ we can always consider $\widetilde{f}_s$ an orthochronous isometry of $\mathbb{L}^3$, consisting of the orthochronous hyperbolic rotation f_s composed with a vertical translation, such that $\widetilde{f}_s(\Pi^+_{\theta t}) = \Pi^+_0$ and $\widetilde{f}_s(\Pi^-_{\theta t}) = \Pi^-_0$ (and so $\widetilde{f}_s(\Pi_{\theta t}) = \Pi_0$).

On the other hand, Σ_α are timelike planes and as before we can find always an orthochronous hyperbolic rotation f_s of $\mathbb{L}^3$ that preserves the forward (backward) light cone $C^+((0,0,0))$ ($C^-((0,0,0))$) and such that $f_s(\Sigma_\alpha) = \Sigma_0$. For more details about these isometries of $\mathbb{L}^3$ we refer the reader to [8].

Now we present the examples of maximal surfaces that we use as barriers.

2.5.1. *Lorentzian catenoids.* Consider $\mathcal{C} = \{\mathcal{C}_a, a \in]0,\infty[\}$ the family of vertical Lorentzian catenoids, it is to say, $\mathcal{C}_a$ is the maximal surface given on $\overline{\mathbb{D}} - \{0\} = \{z \in \mathbb{C} \mid 0 < |z| \leq 1\}$ by the Weierstrass data $g = z$ and $\Phi_3 = a\frac{dz}{z}$ (see Fig. 2).

Lorentzian catenoids can be also seen as graphs of the radially symmetric functions

$$u(r) = -\int_0^r \frac{a}{\sqrt{t^2 + a^2}}dt\ , \quad r > 0\ .$$

It is worth mentioning that Lorentzian catenoids have already been used as barriers for maximum principle application in [1] and [5].

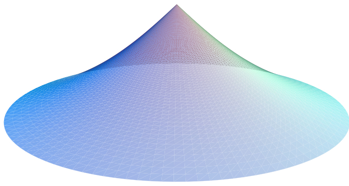

FIGURE 2. Lorentzian catenoid for $a = 1$.

2.5.2. *Riemann type maximal surfaces.* In [15], R. López, F.J. López and R. Souam studied the set of maximal surfaces in $\mathbb{L}^3$ that are foliated by pieces of circles. Between them, we emphasize the one-parameter family of Riemann type maximal surfaces. This is a family of singly-periodic maximal surfaces that plays the same role that Riemann's minimal examples play in the euclidean space and whose fundamental piece is a graph over any spacelike plane, has one planar end and two conelike singularities (see Fig. 3).

FIGURE 3. A Riemann type maximal example.

Let us recall briefly the Weierstrass representation of half fundamental piece of these Riemann type maximal surfaces.

Consider for $r \in]1, \infty[$ the four punctured torus

$$\mathcal{N} = \{(z, w) \in \mathbb{C}^* \times \mathbb{C} \mid w^2 = z(z^2 + 2rz + 1)\}$$

and define in the z-plane

$$s_0 = \{z \in \mathbb{C} \mid |z| = 1\}\ , \quad s_1 = [r_1, 0[\times\{0\}\ , \quad s_2 =]-\infty, r_2] \times \{0\}\ ,$$

where $r_1 = -r + \sqrt{r^2 - 1}$ and $r_2 = -r - \sqrt{r^2 - 1}$. Observe that $r_2 < -1 < r_1 < 0$. Then we label $N \subset \mathcal{N}$ as the connected component of $z^{-1}(\mathbb{C} - \cup_{i=0}^{2} s_i)$ containing the point $\left(\frac{1}{2}, \sqrt{\frac{1}{2}(\frac{5}{4} + r)}\right)$. Finally we define $M = \overline{N}$, where $\overline{N}$ means the closure of N in $\mathcal{N}$.

For the sake of brevity, when $z(z^2 + 2rz + 1) \in \mathbb{R}^+$, we denote:

$$z_+ = (z, +\sqrt{z(z^2 + 2rz + 1)})\ , \quad z_- = (z, -\sqrt{z(z^2 + 2rz + 1)})\ .$$

On M we consider the Weierstrass data $g = z$ and $\Phi_3 = \frac{dz}{w}$ and the 1-forms Φ_j, $j = 1, 2$ given by (2.2). Define γ the lift to M of s_0. Observe that γ generates $\mathcal{H}_1(M, \mathbb{Z})$. It is not difficult to see that Φ_1 is exact and that Φ_2, Φ_3 have no real periods on γ and so we can consider the maximal immersion $X = (X_1, X_2, X_3) = \Re \int_{z_0}^{z} (\Phi_1, \Phi_2, \Phi_3)$.

Denote by γ_1 the lift to M of s_1. It is not hard to prove that $X(\gamma_1)$ is a straight line parallel to $\{x_2 = x_3 = 0\}$. We also observe that the set of singularities of the immersion is the trace of the curve γ and the image of these points by the immersion X is a unique point that we label P^r. Moreover, z_0 can be chosen so that $X(\gamma_1) = \{x_2 = x_3 = 0\}$ and $P^r = (0, P_2^r, P_3^r)$.

Let be $\Theta(r) \in [-\pi, \pi[$ the angle such that $\cos(\Theta(r)) = \frac{P_2^r}{\sqrt{(P_2^r)^2+(P_3^r)^2}}$, $\sin(\Theta(r)) = \frac{P_3^r}{\sqrt{(P_2^r)^2+(P_3^r)^2}}$, it is to say, the angle that forms $(0,1,0)$ and the vector P^r. In order to use the surfaces of this family as barriers, we need to study the function $\Theta(r)$.

First of all, observe that $\Theta(r) = \arctan\left(\frac{h(r)}{d(r)}\right)$, where

$$h(r) = X_3(-1_-) - X_3(r_1) = X_3(1_+) - X_3(0)\ , \quad d(r) = X_2(-1_-) - X_2(r_1)\ .$$

From here we have

$$h(r) = \Re\int_{r_1}^{-1_-} \Phi_3 = \int_{-1}^{r_1} \frac{dt}{\sqrt{t(t^2+2rt+1)}}\ , \tag{2.7}$$

$$h(r) = \Re\int_{0}^{1_+} \Phi_3 = \int_{0}^{1} \frac{dt}{\sqrt{t(t^2+2rt+1)}}\ , \tag{2.8}$$

$$d(r) = \Re\int_{r_1}^{-1_-} \Phi_2 = -\frac{1}{2}\int_{-1}^{r_1} \frac{(1+t^2)dt}{t\sqrt{t(t^2+2rt+1)}}\ . \tag{2.9}$$

Note that h as well as d are positive functions and so $\Theta(r) \in]0, \frac{\pi}{2}[$. Moreover, we observe that

$$d(r) = rh(r) + I(r)\ , \tag{2.10}$$

where

$$I(r) = \frac{1}{2}\int_{-1}^{r_1} \frac{\sqrt{t(t^2+2rt+1)}dt}{t^2}\ . \tag{2.11}$$

Using (2.10) and (2.11) it is not hard to see that

$$\lim_{r\to 1}\Theta(r) = 1\ , \quad \lim_{r\to+\infty}\Theta(r) = 0\ .$$

Moreover, from (2.8) we observe that

$$h'(r) = \int_0^1 \frac{-t^2 dt}{(t(t^2+2rt+1))^{\frac{3}{2}}}\ . \tag{2.12}$$

On the other hand, from (2.10) and (2.11) the derivative of d respect to r is given by

$$d'(r) = rh'(r) + \frac{3}{2}h(r)\ . \tag{2.13}$$

According to (2.10) and (2.13) we have

$$\Theta'(r) = \frac{h'(r)d(r) - h(r)d'(r)}{h(r)^2 + d(r)^2} = \frac{I(r)h'(r) - \frac{3}{2}h(r)^2}{h(r)^2 + d(r)^2}\ .$$

Therefore, taking into account (2.11) and (2.12) we obtain that $\Theta'(r) < 0$ and so Θ is a one-to-one function $\Theta :]1, \infty[\to]0, \frac{\pi}{4}[$.

For convenience of notation, for $\delta \in]0, \frac{\pi}{4}[$ we shall denote by R_δ the maximal surface with boundary defined in $\mathbb{L}^3$ by the above immersion for $r = \Theta^{-1}(\delta)$ (see Fig. 4). We also denote by $\mathcal{R} = \{R_\delta \mid \theta \in]0, \frac{\pi}{4}[\}$.

Finally, we need to prove that $R_\delta \subset \Pi_\delta^- \cap \{x_3 \geq 0\}$. It is not difficult to see that the point $\{0\}$ is a planar end of the surface asymptotic to the plane $\{x_3 = 0\}$. Therefore, X_3

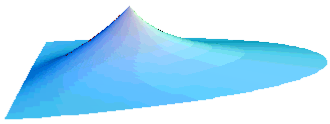

FIGURE 4. R_δ for $\delta = 0.595881$.

is bounded on M. From Corollary 2.8 we deduce that $R_\delta \subset \{x_3 \geq 0\}$. Moreover, from the above facts there exists $t > 0$ such that $R_\delta \subset \Pi^-_{\delta t}$. The maximum principle allows us to assert that $R_\delta \subset \Pi^-_\delta$.

2.5.3. *Scherk's type maximal surfaces.* Now, we describe the one-parameter family of Scherk's type maximal surfaces. This family of singly-periodic maximal surfaces was studied in depth by I. Fernández and F.J. López in [8], although an example of this type of maximal surfaces appeared already in [14]. For $b \in]0, 1[$ we denote by S_b the maximal surface given on $\overline{\mathbb{D}} - \{b, -b\}$ by the Weierstrass data $g(z) = iz$ and $\Phi_3(z) = \frac{zdz}{(z^2-b^2)(b^2z^2-1)}$.

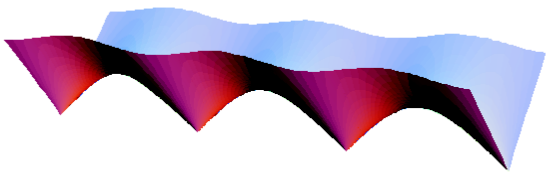

FIGURE 5. A Scherk's type maximal surface.

The surface S_b is a graph over any spacelike plane, it is invariant under the translation along the vector $(0, \frac{\pi}{2b(b^2+1)}, 0)$ and each fundamental piece of this singly-periodic maximal surface has a conelike singularity. Up to translation we can assume one of these singularities is the point $(0, 0, 0)$ and then the set of all conelike singularities lie on the line $\{x_1 = x_3 = 0\}$. Moreover, it is not difficult to see that the ends are asymptotic to the totally geodesic horizontal half cylinder $\partial(W_\delta)$, where

$$W_\delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid -\tan(\delta)x_1 + x_3 \geq 0, \tan(\delta)x_1 + x_3 \geq 0\},$$

for $\delta = \arctan\left(\frac{2b}{1+b^2}\right) \in]0, \frac{\pi}{4}[$ and using Corollary 2.8 we obtain that the surface S_b lies on the region W_δ.

For the sake of simplicity, we denote S_δ the Scherk's type maximal surface that is asymptotic to $\partial(W_\delta)$.

2.6. **The convex hull property.** The objective of this subsection is to prove that a compact maximal surface, even with isolated singularities, satisfies the convex hull property, it is to say, the surface lies in the convex hull of its boundary and singularities.

Let τ_t denote the translation along the vector $(0, 0, t)$, $t \in \mathbb{R}$. In order to demonstrate the convex hull property we need the following version of the maximum principle for maximal surfaces with singularities. We would like to point out that the proof is inspired in the work [10].

Proposition 2.9. *Let $X : D \to \mathbb{L}^3$ be a maximal immersion with an isolated singular point in $q \in D$, $p = X(q)$ and S an embedded maximal surface (without singularities) in $\mathbb{L}^3$ with $p \in S$. Then we have:*

i) If $X(D)$ is above S, then p must be a downward pointing lightlike singularity.
ii) If $X(D)$ is below S, then p must be an upward pointing lightlike singularity.

Proof: We shall prove assertion i). Statement ii) can be proved in a similar way. Suppose first that p is a lightlike singularity but not a downward pointing lightlike singularity. Denote by Π the tangent plane to S at p that is a spacelike plane. From Lemma 2.2 we obtain a curve in $X(D)$ asymptotic to $C^-(p)$. Since S is asymptotic to Π in a neighborhood of p we deduce that there are points of $X(D)$ below S and this contradicts our assumptions.

Now assume p is a spacelike singularity. According to Lemma 2.4 we have that the tangent plane to $X(D)$ at p and the tangent plane to S at p coincide. Denote Π this plane and π the Lorentzian orthogonal projection on Π. Up to a Lorentzian isometry we can suppose that $p = (0, 0, 0)$ and $\Pi = \Pi_0$. Consider a disk Δ in Π centered at $(0, 0)$ such that S is the graph of a function h on Δ and $\Delta \subset \pi(X(D))$. Denote $M = X(V)$, where V is the connected component of $(\pi \circ X)^{-1}(\Delta)$ containing q. If $\partial(M) \cap h(\Delta) \neq \emptyset$, we have an interior regular point in $X(D) \cap S$. By applying the maximum principle we obtain $M = h(\Delta)$ and then $h(\Delta)$ must contain a spacelike singularity. Taking into account Remark 2.5 we infer that this is a contradiction. Suppose then that $\partial(M)$ is strictly above $h(\Delta)$. Then there exists $\theta \in]-\frac{\pi}{4}, 0[$ sufficiently small and f an hyperbolic rotation in $\mathbb{L}^3$ such that $f(\Pi) = \Pi_\theta$ and $f(\partial(M))$ remains strictly above $h(\Delta)$. Note that the tangent plane to the maximal surface $f(M)$ at $(0, 0, 0)$ is Π_θ and thus we can assert that there are points of $f(M)$ below $h(\Delta)$. Translating in the positive x_3-direction, we find a last contact point with $h(\Delta)$ which must be an interior regular point. As in the previous case by using the maximum principle we obtain that $h(\Delta)$ and $\tau_{t_0}(f(M))$ coincide for some $t_0 > 0$. From Lemma 2.6 we have that $\pi^{-1}(0, 0, 0) \cap \tau_{t_0}(f(M)) = (0, 0, t_0)$. Then we can deduce that $(0, 0, t_0) = (0, 0, 0)$ but this contradicts $t_0 > 0$. □

The above result allows us to prove the following proposition.

Proposition 2.10. *Let M be a compact maximal surface with isolated singularities. Then M lies in the convex hull of $\partial(M)$ and its general conelike singularities.*

Proof: Denote by A the set of general conelike singularities. If M is contained in a plane the result is obvious. Assume then that M is not flat and consider $v \in \mathbb{S}^2$ and $y \in \mathbb{R}^3$ such that $(\partial(M) \cup A) \subset H^+(y, v)$. We have to prove that $M \subset H^+(y, v)$ too.

We proceed by contradiction, and suppose that $M \cap (H^-(y, v) - H(y, v)) \neq \emptyset$. Let M' be a connected component of $M \cap H^-(y, v)$. Observe that M' does not contain general conelike singularities.

First, assume v is a timelike vector, it is to say $H(y, v)$ is a spacelike plane. Then, there exists an interior point $p \in M'$ such that M' is contained in the slab determinated by the parallel planes $H(p, v)$ and $H(y, v)$. Therefore, we can use Proposition 2.9 to infer that p is a regular point of M'. Thus, by using the maximum principle we find $M' = H(p, v)$ which is a contradiction.

Analogously, if v is either spacelike or lightlike, it is to say, if $H(y, v)$ is a timelike or a lightlike plane, we can deduce the existence of an interior point $p \in M'$ such that M' is contained in the slab determinated by the parallel planes $H(p, v)$ and $H(y, v)$. If p were a spacelike singularity, from Lemma 2.4 we have that $H(p, v)$ would be the tangent plane to

M' at p, contradicting that $|g(q)| < 1$. Assume now that p is a lightlike singularity. Up a Lorentzian isometry we can assume $p = (0, 0, 0)$ and

- $H(p, v) = \Sigma_\theta$ and $M' \subset \Sigma_\alpha^-$ if v is spacelike,
- $H(p, v) = \Pi_{\frac{\pi}{4}}$ and $M' \subset \Pi_{\frac{\pi}{4}}^+$ if v is lightlike.

By Lemma 2.6 we have that M' is in the exterior of $C((0, 0, 0))$. Consider now π the Lorentzian orthogonal projection on Π_0. It is easy to prove that the preceding conditions imply that $\pi(M') \subset (\Pi_0 - \{(0, y, 0) \mid y \in \mathbb{R}\})$ in a neighborhood of $(0, 0, 0)$. This contradicts Lemma 2.1. Therefore, since p is not a singular point we infer that $H(y, v)$ is the tangent plane to M' at p and so we obtain a contradiction with the fact that M is spacelike. □

Remark 2.11. *Observe that the Proposition 2.10 holds even if M cannot be extended to an open maximal surface M'.*

3. Maximal surfaces whose boundary is contained in a timelike plane

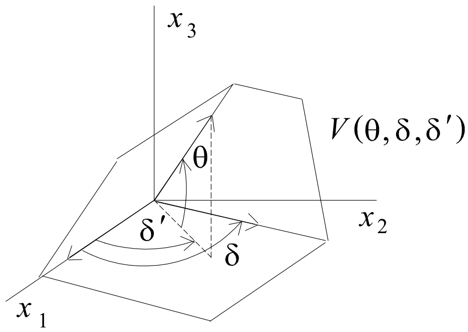

FIGURE 6

We recall that

$$\mathsf{C}^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 \le 0, x_3 \ge 0\} .$$

Furthermore, for $\theta \in]0, \frac{\pi}{4}[$ and $\delta, \delta' \in]0, \pi[$ we had denoted by $V(\theta, \delta, \delta')$ the convex region

$$V(\theta, \delta, \delta') = \Pi_0^+ \cap \Pi_\theta^- \cap H^+((0, 0, 0), (1, -\cot(\delta), \cot(\theta)(\cot(\delta) - \cot(\delta')))) .$$

We note that the region $V(\theta, \delta, \delta')$ is the convex hull of the half lines with origin in $(0, 0, 0)$ and directions $(1, 0, 0)$, $(\cot(\delta), 1, 0)$ and $(\cot(\delta'), 1, \tan(\theta))$ (see Fig. 6).

Proposition 3.1. *For any $\alpha \in] - \frac{\pi}{4}, \frac{\pi}{4}[$, there does not exist a connected properly immersed maximal surface M without downward pointing lightlike singularities in the interior, provided $M \subset \Pi_{\frac{\pi}{4}}^+ \cap \Sigma_\alpha^+$, $\partial(M) \subset \Sigma_\alpha$ and there is a point $p_0 \in \partial(M)$ verifying $x_3(p_0) \le x_3(p)$, for all $p \in \partial(M)$.*

Proof: First of all, observe that up an orthochronous hyperbolic rotation f_s of $\mathbb{L}^3$ we can assume $\alpha = 0$.

Suppose that there exists such an M and define $\hat{t} = x_3(p_0)$. For any $\theta \in [0, \frac{\pi}{4}[$ we can consider the set

$$\mathcal{I}_\theta = \{t \in [0, \hat{t}\,] \mid M \subset \Pi_{\theta t}^+\} .$$

Since $0 \in \mathcal{I}_\theta$ we have $\mathcal{I}_\theta \neq \emptyset$. We shall prove that $\mathcal{I}_\theta = [0, \hat{t}\,]$. If $\hat{t} = 0$ this fact is obvious. Thus, assume $\hat{t} > 0$. Clearly $\mathcal{I}_\theta$ is closed.

Let us see that it is open. Observe that if $t \in \mathcal{I}_\theta$ then $[0,t] \subset \mathcal{I}_\theta$. Now we claim that if $t \in \mathcal{I}_\theta \cap [0,\hat{t}[$ then there exists $\varepsilon > 0$ such that $[t, t+\varepsilon[\subset \mathcal{I}_\theta$. If not, we have two possibilities

- There is p, an interior point of M, in the plane $\Pi_{\theta t}$ or
- M is asymptotic at infinity to $\Pi_{\theta t}$.

Note that in the former case, from our assumptions on the singularities and assertion i) in Proposition 2.9 we deduce that the point p is not a singularity. Then, from the interior maximum principle follows that M and $\Pi_{\theta t}$ coincide. But it leads us to a contradiction with $x_3(p_0) = \hat{t} > t$. Now we shall demonstrate that the second case is not possible either.

Suppose that M is asymptotic to $\Pi_{\theta t}$. In this case we can assume $M \cap \Pi_{\theta t} = \emptyset$. If not, there exists an interior point of M in $\Pi_{\theta t}$ and then we may apply the previous argument. Consider f an orthochronous isometry of $\mathbb{L}^3$ such that $f(\Pi_{\theta t}) = \Pi_0$ and $f(\Pi^+_{\theta t}) = \Pi^+_0$. Denote $\widetilde{M} = f(M)$. Then we have a properly immersed maximal surface $\widetilde{M} \subset \Pi^+_0$, asymptotic to Π_0 with $\widetilde{M} \cap \Pi_0 = \emptyset$.

Since the immersion is proper and $(0,0,0) \notin \widetilde{M}$ we can find $\epsilon > 0$ sufficiently small so that $B((0,0,0),\epsilon) \cap \widetilde{M} = \emptyset$, where $B((0,0,0),\epsilon) = \{x \in \mathbb{R}^3 \mid \langle x, x\rangle_e < \epsilon^2\}$. Hence, there exists $\epsilon' \in]0,\epsilon[$ and $a_0 > 0$ small enough such that $\tau_{\epsilon'}(C_{a_0}) \subset B((0,0,0),\epsilon) \cup \Pi^-_0$. Now we define

$$A = \{a \in]0, a_0] \mid \tau_{\epsilon'}(C_a) \cap \widetilde{M} = \emptyset\} \,.$$

Clearly, $a_0 \in A$ and we can consider $a' = \mathrm{Infimum}(A)$. We claim $a' = 0$. Assume on the contrary that $a' > 0$. Then as $\tau_{\epsilon'}(C_a)$ and $\widetilde{M}$ do not have a contact at infinity, we infer that there is p an interior point of $\widetilde{M}$ in $\tau_{\epsilon'}(C_{a'})$. Furthermore, taking into account our assumptions and statement i) in Proposition 2.9, we can assert that p is a regular point of $\widetilde{M}$. Therefore, by applying the maximum principle, we obtain $\widetilde{M} = \tau_{\epsilon'}(C_{a'})$ which contradicts the fact that $\partial(\widetilde{M}) \subset \Sigma_0 \cap \Pi^+_0$.

Therefore $\mathcal{I}_\theta$ is open and $\mathcal{I}_\theta = [0, \hat{t}\,]$. As a consequence we have $M \subset \Pi^+_{\theta\hat{t}}$ for all $\theta \in [0, \frac{\pi}{4}[$ and thus $M \subset \Pi^+_{\frac{\pi}{4}\hat{t}}$.

Then we have two possibilities:

- p_0 is not a singular point in $\partial(M)$,
- p_0 is a singularity.

Assume that the first case occurs. Then, as $p_0 \in \partial(M) \cap \Pi_{\frac{\pi}{4}\hat{t}}$ and $M \subset \Pi^+_{\frac{\pi}{4}\hat{t}} \cap \Sigma^+_0$ the tangent plane at the point p_0 would be lightlike of timelike, which is a contradiction. In the second case, according to Lemma 2.6 we have that around p_0 the surface is in the exterior of $C(p_0)$, which contradicts again $M \subset \Pi^+_{\frac{\pi}{4}\hat{t}}$. □

Corollary 3.2. *There does not exist a connected properly immersed maximal surface M without downward pointing lightlike singularities in the interior such that $M \subset \mathsf{C}^+ \cap \Sigma^+_\alpha$ and $\partial(M) \subset \Sigma_\alpha$, for any $\alpha \in]-\frac{\pi}{4}, \frac{\pi}{4}[$.*

Proof: The corollary follows immediately from Proposition 3.1. □

Corollary 3.3. *There does not exist a connected properly immersed maximal surface M without downward pointing lightlike singularities in the interior such that $M \subset \mathsf{C}^+$ and $\partial(M)$ lies in the intersection of C^+ with a timelike plane P.*

Proof: Assume that there exists such a maximal surface and consider M' a connected component of $M - (M \cap P)$. Up to an elliptic rotation and a translation we can assume the timelike plane is the plane Σ_α, for $\alpha \in]-\frac{\pi}{4}, \frac{\pi}{4}[$, $M' \subset \mathsf{C}^+ \cap \Sigma_\alpha^+$ and $\partial(M') \subset \Sigma_\alpha$. An immediate application of Corollary 3.2 to M' leads to a contradiction. □

Theorem 3.4. *There does not exist a connected properly immersed maximal surface M without downward pointing lightlike singularities in the interior such that $M \subset W_\delta \cap \Sigma_\alpha^+$ and $\partial(M) \subset \Sigma_\alpha \cap \mathsf{C}^+$, for $\delta \in]0, \frac{\pi}{4}[, \alpha \in]-\frac{\pi}{4}, \frac{\pi}{4}[$.*

Proof: First of all we consider the isometry f_ε of $\mathbb{L}^3$ with $\tanh(\varepsilon) = \tan(\frac{\pi}{8})$. It is not difficult to see that $f_\varepsilon(M) \subset \Pi^+_{\frac{\pi}{8}} \cap W_{\delta'} \cap \Sigma^+_{\alpha'}$ and $\partial(f_\varepsilon(M)) \subset \Sigma_{\alpha'} \cap \mathsf{C}^+$, where

$$\tan(\alpha') = \frac{\tan(\alpha) + \tan(\frac{\pi}{8})}{\tan(\frac{\pi}{8})\tan(\alpha) + 1} ,$$

$$\tan(\delta') = \min\{\tan(\tfrac{\pi}{8}), \cosh(\varepsilon)\tan(\delta)(\tan(\tfrac{\pi}{8})\tan(\alpha) + 1)\} .$$

For the sake of simplicity of notation we consider $M \subset \Pi^+_{\frac{\pi}{8}} \cap W_\delta \cap \Sigma^+_\alpha$ and $\partial(M) \subset \Sigma_\alpha \cap \mathsf{C}^+$.

Furthermore, we claim that $(0,0,0) \notin \partial(M)$ and so $\partial(M) \subset \Pi_0^+ - \Pi_0$. If not, we deduce from Lemma 2.6 that around $(0,0,0)$ the maximal surface M is in the exterior of $C((0,0,0))$ but this contradicts $\partial(M) \subset \mathsf{C}^+$.

Consider now the Scherk's type maximal surface $S_{\frac{\delta}{2}}$ described in 2.5.3 that is asymptotic to the boundary of the region $W_{\frac{\delta}{2}}$. In the following we prove that $S_{\frac{\delta}{2}} \cap M = \emptyset$. It is clear that there exist $t_0 \in]-\infty, 0]$, $t_1 \in]0, \infty[$ such that $\tau_{t_0}(S_{\frac{\delta}{2}}) \cap M = \emptyset$ and $\tau_{t_1}(S_{\frac{\delta}{2}}) \cap M \neq \emptyset$. Therefore, we can define

$$\widehat{t} = \text{Infimum}\{t \in]t_0, \infty[\ | \ \tau_t(S_{\frac{\delta}{2}}) \cap M \neq \emptyset\} .$$

Suppose $\widehat{t} \leq 0$. Observe that, since $M \subset \Pi^+_{\frac{\pi}{8}} \cap W_\delta \cap \Sigma^+_\alpha$, $\partial(M) \subset \mathsf{C}^+ - \{(0,0,0)\}$ and $S_{\frac{\delta}{2}} \cap \mathsf{C}^+ = \{(0,0,0)\}$, then $\tau_{\widehat{t}}(S_{\frac{\delta}{2}})$ and M can have a contact point neither at infinity nor at the boundary. Hence there exists an interior point of M in $\tau_{\widehat{t}}(S_{\frac{\delta}{2}})$. Taking into account our assumptions on the singularities and statement i) in Proposition 2.9 we deduce that this point is not a singularity. Then, by applying the maximum principle we get that M and $\tau_{\widehat{t}}(S_{\frac{\delta}{2}})$ coincide which contradicts the hypothesis on $\partial(M)$. Thus $\widehat{t} > 0$ and $S_{\frac{\delta}{2}} \cap M = \emptyset$.

Consider now $S^\lambda_{\frac{\delta}{2}}$ the homothetic shrinking of $S_{\frac{\delta}{2}}$ by λ, $\lambda \geq 1$. We shall prove that $S^\lambda_{\frac{\delta}{2}} \cap M = \emptyset$ for all $\lambda \geq 1$. Suppose on the contrary that there exists $\lambda' \geq 1$ such that $S^{\lambda'}_{\frac{\delta}{2}} \cap M \neq \emptyset$. We denote by

$$\widehat{\lambda} = \text{Infimum}\{\lambda \in]1, \infty[\ | \ S^\lambda_{\frac{\delta}{2}} \cap M \neq \emptyset\} .$$

Moreover, it is clear that $S^{\widehat{\lambda}}_{\frac{\delta}{2}}$ and M do not contact either at infinity or at the boundary. Therefore there must exist an interior point of M in $S^{\widehat{\lambda}}_{\frac{\delta}{2}}$. Using again statement i) in Proposition 2.9 and our hypothesis on the singularities we deduce that this point is not a singularity and so by applying the maximum principle we obtain that $S^{\widehat{\lambda}}_{\frac{\delta}{2}}$ and M coincide. But this contradicts our assumptions on $\partial(M)$.

Thus $S^\lambda_{\frac{\delta}{2}} \cap M = \emptyset$ for all $\lambda \geq 1$. Taking into account that $S_{\frac{\delta}{2}}$ is asymptotic to $C^+((0,0,0))$ near the conelike singularity $(0,0,0)$, we deduce that $M \subset \mathsf{C}^+$ and the Corollary 3.2 finishes the proof. □

Theorem 3.5. *There does not exist a connected properly immersed maximal surface* M *without downward pointing lightlike singularities in the interior such that* $M \subset \Pi_\theta^+ \cap \Sigma_\alpha^+$ *and* $\partial(M) \subset \Sigma_\alpha \cap \mathsf{C}^+$*, for* $\theta, \alpha \in]-\frac{\pi}{4}, \frac{\pi}{4}[$.

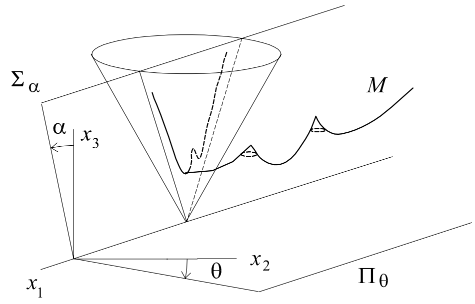

FIGURE 7

Proof: Suppose there exists such an M. We observe that if $\theta \leq 0$ or $\alpha < 0$ we can consider an orthochronous hyperbolic rotation f_s such that $f_s(M) \subset \Pi_{\theta'}^+ \cap \Sigma_{\alpha'}^+$ and $\partial(f_s(M)) \subset \Sigma_{\alpha'} \cap \mathsf{C}^+$ for some $\theta' \in]0, \frac{\pi}{4}[$, $\alpha' \in [0, \frac{\pi}{4}[$. As in the previous theorem, for the sake of simplicity of notation we assume $M \subset \Pi_\theta^+ \cap \Sigma_\alpha^+$ and $\partial(M) \subset \Sigma_\alpha \cap \mathsf{C}^+$, for $\theta \in]0, \frac{\pi}{4}[$, $\alpha \in [0, \frac{\pi}{4}$ [.

Since $\partial(M) \subset \Sigma_\alpha \cap \mathsf{C}^+$ we have that there exists $p_0 \in \partial(M)$ such that $x_3(p_0) \leq x_3(p)$ for all $p \in \partial(M)$. As in the preceding theorem it is easy to see that $p_0 \neq (0,0,0)$ and so $\lambda = x_3(p_0) > 0$. Then, reasoning as in Proposition 3.1 we can conclude $M \subset \Pi_{\theta\lambda}^+ \cap \Sigma_\alpha^+$.

Denote by $\widetilde{R}_\delta$ the Riemann type maximal example that results after applying an elliptic rotation of $\frac{\pi}{2}$ along the axis x_3 on R_δ for any $\delta \in]0, \frac{\pi}{4}[$. We assert that $M \cap \widetilde{R}_\delta = \emptyset$. Observe that we can consider $t_0 \leq 0$ and $t_1 \in \mathbb{R}$ such that $\tau_{t_0}(\widetilde{R}_\delta) \cap M = \emptyset$ and $\tau_{t_1}(\widetilde{R}_\delta) \cap M \neq \emptyset$. Now define

$$\widehat{t} = \text{Infimum}\{t \in]t_0, \infty[\ | \ \tau_t(\widetilde{R}_\delta) \cap M \neq \emptyset\} \ .$$

Suppose $\widehat{t} \leq 0$. Note that $\tau_{\widehat{t}}(\widetilde{R}_\delta)$ and M can have a contact point neither at infinity nor at the boundary. Hence there exists an interior point of M in $\tau_{\widehat{t}}(\widetilde{R}_\delta)$. Making use of statement i) in Proposition 2.9 and taking into account our assumptions on singularities, we deduce that the point is not a singularity. Therefore, by applying the maximum principle we get that $\tau_{\widehat{t}}(\widetilde{R}_\delta)$ and M coincide. But this contradicts our hypothesis on $\partial(M)$. Thus $\widehat{t} > 0$ and $\widetilde{R}_\delta \cap M = \emptyset$.

Consider now $\widetilde{R}_\delta^\lambda$ the homothetic shrinking of $\widetilde{R}_\delta$ by λ, $\lambda > 0$. Next we prove that $\widetilde{R}_\delta^\lambda \cap M = \emptyset$ for all $\lambda \geq 1$. Assume that there exists $\lambda' > 1$ such that $\widetilde{R}_\delta^{\lambda'} \cap M \neq \emptyset$. We denote by

$$\widehat{\lambda} = \text{Infimum}\{\lambda \in]1, \lambda'[\ | \ \widetilde{R}_\delta^\lambda \cap M \neq \emptyset\} \ .$$

Observe that $\widetilde{R}_\delta^{\lambda'}$ and M do not contact either at infinity or at the boundary for all $\lambda \geq 1$. Therefore there is an interior point of M in $\widetilde{R}_\delta^{\lambda'}$. Using again our assumptions on singularities and statement i) in Proposition 2.9 we deduce that the point is not a singularity. Then by applying the maximum principle we obtain that $\widetilde{R}_\delta^{\lambda'}$ and M coincide, which contradicts our hypothesis on $\partial(M)$. The same argument proves that $\widetilde{R}_\delta^\lambda \cap M = \emptyset$ for all $\lambda \leq 1$.

Analogously, considering $\widehat{R}_\delta$ the Riemann type maximal example that results after applying a rotation of $-\frac{\pi}{2}$ along the axis x_3 on R_δ for any $\delta \in]0, \frac{\pi}{4}[$, we obtain $\widehat{R}_\delta^\lambda \cap M = \emptyset$ for all $\lambda \in \mathbb{R}$.

Furthermore, it is not difficult to prove that

$$(\Pi_\theta^+ \cap \Sigma_\alpha^+) - \left(\bigcup_{\lambda\in\mathbb{R}} \widetilde{R}_\delta^\lambda \cup \bigcup_{\lambda\in\mathbb{R}} \widehat{R}_\delta^\lambda \right) \subset \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid -\tan(\delta)x_1 + x_2 + x_3 \geq 0\}$$
$$\cap \quad \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid \tan(\delta)x_1 + x_2 + x_3 \geq 0\}.$$

Taking this into account, we can assert

$M \subset (\Pi_\theta^+ \cap \Sigma_\alpha^+) \cap \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid -\tan(\delta)x_1 + x_2 + x_3 \geq 0, \tan(\delta)x_1 + x_2 + x_3 \geq 0\}$.

A direct computation shows that

$\Pi_\theta^+ \cap \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid -\tan(\delta)x_1 + x_2 + x_3 \geq 0, \tan(\delta)x_1 + x_2 + x_3 \geq 0\} \subset W_{\delta'}$,

where $\delta' \in]0, \frac{\pi}{4}[$ is given by

$$\tan(\delta') = \frac{\tan(\delta)\tan(\theta)}{1+\tan(\theta)} .$$

Then Theorem 3.4 concludes the proof. □

To finish this section, we analyze the case of maximal surfaces whose boundary is contained in a timelike plane but not necessarily in C^+. An easy argument allows us to prove:

Proposition 3.6. *There does not exist a connected properly immersed maximal surface M with at least a connected component of $\partial(M)$ contained in $\Sigma_\alpha^+ \cap \Sigma_{-\alpha}^+ \cap \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1 = 0\}$, where $\alpha \in]0, \frac{\pi}{4}[$.*

Proof: Let B a connected component of $\partial(M)$ satisfying

$$B \subset \Sigma_\alpha^+ \cap \Sigma_{-\alpha}^+ \cap \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1 = 0\} ,$$

where $\alpha \in]0, \frac{\pi}{4}[$. Thus, the function x_2 has at least a minimum on B. We note that the minimum can not be a singularity. Then, the tangent vector to the boundary at this point is vertical and therefore the tangent plane of the maximal surface at this point is timelike, which is contrary to our assumptions. □

Corollary 3.7. *There does not exist a connected properly immersed maximal surface M contained in $V(\theta,\delta,\delta')$ with $\partial(M)$ contained in a timelike plane.*

Proof: From the hypothesis it is not difficult to see that there exists an isometry of $\mathbb{L}^3$ such that sends the timelike plane to the plane $\{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1 = 0\}$ and in particular the image of $\partial(M)$ lies in $\Sigma_\alpha^+ \cap \Sigma_{-\alpha}^+ \cap \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1 = 0\}$ for some $\alpha \in]0, \frac{\pi}{4}[$. The result is then a consequence of Proposition 3.6.

4. Maximal surfaces whose boundary is contained in a spacelike plane

The purpose of this section is to obtain, using the maximum principle, other results about maximal surfaces whose boundary is contained in a spacelike plane but that can not be inferred from the Theorem quoted in Paragraph 2.4. We start with a result similar to Corollary 2.8.

Proposition 4.1. *Let M be a connected properly immersed maximal surface without downward pointing lightlike singularities in the interior such that $M \subset \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid 0 \leq x_3 \leq k\}$ and $\partial(M) \subset \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_3 = k\}$, for $k > 0$. Then M is a planar region.*

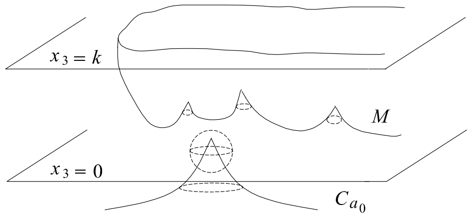

FIGURE 8

Proof: We proceed by contradiction. Assume that there exists $t \geq 0$ such that $M \subset \Pi^+_{0t}$ but $M \not\subset \Pi^+_{0t'}$ for any $t < t'$. Observe that $M \cap \Pi_{0t} = \emptyset$. If not, there exists p an interior point of M in Π_{0t}. Then, from statement i) in Proposition 2.9 and our assumptions on singularities we deduce that p is a regular point. But if p is regular, it follows from the maximum principle that $M = \Pi_{0t}$, contradicting the hypothesis on $\partial(M)$.

Since M is properly immersed and $M \cap \Pi_{0t} = \emptyset$ we can find $\varepsilon > 0$ such that $B((0,0,t),\varepsilon) \cap M = \emptyset$. Hence, there are constants $\varepsilon' \in]0,\varepsilon[$ and $a_0 > 0$ sufficiently small such that $\tau_{t+\varepsilon'}(C_{a_0}) \subset B((0,0,t),\varepsilon) \cup \Pi^-_{0t}$. Now we define

$$A = \{a \in]0,a_0] \mid \tau_{t+\varepsilon'}(C_a) \cap M = \emptyset\} .$$

Clearly, $a_0 \in A$ and we can consider $a' = \text{Infimum}(A)$. We claim $a' = 0$. Assume on the contrary that $a' > 0$. Then as $\tau_{t+\varepsilon'}(C_a)$ and M do not have a contact either at infinity or at the boundary, we infer that there is an interior point of M in $\tau_{t+\varepsilon'}(C_{a'})$. Taking into account statement i) in Proposition 2.9 and our assumptions on the singularities we infer that the interior point is not a singularity and then, by applying the interior maximum principle we obtain $M = \tau_{t+\varepsilon'}(C_{a'})$ which contradicts the hypothesis on $\partial(M)$.

Therefore, $a' = 0$ and so $M \subset \Pi^+_{0t+\varepsilon'}$ contradicting our assumption at the beginning of the proof. $\square$

Theorem 4.2. *Let M be a connected properly immersed maximal surface without upward pointing lightlike singularities in the interior such that $M \subset V(\theta,\delta,\delta')$ and $\partial(M) \subset \Pi_0$. Then M is a planar region.*

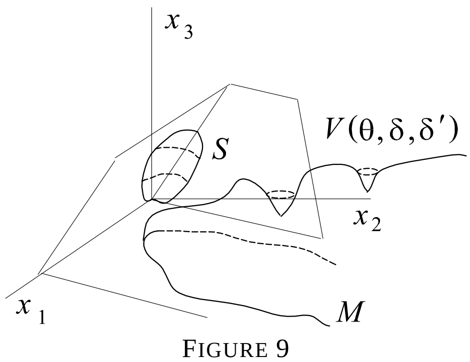

FIGURE 9

Proof: Up a translation we can assume that

$$M \cap (\Pi_\theta \cup H((0,0,0),(1,-\cot(\delta),\cot(\theta)(\cot(\delta)-\cot(\delta')))) = \emptyset .$$

Now we observe that

$$\tau_{-1}(\mathbb{H}^2_+) \cap \Pi_\theta \cap V(\theta, \delta, \delta') = \alpha_1 \ ,$$

$$\tau_{-1}(\mathbb{H}^2_+) \cap H((0,0,0), (1, -\cot(\delta), \cot(\theta)(\cot(\delta) - \cot(\delta')))) \cap V(\theta, \delta, \delta') = \alpha_2 \ ,$$

where α_1 and α_2 are two regular curves. Observe that the union of these curves is a continuous curve of $\tau_{-1}(\mathbb{H}^2_+)$ and that the tangent vectors to these curves at the point $(0,0,0)$ are contained in the plane Π_0 and are linearly independent.

Since $\tau_{-1}(\mathbb{H}^2_+)$ is a spacelike surface, it is well-known (see Theorem 4.1 in [1]) that there exists S a maximal surface (it is even a graph on the x_3-plane) spanned by the curve $\alpha_1 \cup \alpha_2$. Note that the tangent plane of S at $(0,0,0)$ is the plane Π_0. On the other hand, using Proposition 2.10 and Remark 2.11 we have that S is contained in the convex hull of its boundary and thus $S \subset V(\theta, \delta, \delta')$.

Now, we denote by S^λ the homothetic shrinking of S by λ, $\lambda > 0$. As M is properly immersed it is possible to find $\lambda_0 > 0$ such that $S^{\lambda_0} \cap M = \emptyset$. Next we prove that $S^\lambda \cap M = \emptyset$ for all $\lambda > 0$. Assume that there exists $\lambda' > 0$ such that $S^{\lambda'} \cap M \neq \emptyset$. We denote by

$$\widehat{\lambda} = \text{Infimum}\{\lambda \in]\lambda_0, \lambda'] \ \mid S^\lambda \cap M \neq \emptyset\} \ .$$

Observe that S^λ and M do not contact at the boundary for all $\lambda > 0$. Therefore there is an interior point of M in $S^{\widehat{\lambda}}$. It follows from assertion ii) in Proposition 2.9 and the conditions on the singularities that the contact point is not a singularity. Then, by applying the maximum principle we obtain $S^\lambda = M$ which contradicts the assumptions on $\partial(M)$.

Hence, taking into account that the tangent plane of S at $(0,0,0)$ is Π_0 we obtain

$$V(\theta, \delta, \delta') - \bigcup_{\lambda \in \mathbb{R}} S^\lambda \subset \Pi_0 \ ,$$

from which we deduce that $M \subset \Pi_0$. □

Corollary 4.3. *Let M be a connected properly immersed maximal surface without general conelike singularities in the interior such that $M \subset V(\theta, \delta, \delta')$ and $\partial(M)$ is contained in a spacelike plane. Then M is a planar region.*

Proof: Let Π be the spacelike plane such that $\partial(M) \subset \Pi$ and M' a connected component of $M - (M \cap \Pi)$. Denote by Π^+ the half space determined by Π such that $M' \subset \Pi^+$. Then, it is not difficult to see that there exists an isometry of $\mathbb{L}^3$, f, that verifies $f(\Pi) = \Pi_0$ and $f(V(\theta, \delta, \delta') \cap \Pi^+) \subset V(\widehat{\theta}, \widehat{\delta}, \widehat{\delta'})$, for some $\widehat{\theta}, \widehat{\delta}$ and $\widehat{\delta'}$. Therefore, the corollary follows from Theorem 4.2. □

On the other hand it is easy to prove the following result:

Proposition 4.4. *Let M be a connected properly immersed maximal surface without general conelike singularities in the interior such that $M \subset \mathsf{C}^+$ and $\partial(M)$ is contained in a spacelike plane. Then M is a planar region.*

Proof: Observe that our hypothesis imply that $\partial(M)$ is compact. We consider the intersection of M with all the timelike planes $H(y, v)$ such that $\partial(M) \subset H^+(y, v)$. By applying Corollary 3.3 to the connected components of M that are contained in $H^-(y, v)$ we obtain that $M \subset H^+(y, v)$ for all the timelike planes described above. Then M is also compact and Proposition 2.10 proves that M is a planar region. □

5. MAXIMAL SURFACES WHOSE BOUNDARY IS CONTAINED IN A LIGHTLIKE PLANE AND THE CONVEX HULL PROPERTY

As a consequence of the previous sections we deduce the following results for maximal surfaces whose boundary is contained in a lightlike plane.

Proposition 5.1. *There does not exist a connected properly immersed maximal surface M without general conelike singularities in the interior such that $M \subset V(\theta, \delta, \delta')$ and $\partial(M)$ is contained in a lightlike plane.*

Proof: Suppose there exists such an M. Let Π be the lightlike plane such that $\partial(M) \subset \Pi$. Then, we can consider the pencil of planes through the line $L = \Pi \cap \Pi_0$, it is to say the set of planes sharing the line L. Since M cannot be flat, it is possible to find a spacelike or timelike plane in the pencil that intersects M transversally. But Corollaries 4.3 and 3.7 leads to a contradiction in each case. □

Proposition 5.2. *There does not exist a connected properly immersed maximal surface M without general conelike singularities in the interior such that $M \subset \mathsf{C}^+$ and $\partial(M)$ is contained in a lightlike plane.*

Proof: This can be demonstrated as in the preceding proposition using now Proposition 4.4 and Corollary 3.3. □

As we saw in Paragraph 2.6, a compact maximal surface lies in the convex hull of its boundary and the set of its general conelike singularities. This is not true for non-compact maximal surfaces in general. However, Theorem 4.2 and Proposition 4.4 can be seen as a convex hull type property. We have proved that if certain conditions are satisfied then the surfaces lie in the convex hull of their boundary. In the remainder of the section, we use the results obtained in the previous sections to give a generalization of these results. More precisely we have

Proposition 5.3. *Any connected properly immersed maximal surface contained in $V(\theta, \delta, \delta')$ lies in the convex hull of its boundary and its general conelike singularities.*

Proof: Let M be a minimal surface satisfying the hypotheses of the proposition and denote by A the set of general conelike singularities of M. If M is contained in a plane the result is obvious. Assume then that M is not flat and consider $v \in \mathbb{S}^2$ and $y \in \mathbb{R}^3$ such that $(\partial(M) \cup A) \subset H^+(y, v)$. We have to prove that $M \subset H^+(y, v)$ too.

We proceed by contradiction, and suppose that $M \cap (H^-(y, v) - H(y, v)) \neq \emptyset$. Let M' be a connected component of $M \cap H^-(y, v)$.

First, assume v is a spacelike vector, it is to say $H(y, v)$ is a timelike plane. Then, Corollary 3.7 leads to a contradiction.

Suppose now that v is a timelike vector, it is to say $H(y, v)$ is a spacelike plane. In this case, Assumption " M' not flat " contradicts Corollary 4.3.

Finally, if the vector v is a lightlike then the plane $H(y, v)$ is also lightlike. Then Proposition 5.1 gives a contradiction. □

Proposition 5.4. *Any connected properly immersed maximal surface contained in C^+ lies in the convex hull of its boundary and its general conelike singularities.*

Proof: The proof can be obtained as in the above proposition using Corollary 3.3 and Propositions 5.2 and 4.4. □

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DEPARTAMENTO DE MATEMÁTICA,
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
UNIVERSIDADE DE SÃO PAULO,
RUA DO MATÃO, 1010, SÃO PAULO - SP,
CEP 05508–090
BRAZIL

DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA
UNIVERSIDAD DE GRANADA,
18071, GRANADA
SPAIN
E-mail address: rosab@ime.usp.br, lferrer@ugr.es