

Minimal surfaces with helicoidal ends

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1 Introduction and preliminaries

In the last few years the study of minimal surfaces with helicoidal ends has gathered new speed. This is particularly the merit of D. Hoffman, H. Karcher and F. Wei who constructed the first examples of this kind of surfaces different from the helicoid. One of the examples constructed by these authors was the so called singly-periodic genus-one helicoid, [5], that we will denote by \mathcal{H}_1 . The helicoid \mathcal{H}_1 belongs to a continuous family of twisted periodic helicoids with handles that converges to a genus one helicoid. The continuity of this family of surfaces and the subsequent embeddedness of the genus one helicoid was obtained by D. Hoffman, M. Weber and M. Wolf in [6, 14]. In [6] the authors made a careful study of \mathcal{H}_1 , with a new approach to the period problem associated to this surface. A thoughtful reading of this new approach establish a close relationship between \mathcal{H}_1 and an immersed minimal surface with planar ends, constructed by F.J. López, M. Ritoré and F. Wei in [12]. To be more precise, one observe that a fundamental piece of \mathcal{H}_1 can be obtained by deforming a fundamental piece of López-Ritoré-Wei's surface. The deformation consists of moving one of the connected components of the boundary of the surface following a vertical translation (see Fig. 1). López and the second author [10, 11] constructed

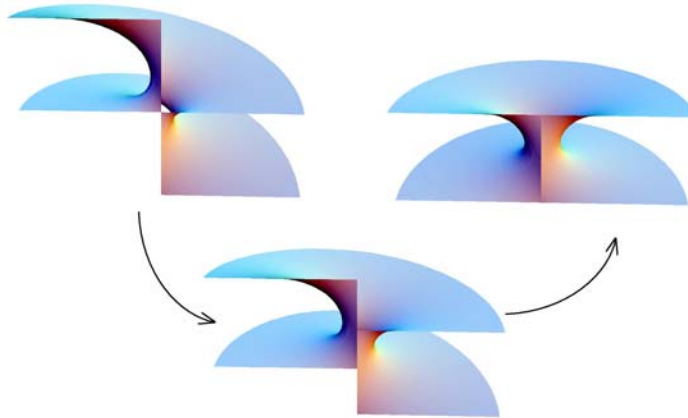


Figure 1: The deformation that connects a fundamental piece of \mathcal{H}_1 to the López-Ritoré-Wei's surface.

a family of minimal surfaces with planar ends based on López-Ritoré-Wei's, by modifying the angle between the horizontal boundary lines of the fundamental piece.

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Figure 2: A non-orientable example for angle $\pi/2$.

Therefore, it is quite natural to ask whether is possible to construct new examples of minimal surfaces with helicoidal ends from the López-Martín examples. The main objective of the present paper is to describe this general deformation that connects López-Martín examples with a family of complete minimal surfaces with helicoidal ends that contains \mathcal{H}_1 .

These new surfaces, except for \mathcal{H}_1 , are not embedded. However, if the angle between horizontal lines is $\frac{\pi}{n}$, with $n \in \mathbb{N}$, $n \geq 2$, then the only self-intersection of our surfaces occurs along the vertical axis. Furthermore, if n is even the examples are non-orientable in \mathbb{R}^3 .

A simple proof of the embeddedness of the fundamental piece of our surfaces is obtained from the study of the above mentioned deformation. In the particular case of angle π this argument provides another proof of the embeddedness for \mathcal{H}_1 (see Theorem 4). We also obtain the following uniqueness result

Any complete, periodic, minimal surface containing a vertical line, whose quotient by vertical translations has genus one, contains two parallel horizontal lines, has two helicoidal ends and total curvature -8π is \mathcal{H}_1 .

This result was essentially obtain in [5, Theorem 1]. Our contribution consists of giving a new approach to the proof of the uniqueness of the period problem (see Remark 4).

The paper is layed out as follows. In Sect. 2 we determine the underlying complex structure and the Weierstrass data of a minimal disk bounded by a polygonal curve as in Fig. 3. Thus we obtain a three-parameter family of Weierstrass data. Sect. 3 is devoted to prove that this family of meromorphic data must contain, for each angle, an example with $q_1^- = q_2^+$ (see Fig. 3). When the angle is a rational multiple of π , the surfaces obtained by successive Schwarz reflections about the straight lines are complete and proper in \mathbb{R}^3 . Finally, Sect. 4 contains the technical details about the geometric functions that appear in Sect. 3. This technical section is not necessary for the new embeddedness proof of \mathcal{H}_1 , because one can also use the methods of the recent paper by Hoffman-Weber-Wolf [6] to construct the deformation of \mathcal{H}_1 to the López-Ritoré-Wei surface.

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2 Determination of the Weierstrass representation

As we announced in Sect. 1, the fundamental piece of the minimal surfaces we wish to construct belongs to a family of minimal disks obtained moving one of the vertical segments upward in the López-Martín examples. Therefore, we are interested in the construction of properly embedded minimal disks whose boundary $\Gamma_{\beta\text{dh}}$ consists of the following configuration of straight lines:

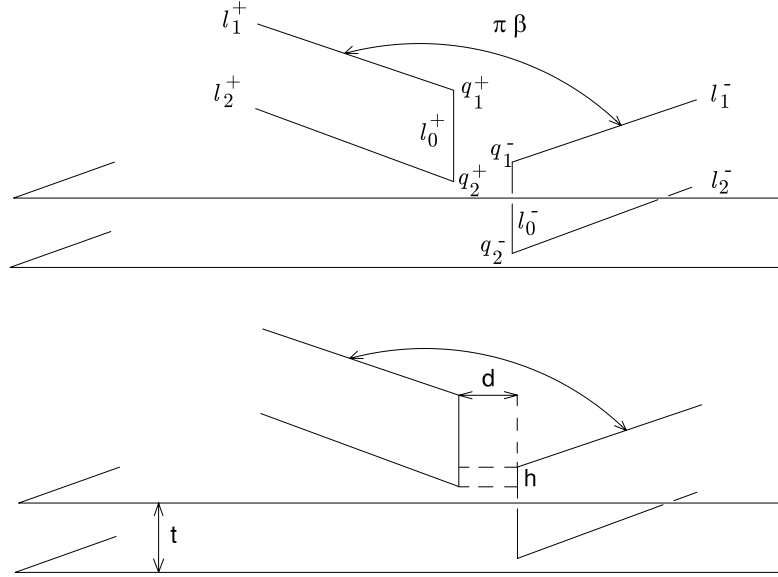


Figure 3: The curve $\Gamma_{\beta\text{dh}}$

Fix $\beta \in]0, 1]$, $d \geq 0$, $h \in \mathbb{R}$. Consider Π a half-plane in \mathbb{R}^3 and denote by ℓ the boundary line of Π . We can use a set of Cartesian coordinates such that ℓ is the x_3 -axis and the half-plane Π coincides with the half-plane $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sin\left(\frac{\pi\beta}{2}\right)x_1 + \cos\left(\frac{\pi\beta}{2}\right)\left(x_2 + \frac{d}{2}\right) = 0, x_2 \leq 0\}$.

Consider the points $q_2^- = (0, -\frac{d}{2}, 0)$ and $q_1^- = (0, -\frac{d}{2}, t - h)$. Denote by ℓ_0^- the segment $[q_1^-, q_2^-]$ and define ℓ_i^- as the horizontal half-line on Π starting at q_i^- , $i = 1, 2$. Finally, we label ℓ_i^+ and q_j^+ as the image of ℓ_i^- and q_j^- for $i = 0, 1, 2$, $j = 1, 2$, respectively, by a screw motion of axis ℓ , angle $\beta\pi$ and vector $(0, 0, t)$, $t \geq 0$. Notice that $d = \text{dist}(\ell_0^+, \ell_0^-)$ and $h = \langle \overrightarrow{q_1^- q_2^+}, (0, 0, 1) \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product of \mathbb{R}^3 .

Denote Π^\pm as the plane that contains $\ell_0^\pm \cup \ell_1^\pm \cup \ell_2^\pm$. Finally, we define

$$\Gamma_{\beta\text{dh}}^+ = \bigcup_{i=0}^2 (\ell_i^+), \quad \Gamma_{\beta\text{dh}}^- = \bigcup_{i=0}^2 (\ell_i^-), \quad \Gamma_{\beta\text{dh}} = \Gamma_{\beta\text{dh}}^+ \cup \Gamma_{\beta\text{dh}}^-.$$

Observe that if $t = 0$ we have exactly the family of examples given by López and Martín. Taking into account the geometric and topological properties of the surfaces we are starting at, we have to construct a properly immersed minimal surface $X = (X_1, X_2, X_3) : M \rightarrow \mathbb{R}^3$ with the following assumptions:

A.1 M is homeomorphic to the closed unit disk \overline{D} minus two boundary points E_1 and E_2 .

A.2 $X(\partial(M)) = \Gamma_{\beta\text{dh}}$.

- A.3 The surface has a symmetry respect to the horizontal line contained in the bisector plane of Π^+ and Π^- that passes through $\frac{1}{2}(q_1^- + q_2^+)$.
- A.4 If $\beta \in]0, 1[$ then $X(M)$ lies in the convex hull, $\mathcal{E}(\Gamma_{\beta\text{dh}})$, of $\Gamma_{\beta\text{dh}}$. If $\beta = 1$, $X(M)$ lies in one of the two half slab determined by the plane $\Pi^+ = \Pi^-$ and the planes orthogonal to ℓ containing ℓ_1^+ and ℓ_2^- .
- A.5 If $\ell_0^+ \cap \ell_0^- = \emptyset$, X is an embedding. In case $\ell_0^+ \cap \ell_0^- \neq \emptyset$, the maps $X|_{M-\partial(M)^+}$ and $X|_{M-\partial(M)^-}$ are injective, where $\partial(M)^+$ and $\partial(M)^-$ are the two connected components of $\partial(M)$.

Using the arguments presented in Sect. 2.1 of [4] with minor changes, it is easy to prove that the boundary has the following behavior:

Lemma 1 *Up to relabelings, any minimal immersion with A.1–A.5 satisfies $X(\partial(M)^+) = \Gamma_{\beta\text{dh}}^+$, $X(\partial(M)^-) = \Gamma_{\beta\text{dh}}^-$, $X^{-1}(\ell_1^+) \cup X^{-1}(\ell_1^-)$ diverges to E_1 and $X^{-1}(\ell_2^+) \cup X^{-1}(\ell_2^-)$ diverges to E_2 .*

Henceforth, g and Φ_3 will denote the Weierstrass data of a immersion $X : M \rightarrow \mathbb{R}^3$ satisfying the preceding five assumptions.

2.1 The underlying complex structure of M

The conformal type of M can be easily determined using a global result on conformal structure of properly immersed minimal surfaces by P. Collin, R. Kusner, W.H. Meeks and H. Rosenberg (see [2]). From Theorem 3.1 of [2] we obtain that M is parabolic and hence, taking into account the topological type of M , M is conformally equivalent to the closed unit disk \mathbb{D} minus two boundary points E_1 and E_2 , where the biholomorphism extends piecewise analytically to the boundary.

Next, we prove that the Gauss map and Weierstrass data extend continuously to the ends.

Lemma 2 *The 1-form Φ_3 extends meromorphically to the ends. Even more, it has simple poles at the ends, with imaginary residues.*

Proof: Let (U_i, z) be a coordinate chart satisfying that U_i is biholomorphic to the upper half disk $\mathbb{D}^+ = \{z \in \mathbb{C} : |z| < 1 \leq 1 + \text{Im}(z)\}$, $z(E_i) = 0$, $z(\gamma_i^+ \cap U_i) = \mathbb{D}^+ \cap \mathbb{R}^+$, and $z(\gamma_i^- \cap U_i) = \mathbb{D}^+ \cap \mathbb{R}^-$, $i = 1, 2$. We know that X_3 is a bounded harmonic function on U_i such that $X_3|_{\gamma_i^-} = C_i$, where C_i is a constant, and $X_3|_{\gamma_i^+} = C_i + t$. So, the function $X_3 + \frac{t}{\pi} \arg(z)$ can be continuously extended to E_i , $i = 1, 2$. Then the function $X_3 + iX_3^* - \frac{it}{\pi} \log(z)$ is a holomorphic function on U_i that extends to E_i , and so $\Phi_3 - \frac{it}{\pi} \frac{dz}{z}$ is a holomorphic 1-form on U_i . This concludes the proof. \square

Lemma 3 *The Gauss map also extends meromorphically and it is vertical at the ends.*

Proof: For the case $0 < \beta < 1$, the arguments used in [10, Theorem 3.12] also work in this setting. The case $\beta = 1$ can be treated as in [4, Proposition 3].

Since $X(M)$ is contained between two horizontal parallel planes, the second assertion follows. \square

The surface $X(M)$ can be extended by 180° rotation about its boundary lines, to a complete surface (without boundary) in \mathbb{R}^3 . Label $\tilde{X} : \tilde{M} \rightarrow \mathbb{R}^3$ the complete minimal immersion obtained in this way, where \tilde{M} is the corresponding Riemann surface without boundary, and let $(\tilde{g}, \tilde{\Phi}_3)$ denote its Weierstrass representation. Let \mathcal{S} be the isometry of \tilde{M} induced by the symmetry described in assumption A.3. We also denote \mathcal{S}_j^\pm , $j = 0, 1, 2$, as the isometry of \tilde{M} induced by the 180° rotation about the straight line containing ℓ_j^\pm . Observe that:

- $\tau_0 = \mathcal{S}_0^+ \circ \mathcal{S}_0^-$ is a horizontal translation whose translation vector, \vec{v}_0 , is orthogonal to ℓ_0^- and ℓ_0^+ . Furthermore, the length of \vec{v}_0 is $2d$;

- $\tau_1 = \mathcal{S}_1^+ \circ \mathcal{S}_2^+$ is a vertical translation of translation vector $\vec{v}_1 = (0, 0, 2(\mathbf{t} - \mathbf{h}))$;
- $\tau_2 = \mathcal{S}_2^+ \circ \mathcal{S}_2^-$ is a screw motion about the x_3 -axis of angle $2\beta\pi$ and translation vector $\vec{v}_2 = (0, 0, 2\mathbf{t})$.

Let \mathcal{G} be the subgroup of $\text{Iso}(\widetilde{M})$ generated by $\{\tau_0, \tau_1, \tau_2\}$. As \mathcal{G} acts freely and properly discontinuously on \widetilde{M} , the quotient $\mathcal{T} = \widetilde{M}/\mathcal{G}$ is a Riemann surface. Observe that $(d\widetilde{g})/\widetilde{g}$ and $\widetilde{\Phi}_3$ can be induced in the quotient. We label $(dg)/g$ and Φ_3 as the induced one-forms.

By Lemma 1 it is not hard to see that \mathcal{T} has the topology of a torus minus four points. Moreover, this torus consists of four copies of M : $M \cup \mathcal{S}_0^+(M) \cup \mathcal{S}_2^+(M) \cup (\mathcal{S}_2^+ \circ \mathcal{S}_0^+)(M)$ where the boundary is identified according to the symmetries in \mathcal{G} . The compact torus is labeled as $\overline{\mathcal{T}}$. Note that $(dg)/g$ and Φ_3 extend meromorphically to $\overline{\mathcal{T}}$.

Now, we need to determine the underlying complex structure of $\overline{\mathcal{T}}$. In order to do this, we will consider the symmetry $\mathcal{A} = \mathcal{S}_2^+ \circ \mathcal{S}_0^+$. This symmetry is induced by a 180° rotation in \mathbb{R}^3 about the orthogonal line to Π^+ passing through q_2^+ . \mathcal{A} is a holomorphic involution that can be induced in the quotient \mathcal{T} . It can be also extended to $\overline{\mathcal{T}}$. We label A as the induced involution in $\overline{\mathcal{T}}$. Note that A fixes exactly four points $\{[q_i^+], [q_i^-]\}_{i=1,2}$ where $[p]$ denotes the class in $\overline{\mathcal{T}}$ of a point $p \in \widetilde{M}$. Using Riemann-Hurwitz formula, it is straightforward to check that $\overline{\mathcal{T}}/\langle A \rangle$ is conformally equivalent to the Riemann sphere $\overline{\mathbb{C}}$. If we label $u : \overline{\mathcal{T}} \rightarrow \overline{\mathcal{T}}/\langle A \rangle$ the canonical projection, then (after an identification of $\overline{\mathcal{T}}/\langle A \rangle$ with $\overline{\mathbb{C}}$) u can be seen as an elliptic function on $\overline{\mathcal{T}}$. Furthermore, the branch points of u coincide with $\{[q_i^+], [q_i^-]\}_{i=1,2}$. Up to a Möbius transformation we can assume that $u([q_1^+]) = 0$, $u([q_2^-]) = \infty$ and $u([q_1^-]) = r \in \mathbb{R}^+$. Moreover, we label $s = u([q_2^+])$. Hence, $\overline{\mathcal{T}}$ is conformally equivalent to the algebraic elliptic curve $\{(u, v) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}} \mid v^2 = u(u-r)(u-s)\}$.

Let $S, S_0^+, S_2^+ : \overline{\mathcal{T}} \rightarrow \overline{\mathcal{T}}$ be the maps induced by $\mathcal{S}, \mathcal{S}_0^+$ and \mathcal{S}_2^+ , respectively. Observe that S_0^+ is an antiholomorphic involution that fixes the branch points of u . This means that $u \circ S_0^+ = \bar{u}$ and so $s \in \mathbb{R}$. On the other hand, S is a holomorphic involution verifying $S([q_1^+]) = [q_2^-]$ and $S([q_1^-]) = [q_2^+]$. Furthermore, as the fixed points of S are not at the boundary then $u \circ S = \frac{k}{u}$, where $k < 0$. Up to the change $u \mapsto \frac{u}{\sqrt{-k}}$, we can assume that $u \circ S = -\frac{1}{u}$. In particular $s = -\frac{1}{r}$. It is important to note that the existence of such symmetries implies that $\overline{\mathcal{T}}$ is rhombic. Summarizing,

- (a) $\overline{\mathcal{T}} \equiv \{(u, v) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}} \mid v^2 = u(u-r)(ru+1)\}$,
(b) $S(u, v) = (-\frac{1}{u}, \frac{v}{u^2})$, $S_0^+(u, v) = (\bar{u}, \bar{v})$, and $S_2^+(u, v) = (\bar{u}, -\bar{v})$.

Next, we will write our torus in a new way which is more suitable for our computations. Consider, for $\rho = 2 \operatorname{arccot}(r) \in]0, \pi[$, the following torus

$$\overline{\mathcal{N}} = \{(z, w) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}} \mid w^2 = z^4 + 1 - 2z^2 \cos \rho\} .$$

It is not difficult to see that the map $(u, v) = B(z, w) : \overline{\mathcal{N}} \rightarrow \overline{\mathcal{T}}$, given by

$$u(z, w) = \frac{(e^{-i\frac{\rho}{2}} + z) (-e^{i\frac{\rho}{2}} + z) \cos(\frac{\rho}{2}) + (1 - \sin(\frac{\rho}{2})) w}{\cos(\frac{\rho}{2}) w + (-e^{-i\frac{\rho}{2}} - z) (-e^{i\frac{\rho}{2}} + z) (1 - \sin(\frac{\rho}{2}))} ,$$

$$v(z, w) = \frac{\frac{-i}{2} (-i + e^{i\frac{\rho}{2}})^6 (e^{i\frac{\rho}{2}} - z) (1 + e^{i\frac{\rho}{2}} z) (1 + z^2)}{e^{4i\frac{\rho}{2}} (-3 + \cos(\rho) + 4 \sin(\frac{\rho}{2})) (\cos(\frac{\rho}{2}) w + (-1 + \sin(\frac{\rho}{2})) (-1 + z^2 - 2iz \sin(\frac{\rho}{2})))^2} ,$$

is a biholomorphism. It is worth mentioning that B does not map the brach points of z into the brach points of u . To be more precise, the brach points of the function $u \circ B$ in the torus $\overline{\mathcal{N}}$ are $z^{-1}(\{i, -i\})$.

Note that the torus $\overline{\mathcal{N}}$ is a two-fold covering of the rhombic torus $\{(x, y) \in \overline{\mathbb{C}}^2 \mid y^2 = x + \frac{1}{x} - 2 \cos \rho\}$. The covering map is given by $(z, w) \mapsto (z^2, \frac{w}{z})$ (see Fig. 4). This family of rhombic tori (depending on

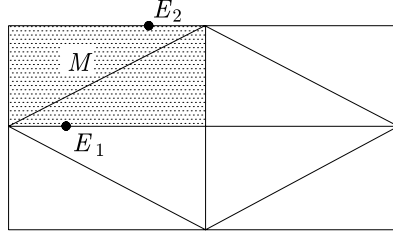


Figure 4: The torus $\overline{\mathcal{N}}$ and the fundamental piece M .

ρ) coincides with those used by Hoffman, Karcher and Wei to construct the singly-periodic helicoids in [5, 7].

We will continue denoting by S , S_0^+ and S_2^+ the symmetries on the new torus $\overline{\mathcal{N}}$. According to (b) the expressions of these symmetries on $\overline{\mathcal{N}}$ are given by

$$S(z, w) = \left(\frac{1}{z}, \frac{w}{z^2} \right), \quad S_0^+(z, w) = \left(\frac{1}{\bar{z}}, -\frac{\bar{w}}{\bar{z}^2} \right), \quad S_2^+(z, w) = (-\bar{z}, \bar{w}). \quad (1)$$

For the sake of brevity, when $z^4 + 1 - 2z^2 \cos \rho \in \mathbb{R}^+$ we denote:

$$z_+ = \left(z, +\sqrt{z^4 + 1 - 2z^2 \cos \rho} \right), \quad z_- = \left(z, -\sqrt{z^4 + 1 - 2z^2 \cos \rho} \right).$$

Now we need to identify the punctures in this torus. Note that S_2^+ fixes the ends and S_0^+ interchanges them. Taking (1) into account we deduce that the ends are $E = \{ia_+, -ia_-, \frac{i}{a_+}, -\frac{i}{a_-}\}$, where $a \in \mathbb{R}$. Up to relabeling, we can assume $a \in [0, 1[$. We denote $\mathcal{N} = \overline{\mathcal{N}} - E$.

On $\overline{\mathcal{C}}$ define the following set of curves:

$$s_0^+ = \left\{ e^{i\frac{t}{2}} \mid t \in [\rho, \pi] \right\}, \quad s_1^+ = \{\lambda i \mid \lambda \in]a, 1]\}, \quad s_2^+ = \{\lambda i \mid \lambda \in [1, \infty[\cup]-\infty, -\frac{1}{a}]\} \cup \{\infty\},$$

$$s_0^- = \left\{ e^{i\frac{t}{2}} \mid t \in [-\pi, -\rho] \right\}, \quad s_1^- = \{\lambda i \mid \lambda \in [-1, a]\}, \quad s_2^- = \{\lambda i \mid \lambda \in]-\frac{1}{a}, -1]\}.$$

Label $\gamma_i^+ = z^{-1}(s_i^+)$, $\gamma_i^- = z^{-1}(s_i^-)$, $i = 0, 1, 2$. In order to determine the domain in \mathcal{N} that corresponds to our fundamental piece we observe that the set of fixed points of S_0^+ is $\gamma_0^+ \cup \gamma_0^- \cup S_2^+(\gamma_0^+ \cup \gamma_0^-)$ and the set of fixed points of S_2^+ is $\gamma_1^+ \cup \gamma_1^- \cup \gamma_2^+ \cup \gamma_2^- \cup S_0^+(\gamma_1^+ \cup \gamma_1^- \cup \gamma_2^+ \cup \gamma_2^-)$. According to this we can identify M with the closure in \mathcal{N} of the connected component of $z^{-1}(\overline{\mathcal{C}} - (\bigcup_{i=0}^2 (s_i^+ \cup s_i^-)))$ containing the point $P_0 = 1_+$. We will label this domain as $M_{a, \rho}$.

Define γ^+ and γ^- by:

$$\gamma^+ = \bigcup_{i=0}^2 \gamma_i^+, \quad \gamma^- = \bigcup_{i=0}^2 \gamma_i^-.$$

It is clear that $\partial(M) = \gamma^+ \cup \gamma^-$. Furthermore, note that $z|_{\gamma_i^+}$ and $z|_{\gamma_i^-}$ are bijective maps onto s_i^+ and s_i^- , respectively, $i = 1, 2$. However, γ_0^+ and γ_0^- consist of two copies of s_0^+ and s_0^- , respectively.

2.2 The complex height differential

According to Lemma 2 the height differential Φ_3 has a simple pole with imaginary residue at the ends. As $\overline{\mathcal{N}}$ is a torus, then Φ_3 has as many zeros as poles. Moreover, it is easy to see that the symmetries act on Φ_3 as follows

$$S^* \Phi_3 = -\Phi_3, \quad (S_0^+)^* \Phi_3 = \overline{\Phi_3}, \quad (S_2^+)^* \Phi_3 = -\overline{\Phi_3}. \quad (2)$$

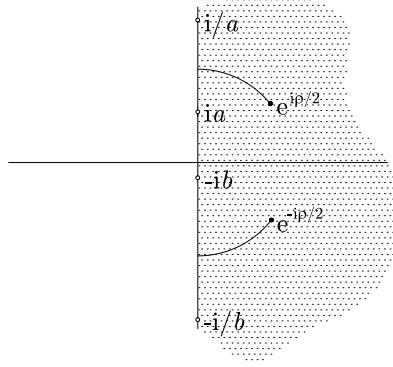


Figure 5: The z -projection of the domain $M_{a,\rho}$.

Both facts imply that Φ_3 has four zeros of order one and they have this form $V = \{-ib_+, ib_-, -\frac{i}{b_+}, \frac{i}{b_-}\}$, where $b \in]-1, 1[$. All this information lead us to:

$$\Phi_3 = \lambda \frac{w + c_2 iz dz}{w + c_1 iz w}, \quad (3)$$

where $\lambda > 0$ and

$$c_1 = \frac{1}{a} w(ia_+) = \frac{\sqrt{a^4 + 1 + 2a^2 \cos \rho}}{a}, c_2 = -\frac{1}{b} w(-ib_+) = -\frac{\sqrt{b^4 + 1 + 2b^2 \cos \rho}}{b}. \quad (4)$$

2.3 The Gauss map

The objective of this subsection is to find the expression of the Gauss map of our examples. From Lemma 3, we have that the normal vector at the ends must be vertical. Then we can assume $g(ia_+) = 0$. Furthermore, taking A.2 into account we deduce that the behavior of g in a neighborhood of ia_+ is given by $g(z) = z^\beta$. Since Φ_3 has zeros of order one at the points in V we deduce that g has at these points either simple poles or zeros of order one, in any case the points in V are points where the normal vector is vertical. To obtain more information about the normal vector at the point $-ib_+$ we need to return to our initial conditions.

In fact, using A.4 and the interior maximum principle one can prove that $X(M) \cap \{x_3 = 0\} = \ell_2^-$, $X(M) \cap \{x_3 = 2t - h\} = \ell_1^+$, $X(M) \cap \{\sin\left(\frac{\pi\beta}{2}\right) x_1 + \cos\left(\frac{\pi\beta}{2}\right) (x_2 + \frac{d}{2}) = 0\} = \Gamma_{\beta dh}^-$ and $X(M) \cap \{\sin\left(\frac{\pi\beta}{2}\right) x_1 - \cos\left(\frac{\pi\beta}{2}\right) (x_2 - \frac{d}{2}) = 0\} = \Gamma_{\beta dh}^+$. By studying the intersection of $X(M)$ with a horizontal plane containing ℓ_1^- , we deduce the existence of a point with vertical normal at ℓ_1^- . Clearly, this point must be $-ib_+$. Suppose that g has at the point $-ib_+$ a simple pole. Then, as $X(-ib_+) \in \ell_1^-$ and $g(ia_+) = 0$, should exists a point in ℓ_1^- , different from the point q_1^- , whose tangent plane is $\{\sin\left(\frac{\pi\beta}{2}\right) x_1 + \cos\left(\frac{\pi\beta}{2}\right) (x_2 + \frac{d}{2}) = 0\}$, but this contradicts what we have obtain previously. So, g has at $-ib_+$ a zero of order one.

Finally, the behavior of the symmetries at the points in $E \cup V$, allows us to deduce that distribution of poles and zeros of the multivalued function g using non-integral exponents must be as follows

ia_+	$-ia_-$	$\frac{i}{a_+}$	$-\frac{i}{a_-}$	$-ib_+$	ib_-	$-\frac{i}{b_+}$	$\frac{i}{b_-}$
0^β	0^β	∞^β	∞^β	0^1	0^1	∞^1	∞^1

Thus, $\frac{dg}{g}$ (recall that this is a meromorphic function on $\overline{\mathcal{N}}$), have only simple poles at the points in $E \cup V$ and the residues of $\frac{dg}{g}$ at these points are given in the following table

p	ia_+	$-ia_-$	$\frac{i}{a_+}$	$-\frac{i}{a_-}$	$-ib_+$	ib_-	$-\frac{i}{b_+}$	$\frac{i}{b_-}$
Residue($\frac{dg}{g}, p$)	β	β	$-\beta$	$-\beta$	1	1	-1	-1

Furthermore, from the definitions of the symmetries in (1) we have

$$S^*(\Phi_1 - i\Phi_2) = e^{-i\beta\pi}(\Phi_1 + i\Phi_2), (S_0^+)^*(\Phi_1 - i\Phi_2) = -(\overline{\Phi_1 - i\Phi_2}), (S_2^+)^*(\Phi_1 - i\Phi_2) = e^{-2i\beta\pi}(\overline{\Phi_1 + i\Phi_2}).$$

Now, taking into account (2) and that $g = \frac{\Phi_3}{\Phi_1 - i\Phi_2}$ we infer

$$S^*\left(\frac{dg}{g}\right) = -\frac{dg}{g}, (S_0^+)^*\left(\frac{dg}{g}\right) = -\overline{\left(\frac{dg}{g}\right)}, (S_2^+)^*\left(\frac{dg}{g}\right) = \overline{\left(\frac{dg}{g}\right)}. \quad (5)$$

All the facts above presented imply that $\frac{dg}{g}$ can be written as

$$\frac{dg}{g} = \eta_1 + a_3\eta_2,$$

where $a_3 \in \mathbb{R}$ and

$$\eta_1 = \frac{i\beta}{a_1} \frac{dz}{w + c_1 iz} + \frac{i}{a_2} \frac{dz}{w + c_2 iz}, \quad \eta_2 = i \frac{dz}{w},$$

$$a_1 = \text{Residue}\left(\frac{dz}{w + c_1 iz}, ia_+\right) = \frac{a\sqrt{a^4 + 1 + 2a^2 \cos(\rho)}}{1 - a^4}, \quad (6)$$

$$a_2 = \text{Residue}\left(\frac{dz}{w + c_2 iz}, -ib_+\right) = -\frac{b\sqrt{b^4 + 1 + 2b^2 \cos(\rho)}}{1 - b^4}. \quad (7)$$

2.4 Construction of the Plateau solution

This section is devoted to show that any Plateau solution is given by Weierstrass data on our punctured torus given by real parameters a, b and β , where b will be uniquely and analytically determined by ρ, a and β .

First of all, we must prove that there exist $b(a, \rho, \beta)$ and $a_3(a, \rho, \beta)$ so that the Gauss map $g = \exp\left(\int_{1_+} \frac{dg}{g}\right)$ verifies on M the other required conditions. Taking into account the symmetry S it is sufficient to pay attention to $\{(z, w) \in M \mid \text{Re}(z) \geq 0\}$. To translate these conditions into equations we need some terminology.

The curve γ_0^+ consists of two copies, δ_1 and δ_2 , of s_0^+ . We can assume that $\delta_1(t)$ and $\delta_2(t)$ are the two lifts to M of the curve $e^{i\frac{t}{2}}$, $t \in [\rho, \pi]$, in the z -plane, satisfying $\delta_1(\pi) \in \gamma_1^+$ and $\delta_2(\pi) \in \gamma_2^+$, respectively. Let $\delta(t)$ be the lift to M of the curve $e^{i\frac{t}{2}}$, $t \in [0, \rho]$, in the z -plane. Observe $\delta(0) = 1_+$ and $\delta(\rho) = \delta_1(\rho) = \delta_2(\rho)$. With this notation we have:

- As we wish that $|g| = 1$ on γ_0^+ we have to impose

$$\text{Re}\left(\int_{\delta} \frac{dg}{g}\right) = 0. \quad (8)$$

- We also have to impose that $g(i_+) = g(i_-)$. Then we have the condition

$$\text{Im}\left(\int_{\tilde{\delta}} \frac{dg}{g}\right) = 0, \quad (9)$$

where $\tilde{\delta} = -\delta_1 + \delta_2$, with $-\delta_1(t) = \delta_1(\pi - t)$.

On \mathcal{N} we consider the curves γ_1 and γ_2 below described:

- γ_1 is the curve in \mathcal{N} given by $\gamma_1 = \alpha_1 - (S_0^+)_*(\alpha_1)$, where $\alpha_1 = -S_*(\delta) + \delta$.
- γ_2 is the curve in \mathcal{N} given by $\gamma_2 = -\alpha_2 + \alpha_3$, where $\alpha_2 = \delta_1 - (S_2^+)_*(\delta_1)$ and $\alpha_3 = \delta_2 - (S_2^+)_*(\delta_2)$.

Observe that $\{\gamma_1, \gamma_2\}$ is a canonical homology base of $\overline{\mathcal{N}}$ and $\gamma_2 = \tilde{\delta} - (S_2^+)_*(\tilde{\delta})$. Thus, from (5) we have

$$\begin{aligned} \int_{\gamma_1} \frac{dg}{g} &= \int_{\alpha_1} \frac{dg}{g} + \overline{\int_{\alpha_1} \frac{dg}{g}} = 2 \operatorname{Re} \left(\int_{-S_*(\delta)+\delta} \frac{dg}{g} \right) = 4 \operatorname{Re} \left(\int_{\delta} \frac{dg}{g} \right), \\ \int_{\gamma_2} \frac{dg}{g} &= \int_{\tilde{\delta}} \frac{dg}{g} - \overline{\int_{\tilde{\delta}} \frac{dg}{g}} = 2i \operatorname{Im} \left(\int_{\tilde{\delta}} \frac{dg}{g} \right). \end{aligned}$$

Therefore the equations (8) and (9) are equivalent to the system:

$$\int_{\gamma_i} \frac{dg}{g} = \int_{\gamma_i} \eta_1 + a_3 \int_{\gamma_i} \eta_2 = 0 \quad \text{for } i = 1, 2. \quad (10)$$

In order to solve (10) it is sufficient to prove that there exists $b(a, \rho, \beta)$ such that

$$\det \begin{pmatrix} \int_{\gamma_1} \eta_1 & \int_{\gamma_1} \eta_2 \\ \int_{\gamma_2} \eta_1 & \int_{\gamma_2} \eta_2 \end{pmatrix} = \int_{\gamma_1} \eta_1 \int_{\gamma_2} \eta_2 - \int_{\gamma_1} \eta_2 \int_{\gamma_2} \eta_1 = 0. \quad (11)$$

Applying the bilinear relations of Riemann to the 1-forms η_1 and η_2 we obtain:

$$\int_{\gamma_1} \eta_1 \int_{\gamma_2} \eta_2 - \int_{\gamma_1} \eta_2 \int_{\gamma_2} \eta_1 = 2\pi i \sum_{p \in E \cup V} f(p) \operatorname{Residue}(\eta_1, p), \quad (12)$$

where f is a primitive of η_2 on the simply connected domain of \mathcal{N} , Ω , obtained by removing the curves γ_1 and γ_2 (see [3]). We choose f so that $f(e^{i\frac{\rho}{2}}) = 0$. From (1) we obtain that $(S_0^+)^*(\eta_2) = -\overline{\eta_2}$. Then we get

$$f(S_0^+(p)) = \int_{e^{i\frac{\rho}{2}}}^{S_0^+(p)} \eta_2 = \int_{e^{i\frac{\rho}{2}}}^p (S_0^+)^*(\eta_2) = -\overline{\int_{e^{i\frac{\rho}{2}}}^p \eta_2} = -\overline{f(p)}.$$

On the other hand we can consider the holomorphic transformation on \mathcal{N} given by $T(z, w) = (z, -w)$. As $T^*(\eta_2) = -\eta_2$ we also obtain

$$f(T(p)) = \int_{e^{i\frac{\rho}{2}}}^{T(p)} \eta_2 = \int_{e^{i\frac{\rho}{2}}}^p T^*(\eta_2) = -\int_{e^{i\frac{\rho}{2}}}^p \eta_2 = -f(p).$$

Notice that in the above computations we have used that $S_0^+(\Omega) = T(\Omega) = \Omega$.

Therefore, the equality in (12) can be written as

$$\begin{aligned} \int_{\gamma_1} \eta_1 \int_{\gamma_2} \eta_2 - \int_{\gamma_1} \eta_2 \int_{\gamma_2} \eta_1 &= 4\pi i \operatorname{Re} (\beta(f(ia_+) - f(-ia_+)) + f(-ib_+) - f(ib_+)) = \\ 4\pi i \operatorname{Re} \left(\beta \int_{-ia_+}^{ia_+} \eta_2 - \int_{-ib_+}^{ib_+} \eta_2 \right) &= -8\pi i \left(\beta \int_0^a \frac{dt}{\sqrt{t^4 + 1 + 2t^2 \cos \rho}} - \int_0^b \frac{dt}{\sqrt{t^4 + 1 + 2t^2 \cos \rho}} \right) \end{aligned} \quad (13)$$

Taking into account (13), the equation (11) is satisfied if $F(a, b, \rho, \beta) = 0$, where

$$F(a, b, \rho, \beta) = \beta \int_0^a \frac{dt}{\sqrt{t^4 + 1 + 2t^2 \cos \rho}} - \int_0^b \frac{dt}{\sqrt{t^4 + 1 + 2t^2 \cos \rho}}. \quad (14)$$

Since $0 < \beta \leq 1$ we have that $F(a, a, \rho, \beta) \leq 0$. Moreover, it is easy to check that $\lim_{b \rightarrow 0} F(a, b, \rho, \beta) \geq 0$ and $\frac{\partial F}{\partial b}(a, b, \rho, \beta) < 0$. From the above settings, we infer that there exists a unique $b(a, \rho, \beta) \in [0, a]$ such that $F(a, b(a, \rho, \beta), \rho, \beta) = 0$ and therefore verifies equation (11).

Moreover, one can give an explicit, but rather long, formula for the function b in terms of elliptic functions and so b is a real analytic function. Consequently, there exists $a_3(a, \rho, \beta) \in \mathbb{R}$ solution of the system (10). From equation (8) we obtain

$$a_3(a, \rho, \beta) = -\frac{\operatorname{Re} \int_{\delta}^{\rho} \eta_1}{\operatorname{Re} \int_{\delta}^{\rho} \eta_2} = -\frac{\int_0^{\rho} \left(\frac{\beta}{a_1} \frac{1}{a^2 + \frac{1}{a^2} + 2 \cos(t)} + \frac{1}{a_2} \frac{1}{b^2 + \frac{1}{b^2} + 2 \cos(t)} \right) \sqrt{2(\cos(t) - \cos(\rho))} dt}{\int_0^{\rho} \frac{1}{\sqrt{2(\cos(t) - \cos(\rho))}} dt}. \quad (15)$$

Analogously, from (9) we obtain an alternative definition for a_3 given by

$$a_3(a, \rho, \beta) = -\frac{\operatorname{Im} \int_{\tilde{\delta}}^{\rho} \eta_1}{\operatorname{Im} \int_{\tilde{\delta}}^{\rho} \eta_2} = \frac{\int_{\rho}^{\pi} \left(\frac{\beta}{a_1} \frac{1}{a^2 + \frac{1}{a^2} + 2 \cos(t)} + \frac{1}{a_2} \frac{1}{b^2 + \frac{1}{b^2} + 2 \cos(t)} \right) \sqrt{2(\cos(\rho) - \cos(t))} dt}{\int_{\rho}^{\pi} \frac{1}{\sqrt{2(\cos(\rho) - \cos(t))}} dt}. \quad (16)$$

As a consequence of the above arguments, we can state the following theorem:

Theorem 1 *A minimal immersion satisfying assumptions A.1–A.5 has Weierstrass data of the form:*

$$g = \exp \left(\int_{1_+} \frac{i\beta}{a_1} \frac{dz}{w + c_1 iz} + \frac{i}{a_2} \frac{dz}{w + c_2 iz} + ia_3 \frac{dz}{w} \right), \quad \Phi_3 = \lambda \frac{w + c_2 iz}{w + c_1 iz} \frac{dz}{w}, \quad (17)$$

on the disk $M_{a,\rho}$ defined at the end of paragraph 2.1, where $a \in [0, 1[$, $\rho \in]0, \pi[$, $\beta \in]0, 1[$, b is the function satisfying $F(a, b(a, \rho, \beta), \rho, \beta) = 0$ with F defined in (14), a_3 is given by either (15) or (16), $\lambda > 0$ and c_i, a_i are given in (4), (6) and (7).

Our geometric assumptions of the minimal disk have led us to an explicit tree-parameter-family of Weierstrass data. Now we want to check that these meromorphic data solve our Plateau problem.

Theorem 2 *For $a \in [0, 1[$, $\rho \in]0, \pi[$ and $\beta \in]0, 1[$ the Weierstrass data (17) define a proper minimal immersion $X_{a,\rho,\beta} : M_{a,\rho} \rightarrow \mathbb{R}^3$ that fulfills the assumptions A.1–A.4. Furthermore, $X|_{\gamma^+}$ and $X|_{\gamma^-}$ are injective.*

Proof: For the sake of brevity, throughout the proof we denote $M = M_{a,\rho}$ and $X = X_{a,\rho,\beta}$. Clearly, a surface represented by the data as given in (17) on M satisfies A.1 and A.3. To see A.2 we parametrize the curves γ_1^{\pm} as $\gamma_1^+(t) = it_+$, $t \in]a, 1[$ and $\gamma_1^-(t) = it_+$, $t \in [-1, a[$. At this point, we are interested in compute $g(\pm i_+)$. In order to do this, we consider the curve $\gamma = \alpha_1 + \alpha_2 - (S_2^+)_*(\alpha_1) - S_*(\alpha_2)$, where α_i , $i = 1, 2$ are the curves defined in paragraph 2.3. Observe that the curve $\gamma - S_*(\gamma_2)$ bounds a disk in $\overline{\mathcal{N}}$ whose projection in the z -plane is the unit disk. So, we have

$$\int_{\gamma - S_*(\gamma_2)} \frac{dg}{g} = 2\pi i \left(\operatorname{Residue} \left(\frac{dg}{g}, ia_+ \right) + \operatorname{Residue} \left(\frac{dg}{g}, -ib_+ \right) \right) = 2\pi i (\beta + 1). \quad (18)$$

Using again the notation of paragraph 2.3 and taking into account (5), $i \frac{dg}{g} \left(\frac{d\delta_1}{dt} \right) \in \mathbb{R}$ and that b and a_3 have been chosen to satisfy (10) (or equivalently equations (8) and (9)), we also obtain

$$\int_{\gamma - S_*(\gamma_2)} \frac{dg}{g} = \int_{\gamma + \gamma_2} \frac{dg}{g} = \int_{\gamma} \frac{dg}{g} = 2 \int_{\delta - \delta_1 - (S_2^+)_*(\delta - \delta_1)} \frac{dg}{g} = 2 \int_{\delta - \delta_1} \frac{dg}{g} - \overline{\frac{dg}{g}} = 4 \int_{\delta - \delta_1} \frac{dg}{g}. \quad (19)$$

Then, from (18) and (19) we get

$$g(i_+) = \exp\left(\int_{\delta-\delta_1} \frac{dg}{g}\right) = e^{\frac{i(\beta+1)}{2}}.$$

Analogously, we can prove that $g(-i_+) = e^{-\frac{i(\beta+1)}{2}}$. By using these computations and (17), it is not difficult to see that on the curves γ_1^\pm we have

$$g(\gamma_1^\pm(t)) = e^{\pm \frac{i(\beta+1)}{2}} \psi(t), \quad \Phi_3\left(\frac{d\gamma_1^\pm}{dt}\right) = i\lambda \varphi(t) dt,$$

where $\psi : [-1, -b[\cup] - b, a[\cup] a, 1] \rightarrow \mathbb{R}$ and $\varphi : [-1, a[\cup] a, 1] \rightarrow \mathbb{R}$ have the following properties

- $\psi > 0$ and $\varphi < 0$ on $]a, 1]$,
- $\psi > 0$ on $[-1, -b[$, $\psi < 0$ on $] - b, a[$, $\varphi < 0$ on $[-1, -b[$, $\varphi > 0$ on $] - b, -a[$, $\varphi(b) = 0$, $\lim_{t \rightarrow -b^-} \psi = +\infty$, $\lim_{t \rightarrow -b^+} \psi = -\infty$ and $\psi\varphi$ is well defined on $[-1, a[$ and in this interval $\psi\varphi < 0$.
- $\lim_{t \rightarrow a^+} \varphi = -\infty$, $\lim_{t \rightarrow a^-} \varphi = \infty$, $\lim_{t \rightarrow a^+} \psi = \infty$ and $\lim_{t \rightarrow a^-} \psi = -\infty$.

Hence we have

$$\begin{aligned} \Phi_1\left(\frac{d\gamma_1^\pm}{dt}\right) &= \frac{i\lambda}{2} \left(e^{\mp \frac{i(\beta+1)}{2}} \frac{1}{\psi(t)} - e^{\pm \frac{i(\beta+1)}{2}} \psi(t) \right) \varphi(t), \\ \Phi_2\left(\frac{d\gamma_1^\pm}{dt}\right) &= -\frac{\lambda}{2} \left(e^{\mp \frac{i(\beta+1)}{2}} \frac{1}{\psi(t)} + e^{\pm \frac{i(\beta+1)}{2}} \psi(t) \right) \varphi(t). \end{aligned}$$

The expressions for Φ_i , $i = 1, 2, 3$, imply that ℓ_1^\pm is a half-line contained in a straight line $\{x_3 = k^\pm, \mp \sin\left(\frac{\beta\pi}{2}\right)x_1 + \cos\left(\frac{\beta\pi}{2}\right)x_2 = K^\pm\}$, for suitable $k^\pm, K^\pm \in \mathbb{R}$. Observe that these straight lines meet the straight line $\{x_3 = k^\pm, x_1 = 0\}$ at an angle $\frac{\beta\pi}{2}$. Note also that

$$\begin{aligned} \operatorname{Re}\left(\Phi_1\left(\frac{d\gamma_1^\pm}{dt}\right)\right) &= \pm \frac{\lambda}{2} \sin\left(\frac{(\beta+1)\pi}{2}\right) \left(\frac{1}{\psi(t)} + \psi(t)\right) \varphi(t), \\ \operatorname{Re}\left(\Phi_2\left(\frac{d\gamma_1^\pm}{dt}\right)\right) &= -\frac{\lambda}{2} \cos\left(\frac{(\beta+1)\pi}{2}\right) \left(\frac{1}{\psi(t)} + \psi(t)\right) \varphi(t). \end{aligned}$$

Taking into account the properties of the functions ψ and φ and that $\beta \in]0, 1[$, we deduce that $\operatorname{Re}\left(\Phi_2\left(\frac{d\gamma_1^\pm}{dt}\right)\right) < 0$ and so $X|_{\gamma_1^\pm}$ is injective.

Moreover, it is clear that $x_2|_{\ell_1^\pm}$ diverges to $\pm\infty$. If $\beta \in]0, 1[$, then $x_1|_{\ell_1^\pm}$ diverges to $+\infty$, while $\beta = 1$ implies that $x_1|_{\ell_1^\pm}$ is constant.

On the other hand, it is straightforward to check that $\Phi_3\left(\frac{d\delta_i}{dt}\right) \in \mathbb{R}$ and $i\frac{dg}{g}\left(\frac{d\delta_i}{dt}\right) \in \mathbb{R}$, for $i = 1, 2$. Moreover, since b and a_3 had been selected to satisfy the system (10) we have that the Weierstrass data verify equation (8). All these facts imply that

$$\operatorname{Re}\left(\Phi_j\left(\frac{d\delta_i}{dt}\right)\right) = 0, \quad i = 1, 2, \quad j = 1, 2,$$

and so ℓ_0^+ is a vertical segment.

Furthermore, note that

$$\Phi_3\left(\frac{d\delta_1}{dt}\right) \in \mathbb{R}^+, \quad \Phi_3\left(\frac{d\delta_2}{dt}\right) \in \mathbb{R}^-,$$

and this implies that $X_3(i_+) - X_3(i_-) > 0$ and $X|_{\gamma_0^+}$ is injective. On the other hand, since $i_+ \in \gamma_1^+$ and $0 \in \gamma_1^-$ we have

$$t = k^+ - k^- = X_3(i_+) - X_3(0_+) = \operatorname{Re} \left(\int_{0_+}^{i_+} \Phi_3 \right) = i \pi \operatorname{Residue}(\Phi_3, ia_+) = \pi \lambda R, \quad (20)$$

where $R = \frac{a^2}{1-a^4} \left(\sqrt{a^2 + \frac{1}{a^2} + 2 \cos(\rho)} + \sqrt{b^2 + \frac{1}{b^2} + 2 \cos(\rho)} \right)$. Thus, $t \geq 0$. Taking into account that symmetry S given in (1) verifies $S(\gamma_1^+) = \gamma_2^-$, $S(\gamma_1^-) = \gamma_2^+$ and $S(\gamma_0^+) = \gamma_0^-$, it is not hard to conclude A.2 and that $X|_{\gamma^+}$ and $X|_{\gamma^-}$ are injective.

Next, we prove that X is a proper immersion that satisfies A.4. In order to do this we recall that $\frac{dg}{g}$ has a simple pole at ia_+ with residue β . Therefore, one has that the behavior of g in a neighborhood of ia_+ in M is given by

$$g = (z - ia)^\beta H_0(z),$$

where H_0 is a non-zero holomorphic function in that neighborhood and the branch of $(z - ia)^\beta$ satisfies $1^\beta = 1$. On the other hand, since Φ_3 has a simple pole at ia_+ one has that in a neighborhood of ia_+ in M

$$\Phi_3 = \left(\frac{-i \lambda R}{z - ia} + H_1(z) \right) dz,$$

where H_1 is a holomorphic function in that neighborhood. Hence we deduce that

$$\frac{1}{g} \Phi_3(z) = \frac{1}{(z - ia)^{1+\beta}} H_2(z) dz, \quad g \Phi_3(z) = \frac{1}{(z - ia)^{1-\beta}} H_3(z) dz,$$

where as before H_2 and H_3 are non-zero holomorphic functions at ia_+ . From these expressions, in a neighborhood of ia_+ we have

$$\begin{aligned} X(z) &= (X_1(z) + iX_2(z), X_3(z)) = \left(\frac{1}{2} \int^z \frac{1}{g} \Phi_3 - \frac{1}{2} \int^z g \Phi_3, X_3(z) \right) \\ &= \left(\frac{\overline{B}}{(z - ia)^\beta} + C + \overline{(z - ia)^{1-\beta} H_4(z)} + (z - ia)^\beta H_5(z), \operatorname{Re}(-i\lambda R \log(z - ia) + H_6(z)) \right) \\ &= \left(\frac{\overline{B}}{(z - ia)^\beta}, 0 \right) + (O_1(1) + iO_2(1), O_3(1)), \end{aligned}$$

where $B, C \in \mathbb{C}$, $B \neq 0$, $O_i(1)$ is a real function bounded in a neighborhood of ia_+ and H_i , for $i = 4, 5, 6$ are holomorphic functions at ia_+ . Firstly, from the above expression it follows that X is proper. We also have that $\arg \left(\frac{\overline{B}}{(z - ia)^\beta} \right) \in [\arg(\overline{B}) - \frac{\pi}{2}, \arg(\overline{B}) + \frac{\pi}{2}]$ and then $X(M) \subset \Sigma^+$, where Σ^+ is a half space of \mathbb{R}^3 determined by a plane Σ orthogonal to $\{x_3 = 0\}$ that verifies $X(\partial(M)) \subset \Sigma^+$. Furthermore, we can infer from the preceding expression that $X(M) \subset \{(x_1, x_2, x_3) \mid k \leq x_3 \leq k'\}$, where $k < k'$. In the case $\beta = 1$, we can use a result by Meeks and Rosenberg (see Lemma 2.1 in [13]) to obtain that $X(M)$ lies in the half slab determined by two horizontal planes containing ℓ_1^+ and ℓ_2^- , respectively, and Σ_0^+ , where Σ_0^+ is the half space determined by the plane parallel to Σ that contains $\partial(M)$ and $\Sigma_0^+ \subset \Sigma^+$. This concludes the proof of A.4 in the case $\beta = 1$. If $\beta \in]0, 1[$ we have that $X(M)$ is in a half slab and $X(\partial(M))$ is contained in a wedge of angle $\pi\beta \in]0, \pi[$. Then a result by López and Martín (see Corollary 2 in [11]) asserts that with this conditions $X(M)$ lies in the convex hull of its boundary. \square

Remark 1 *Observe that the case $a = 0$ in our family corresponds with the López-Martín examples (see [11]).*

We would like to point out that the range of geometric Plateau boundaries is not arbitrary, in the sense that the quotients h/t and $d/(2t - h)$ are bounded from above. To be more precise, we can prove the following:

Proposition 1 *For $a \in [0, 1[$, $\rho \in]0, \pi[$ and $\beta \in]0, 1]$ the proper minimal immersion $X_{a,\rho,\beta} : M_{a,\rho} \rightarrow \mathbb{R}^3$ given by Weierstrass data (17) satisfies:*

$$i) \frac{h(a, \rho, \beta)}{t(a, \rho, \beta)} < \frac{1}{2},$$

$$ii) \frac{d(a, \rho, \beta)}{2t(a, \rho, \beta) - h(a, \rho, \beta)} \leq \mathcal{D}_\beta, \text{ where } \mathcal{D}_\beta \text{ is a constant depending only on } \beta.$$

Proof: The proof of item *i)* is based on several technical properties of h and t that will be obtained in Sect. 4. Thereby, we will postpone the proof of this fact till the end of that section (see Lemma 8.)

Regarding item *ii)*, observe that $X_{a,\rho,\beta}(M_{a,\rho})$ is contained in a wedge of angle $\pi\beta$ in a slab of height $2t(a, \rho, \beta) - h(a, \rho, \beta)$. Thus, this statement is a direct consequence of Theorem 7 in [11]. \square

Remark 2 *If $\beta \in \mathbb{Q}$, $d(a, \rho, \beta) = 0$ and $h(a, \rho, \beta) = n t(a, \rho, \beta)$, $n \in \mathbb{N}$, the minimal surface (without boundary) obtained by successive Schwarz reflections of $X_{a,\rho,\beta}(M_{a,\rho})$ about its boundary is a complete proper singly periodic minimal surface in \mathbb{R}^3 . Moreover, if $\beta = 1$ one can prove that this surface is embedded (see Theorem 4.) Proposition 1 says us that the above situation only occurs for $n = 0$.*

3 The complete examples

As we mentioned before, the most interesting cases occur when $q_1^- = q_2^+$ and $\beta \in \mathbb{Q}$. In this case our minimal disks can be employed to construct complete minimal surfaces in \mathbb{R}^3 which are singly periodic.



Figure 6: A complete surface constructed with a fundamental piece of angle $\pi/3$.

Hence, the first objective of the present section is to prove the following result:

Theorem 3 *For each $\beta \in]0, 1]$ there exists $(a(\beta), \rho(\beta))$, such that*

$$h(a(\beta), \rho(\beta), \beta) = 0, \quad d(a(\beta), \rho(\beta), \beta) = 0. \quad (21)$$

Assume $\beta \in]0, 1]$ fixed. First we try to write the system (21) in terms of integrals of the Weierstrass data. Observe that $0_+ \in \gamma_1^-$ and $\infty_- \in \gamma_2^+$. On the other hand we have the symmetry S that satisfies $S(0_+) = S(\infty_-)$ and $S(1_+) = 1_+$. Thus, taking into account (2) we get the following expression for the function h

$$h(a, \rho, \beta) = x_3(q_2^+) - x_3(q_1^-) = X_3(\infty_-) - X_3(0_+) = \operatorname{Re} \left(\int_{0_+}^{\infty_-} \Phi_3 \right) = 2 \operatorname{Re} \left(\int_{0_+}^{1_+} \Phi_3 \right). \quad (22)$$

So the first equation in (21) is equivalent to

$$h(a, \rho, \beta) = \operatorname{Re} \left(\int_{0_+}^{1_+} \frac{1}{\lambda} \Phi_3 \right) = 0. \quad (23)$$

Regarding the function d , it is clear that $e^{i\frac{\rho}{2}} \in \gamma_0^+$ and $e^{-i\frac{\rho}{2}} \in \gamma_0^-$, thus we have

$$d(a, \rho, \beta) = X_2(e^{i\frac{\rho}{2}}) - X_2(e^{-i\frac{\rho}{2}}) = -\frac{1}{2} \operatorname{Im} \left(\int_{e^{-i\frac{\rho}{2}}}^{e^{i\frac{\rho}{2}}} \left(g + \frac{1}{g} \right) \Phi_3 \right) = -\frac{1}{2} \operatorname{Im} \left(\int_{\alpha_1} \left(g + \frac{1}{g} \right) \Phi_3 \right), \quad (24)$$

where α_1 is the curve defined in paragraph 2.3. We recall that $\alpha_1 = -S_*(\delta) + \delta$ and thus $S_*(\alpha_1) = -\alpha_1$. Then, taking into account (2) and (5) we deduce that

$$\int_{\alpha_1} \frac{1}{g} \Phi_3 = \int_{\alpha_1} g \Phi_3. \quad (25)$$

From here and (24) we obtain that the second equation in (21) is equivalent to

$$d(a, \rho, \beta) = -\frac{1}{\lambda} \operatorname{Im} \left(\int_{\alpha_1} g \Phi_3 \right) = -\frac{1}{\lambda} \operatorname{Im} \left(\int_{e^{-i\frac{\rho}{2}}}^{e^{i\frac{\rho}{2}}} g \Phi_3 \right) = 0. \quad (26)$$

Thus, to prove Theorem 3 it suffices to prove that there exists $(a(\beta), \rho(\beta))$ such that

$$h(a(\beta), \rho(\beta), \beta) = 0, \quad d(a(\beta), \rho(\beta), \beta) = 0.$$

Unfortunately, the proof of the above assertion is quite long and technical. For the sake of clarity, we develop here a sketch of the proof and we present the complete details in Sect. 4.

We study first the function h . From (23) we have that

$$h(a, \rho, \beta) = \int_0^1 \frac{t^4 + 1 + (c_1 c_2 - 2 \cos \rho) t^2}{(t^2 + a^2)(t^2 + \frac{1}{a^2}) \sqrt{t^4 + 1 - 2t^2 \cos \rho}} dt. \quad (27)$$

Observe that the function h can be extended to $[0, 1] \times]0, \pi] \times]0, 1]$. We obtain the following properties for h :

CLAIM H.1 $h(0, \rho, \beta) < 0$ for $\rho \in]0, \pi]$,

CLAIM H.2 $\lim_{\rho \rightarrow 0^+} h(a, \rho, \beta) = -\infty$ for $a \in [0, 1]$,

CLAIM H.3 $h(1, \pi, \beta) = \frac{\pi}{4}$,

CLAIM H.4 $\frac{\partial h}{\partial \rho}(a, \rho, \beta) > 0$ for $(a, \rho, \beta) \in]0, 1] \times]0, \pi] \times]0, 1]$,

CLAIM H.5 $\frac{\partial h}{\partial a}(a, \rho, \beta) > 0$ for $(a, \rho, \beta) \in]0, 1[\times]0, \pi[\times]0, 1[$.

From the preceding assertions we have that there exist $\rho_0(\beta) \in]0, \pi[$ and $a_0(\beta) \in]0, 1[$ such that $h(1, \rho_0(\beta), \beta) = h(a_0(\beta), \pi, \beta) = 0$. Furthermore, the set $C_1 = \{(a, \rho) \in [0, 1] \times [0, \pi] \mid h(a, \rho, \beta) = 0\}$ is a differentiable embedded curve in $]0, 1[\times]0, \pi[$ from $(a_0(\beta), \pi)$ to $(1, \rho_0(\beta))$.

However, the study of the function d is much more complicated due to the expression of the Gauss map g . For this function the following facts can be proved:

CLAIM D.1 $d(a, 0, \beta) > 0$,

CLAIM D.2 $\lim_{\rho \rightarrow \pi} d(a, \rho, \beta) = -\infty$,

CLAIM D.3 There exists a unique $\rho_1(\beta) \in]0, \pi[$ such that $d(0, \rho_1(\beta), \beta) = 0$. Furthermore, d is positive for $\rho \in [0, \rho_1(\beta)[$ and negative for $\rho \in]\rho_1(\beta), 1]$,

CLAIM D.4 $d(1, \rho, \beta) > 0$ for $\rho \in [0, \rho_0(\beta)]$.

All the preceding claims allow us to assert that there exists a connected component C of the set $\{(a, \rho) \in [0, 1] \times [0, \pi] \mid d(a, \rho, \beta) = 0\}$ that contains the point $(0, \rho_1(\beta))$ and a point in the segment $\{(1, \rho) \mid \rho \in [\rho_0(\beta), \pi]\}$. Since d is a real analytic function in the interior of $[0, 1] \times [0, \pi]$, we have that $d^{-1}(0)$ is locally path-connected and thus C is also a path component. We denote by C_2 a path in C starting at $(0, \rho_1(\beta))$ and finishing at a point in the segment $\{(1, \rho) \mid \rho \in [\rho_0(\beta), \pi]\}$. Therefore, $C_1 \cap C_2 \neq \emptyset$ and this concludes the proof of Theorem 3.

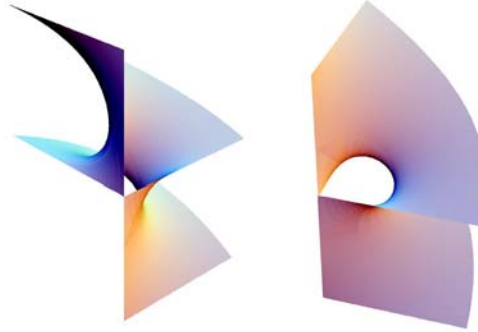


Figure 7: A fundamental piece ($d = h = 0$) of angle $\pi/2$.

Theorem 4 *If $(a, \rho) \in C_2$, then the surface given by the Weierstrass data (17) also fulfills A.5.*

Proof: Let $C_2(s) = (a(s), \rho(s))$, $s \in [0, 1]$ a parametrization of the curve C_2 . We label $M_s = M_{a(s), \rho(s)}$ and $X_s = X_{a(s), \rho(s), \beta}$. We also define the set $\mathcal{I} = \{s \in [0, 1] \mid X_s \text{ satisfies A.5}\}$. To conclude it is sufficient to see that $\mathcal{I} = [0, 1]$.

Recall that $a = 0$ corresponds to the López-Martín example of angle $\pi\beta$. This surface verifies A.5 when $d = 0$ (see [11]). So, $0 \in \mathcal{I} \neq \emptyset$. Consider $s_0 \in \mathcal{I}$. Taking into account the Weierstrass representation, one has that the ends of X_s are asymptotic to two pieces of helicoid and the distance between both ends is positive, $\forall s$. Then, there exists $\epsilon, r > 0$ such that $X_s(M) \cap (\mathbb{R}^3 - B(0, r))$ is embedded, $\forall s \in]s_0 - \epsilon, s_0 + \epsilon[$. If X_s were not injective for some $s \in]s_0 - \epsilon, s_0 + \epsilon[$, then self-intersections of $X_s(M_s)$ would be in $\overline{B}(0, r)$. As $X_s(M_s)$ is contained in the convex hull of its boundary, there are

no contacts of interior points with points at the boundary. So we would arrive to a contradiction, by using either the classical maximum principle or the maximum principle at the boundary. Hence, $]s_0 - \epsilon, s_0 + \epsilon[\subset \mathcal{I}$ which implies that \mathcal{I} is open.

Now, take $\{s_n\}_{n \in \mathbb{N}}$ a sequence in \mathcal{I} converging to $s_0 > 0$. Assume that X_{s_0} is not injective. Then, there are two points $x, y \in M_{s_0}$ satisfying $X_{s_0}(x) = X_{s_0}(y)$. The convergence of $\{X_{s_n}\}_{n \in \mathbb{N}}$ to X_{s_0} uniformly over compact subsets of \mathbb{R}^3 and the interior maximum principle assure that there exist neighborhoods $N(x), N(y)$, of x and y , respectively, such that $X_{s_0}(N(x)) = X_{s_0}(N(y))$. So, the image set $X_{s_0}(M)$ is an embedded minimal surface with finite total curvature and $X_{s_0} : M_{s_0} \rightarrow X_{s_0}(M_{s_0})$ is a finitely sheeted covering map. As X_{s_0} is one-to-one in a neighborhood of the end, then we deduce that X_{s_0} is injective, which is contrary to our assumption. This contradiction proves that \mathcal{I} is closed.

Thus, an elementary connectedness argument gives that $\mathcal{I} = [0, 1]$, which concludes the proof. \square

We can describe our family of surfaces \mathcal{M} as follows:

$$\mathcal{M} = \{X_{a,\rho,\beta} : M_{a,\rho} \rightarrow \mathbb{R}^3 \mid a \in [0, 1[, \rho \in]0, \pi[, \beta \in]0, 1[\} .$$

Finally, we are interested in the complete surfaces obtained from the minimal immersion $X_{a,\rho,\beta} : M_{a,\rho} \rightarrow \mathbb{R}^3$ when the parameters (a, ρ, β) satisfy (21). We summarize the properties of these complete surfaces in the following remark.

Remark 3 *As we mentioned in the paragraph 2.1, the complete orientable minimal surface without boundary*

$$\tilde{X}_{a,\rho,\beta} : \tilde{M}_{a,\rho,\beta} \rightarrow \mathbb{R}^3$$

obtained from $X_{a,\rho,\beta}(M_{a,\rho})$ by successive Schwarz reflections about straight lines is invariant under the vertical translation τ_1 .

The case $\beta \in \mathbb{Q}$ is specially interesting. The immersion $\tilde{X}_{a,\rho,\beta}$ is singly periodic and the induced immersion

$$Y_{a,\rho,\beta} : (\tilde{M}_{a,\rho,\beta}) / \langle \tau_1 \rangle \rightarrow \mathbb{R}^3 / \langle \tau_1 \rangle$$

has four ends and finite total curvature. If we write $\beta = q/p$, $p, q \in \mathbb{N}$, $\gcd(p, q) = 1$, then it is not hard to check that:

- *If p is even the surface $\tilde{M}_{a,\rho,\beta}$ is the two sheeted orientable covering of a nonorientable minimal surface properly immersed in \mathbb{R}^3 . Moreover, $Y_{a,\rho,\beta}$ has four ends, its total curvature is $-8\pi(p+q)$ and $\tilde{M}_{a,\rho,\beta} / \langle \tau_1 \rangle$ has genus $2p - 1$. A fundamental piece bounded by straight lines of a surface $\tilde{X}_{a,\rho,\beta}$, $\beta = 1/2$, is illustrated in Figure 2.*
- *If p is odd the induced immersion $Y_{a,\rho,\beta}$ has two ends. Moreover, if q is even (resp. q is odd), $Y_{a,\rho,\beta}$ has total curvature $-8\pi(p+q)$ (resp. $-4\pi(p+q)$) and $\tilde{M}_{a,\rho,\beta} / \langle \tau_1 \rangle$ has genus $2p$ (resp. p). Figures 6 and 8 shows a fundamental piece of a surface $\tilde{X}_{a,\rho,\beta}(\tilde{M}_{a,\rho,\beta})$, $\beta = 1/3$ and $\beta = 2/3$, respectively.*
- *The ends of $Y_{a,\rho,\beta}$ are embedded if, and only if, $\beta = \frac{1}{n}$, $n \in \mathbb{N}$, $n > 1$. In this case the only self-intersection of $\tilde{X}_{a,\rho,\beta}(\tilde{M}_{a,\rho,\beta})$ occurs along the x_3 -axis.*

Remark 4 (Uniqueness of \mathcal{H}_1) *Observe that the case $\beta = 1$ (i.e., angle π) corresponds to the simply periodic genus-one helicoid whose existence and embeddedness were proved by Hoffman-Karcher-Wei in [5].*

From (14), it is clear that if $\beta = 1$ we have $b(a, \rho, 1) = a$. Hence $a_2 = -a_1$, $c_2 = -c_1$ and $a_3 = 0$. Thus we obtain the following expressions for the Weierstrass data in (17)

$$g = \frac{z^2 + a^2}{a^2 z^2 + 1}, \quad \Phi_3 = \lambda \frac{w - c_1 i z dz}{w + c_1 i z w} .$$

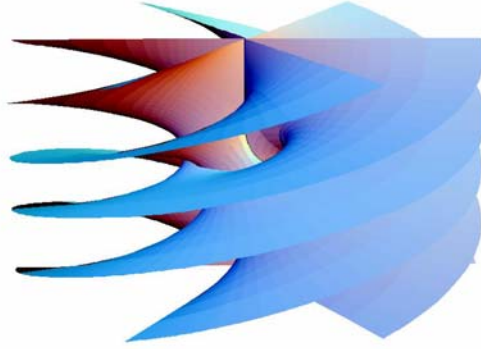


Figure 8: A complete surface constructed with a fundamental piece of angle $2\pi/3$.

It is easy to check that

$$\frac{1}{\lambda}g\Phi_3 = d\left(\frac{a^2zw + ic_1}{a^2z^2 + 1}\right) - \frac{z^2}{w}dz .$$

Thus, from the expression (34), that we will prove in the following section, we have

$$d(a, \rho, 1) = -\frac{i}{2} \int_{\gamma_1} \frac{z^2}{w} dz .$$

Therefore, the function d in this case does not depend on a . Then, taking into account CLAIM H.3, the set $\{(a, \rho) \in [0, 1] \times [0, \pi] \mid d(a, \rho, 1) = 0\} = \{(a, \rho_1(1)) \mid a \in [0, 1]\} = C_2$. Finally, using CLAIM H.5 and that C_1 is a graph in ρ we obtain that $C_1 \cap C_2$ is a unique point and so we have the uniqueness of the example when the angle is π .

Remark 5 (Limit case $\beta = 0$) Regarding to the case $\beta = 0$, we observe that $\lim_{\beta \rightarrow 0} b(a, \rho, \beta) = 0$ and therefore $\lim_{\beta \rightarrow 0} a_3(a, \rho, \beta) = 0$ (see Lemma 4 and (15)). Furthermore, if we impose, in order to take limits, that the length of the vertical segments were 1 we obtain that

$$\frac{1}{\lambda} = \operatorname{Re} \left(\int_{\infty_-}^{i+} \frac{w + c_2 iz}{w + c_1 iz} \frac{dz}{w} \right) = -2h(a, \rho, \beta) + \operatorname{Re} \left(\int_0^{i+} \frac{w + c_2 iz}{w + c_1 iz} \frac{dz}{w} \right) = -2h(a, \rho, \beta) + \pi R ,$$

where R was defined in the proof of Theorem 3. Hence, if we study the limits of the Weierstrass data given in (17) as $\beta \rightarrow 0$ we obtain

$$g = z , \quad \Phi_3 = \mu \frac{iz}{w + c_1 iz} \frac{dz}{w} ,$$

where $\mu \in \mathbb{R}$. One can see, using similar arguments as in Theorem 2, that the minimal surface with this Weierstrass data satisfies assumptions A.1 to A.4. Taking into account the expression of the Gauss map, it is easy to prove that this surface is a graph over the plane $\{x_1 = 0\}$. Therefore, this Weierstrass data corresponds to a Jenkins-Serrin graph defined on a rhomboid (see Fig. 9).

4 Appendix: Technical computations

The aim of this section is to present in detail the proofs of the claims in Sect. 3. Firstly, we have to study in deep the function $b(a, \rho, \beta)$ defined in the paragraph 2.3. More precisely, we prove:

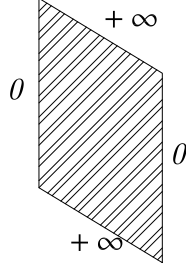


Figure 9: The rhomboid with the boundary values for the Jenkins-Serrin graph.

Lemma 4 *The function $b : [0, 1] \times]0, \pi[\times [0, 1] \longrightarrow [0, a]$ fulfills the following properties:*

- a) $b(a, \rho, \beta) = 0$ if and only if $a = 0$ or $\beta = 0$,
- b) $b(a, 0, \beta) = \tan(\beta \arctan(a))$ and $b(a, \pi, \beta) = \tanh(\beta \operatorname{arctanh}(a))$,
- c) $\lim_{a \rightarrow 0} \frac{b(a, \rho, \beta)}{a} = \beta$,
- d) $\frac{\partial b}{\partial \rho}(a, \rho, \beta) \geq 0$ for $(a, \rho, \beta) \in]0, 1[\times]0, \pi[\times [0, 1]$,

Proof: The assertions a) and b) are obtained straightforward from (14). It is obvious that F (given by (14)) can be expressed as

$$F(a, b, \rho, \beta) = \int_0^1 \left(\frac{\beta a}{\sqrt{a^4 t^4 + 1 + 2a^2 t^2 \cos \rho}} - \frac{b}{\sqrt{b^4 t^4 + 1 + 2b^2 t^2 \cos \rho}} \right) dt.$$

Using that F vanishes for $b = b(a, \rho, \beta)$ and dividing by a , for $a \neq 0$, we get

$$\int_0^1 \frac{\frac{b}{a}}{\sqrt{b^4 t^4 + 1 + 2b^2 t^2 \cos \rho}} dt = \int_0^1 \frac{\beta}{\sqrt{a^4 t^4 + 1 + 2a^2 t^2 \cos \rho}} dt.$$

Since $b(0, \rho, \beta) = 0$, taking limit in the above equation we obtain c).

Now we prove assertion d). For the sake of brevity, we will denote $b_\rho = \frac{\partial b}{\partial \rho}$, $b_\beta = \frac{\partial b}{\partial \beta}$, $b_{\rho\beta} = \frac{\partial^2 b}{\partial \rho \partial \beta}$ and $b_{\rho\beta\beta} = \frac{\partial^3 b}{\partial \rho \partial \beta \partial \beta}$. By differentiating in (14) we obtain the following expressions

$$b_\rho = \sin \rho \sqrt{b^4 + 1 + 2b^2 \cos \rho} \left(\beta \int_0^a \frac{t^2 dt}{(t^4 + 1 + 2t^2 \cos \rho)^{\frac{3}{2}}} - \int_0^b \frac{t^2 dt}{(t^4 + 1 + 2t^2 \cos \rho)^{\frac{3}{2}}} \right), \quad (28)$$

$$b_\beta = \sqrt{b^4 + 1 + 2b^2 \cos \rho} \int_0^a \frac{dt}{t^4 + 1 + 2t^2 \cos \rho}. \quad (29)$$

Differentiating again in the preceding expressions we get the following differential equations

$$b_{\rho\beta} = A + B b_\rho, \quad (30)$$

$$b_{\rho\beta\beta} = C b_\beta b_{\rho\beta} + D b_\beta^2 b_\rho + E b_\beta^2, \quad (31)$$

where

$$\begin{aligned}
A &= \sin \rho \sqrt{b^4 + 1 + 2b^2 \cos \rho} \int_0^a \frac{t^2 dt}{\sqrt{t^4 + 1 + 2t^2 \cos \rho}} - \frac{b^2 \sin \rho}{\sqrt{b^4 + 1 + 2b^2 \cos \rho}} \int_0^a \frac{dt}{\sqrt{t^4 + 1 + 2t^2 \cos \rho}}, \\
B &= \frac{2b(b^2 + \cos \rho)}{\sqrt{b^4 + 1 + 2b^2 \cos \rho}} \int_0^a \frac{dt}{\sqrt{t^4 + 1 + 2t^2 \cos \rho}}, \quad C = \frac{4b(b^2 + \cos \rho)}{b^4 + 1 + 2b^2 \cos \rho}, \\
D &= \frac{2(1 - b^4)(b^2 + \cos(\rho)) + 4b^2 \sin(\rho)^2}{(b^4 + 1 + 2b^2 \cos \rho)^2}, \quad E = -\frac{2b \sin(\rho)(1 - b^4)}{(b^4 + 1 + 2b^2 \cos \rho)^2}.
\end{aligned}$$

Let us prove that there does not exist any point $(a, \rho, \beta) \in]0, 1] \times]0, \pi[\times]0, 1]$ such that

$$b_\rho(a, \rho, \beta) < 0, \quad b_{\rho\beta}(a, \rho, \beta) = 0 \quad \text{and} \quad b_{\rho\beta\beta}(a, \rho, \beta) \geq 0. \quad (32)$$

Assume we have a point (a, ρ, β) with $b_\rho(a, \rho, \beta) < 0$ and $b_{\rho\beta}(a, \rho, \beta) = 0$. If $\rho \in]0, \frac{\pi}{2}]$ we obtain from (29), (31) and the definitions of D and E that $b_{\rho\beta\beta}(a, \rho, \beta) < 0$.

Suppose now that $\rho \in [\frac{\pi}{2}, \pi]$. From our assumptions and (30) we have

$$b(a, \rho, \beta) \sin \rho \geq 2b_\rho(a, \rho, \beta)(b(a, \rho, \beta)^2 + \cos \rho). \quad (33)$$

Then, taking into account (31), (33) and that $b \in [0, 1]$ it is clear that

$$b_{\rho\beta\beta}(a, \rho, \beta) \leq \left(\frac{2(b^2 - \cos \rho)}{b^4 + 1 + 2b^2 \cos \rho} b_\beta^2 b_\rho \right) (a, \rho, \beta) < 0.$$

On the other hand, from (28) we have $b_\rho(a, \rho, 0) = b_\rho(a, \rho, 1) = 0$. Since there are no points satisfying (32) we deduce that b_ρ as a function of β have no minimums corresponding to negative values of b_ρ . Hence $b_\rho \geq 0$ and this concludes the proof. \square

PROOF OF CLAIM H.1 . This assertion can be inferred straightforward from (4), (27) and parts *a*) and *c*) of Lemma 4.

PROOF OF CLAIM H.2 . Note that if $\rho = 0$, the denominator of (27) has a zero of order one at $t = 1$ while the numerator is strictly negative at this point. This gives the claim.

PROOF OF CLAIM H.3 . From part *a*) in Lemma 4 and (4) we have that $c_1 \cdot c_2(1, \pi, \beta) = 0$. Therefore the integral (27) becomes

$$h(1, \pi, \beta) = \int_0^1 \frac{dt}{t^2 + 1} = \frac{\pi}{4}.$$

PROOF OF CLAIM H.4 . To prove this claim we will consider the function $\tilde{h}(a, b, \rho) = h(a, \rho, \beta)$, it is to say we consider \tilde{h} as a function of the independent variables (a, b, ρ) . Then

$$\frac{\partial h}{\partial \rho} = \frac{\partial \tilde{h}}{\partial \rho} + \frac{\partial \tilde{h}}{\partial b} b_\rho.$$

By differentiating in (27) we get

$$\begin{aligned}
\frac{\partial \tilde{h}}{\partial \rho} &= \frac{-\sin \rho}{c_1 c_2} \int_0^1 \frac{t^2 ((c_1^2 + c_2^2 - c_1 c_2)(t^4 + 1 - 2t^2 \cos \rho) + c_1^2 c_2^2 t^2)}{(t^2 + a^2)(t^2 + \frac{1}{a^2})(t^4 + 1 - 2t^2 \cos \rho)^{\frac{3}{2}}} dt, \\
\frac{\partial \tilde{h}}{\partial b} &= \frac{-c_1(1 - b^4)}{c_2 b^3} \int_0^1 \frac{t^2}{(t^2 + a^2)(t^2 + \frac{1}{a^2})\sqrt{t^4 + 1 - 2t^2 \cos \rho}} dt.
\end{aligned}$$

From here and part *a*) in Lemma 4 we have $\frac{\partial \tilde{h}}{\partial \rho} > 0$ and $\frac{\partial \tilde{h}}{\partial b} \geq 0$ for $(a, \rho, \beta) \in]0, 1[\times]0, \pi[\times]0, 1[$. Therefore, taking into account part *d*) in Lemma 4 we obtain CLAIM H.4.
PROOF OF CLAIM H.5 . By differentiating in (27) and simplifying we obtain

$$\frac{\partial h}{\partial a} = \left(\frac{1}{a^3} - a \right) \int_0^1 \frac{t^2 \left(2 - \frac{c_2}{c_1} t^2 \right) \sqrt{t^4 + 1 - 2t^2 \cos \rho}}{(t^2 + a^2)^2 \left(t^2 + \frac{1}{a^2} \right)^2} dt + \int_0^1 \frac{t^2 P(t, a, \rho, \beta)}{(t^2 + a^2)^2 \left(t^2 + \frac{1}{a^2} \right)^2 \sqrt{t^4 + 1 - 2t^2 \cos \rho}} dt ,$$

where $P(t, a, \rho, \beta) = At^4 + Bt^2 + A$ and $A = \frac{\beta b}{a} \left(\frac{1}{b^3} - b \right)$, $B = c_1 c_2 \left(\frac{1}{a^3} - a \right) + A \left(\frac{1}{a^2} + a^2 \right)$. Evidently, the first summand in the above expression is positive for $a \in]0, 1[$. Then it suffices to see $P(t, a, \rho, \beta) \geq 0$. Since $A \geq 0$, if $B \geq 0$ this is obvious. Therefore, assume $B < 0$. In order to see $P(t, a, \rho, \beta) \geq 0$ we observe that the discriminant of P as a polynomial in t is given by $\Delta = B^2 - 4A^2 = (B - 2A)(B + 2A)$. From our assumption we know that the first factor of Δ is non positive. Let us analyze the second one. We have

$$B + 2A = c_1 c_2 \left(\frac{1}{a^3} - a \right) + \frac{\beta b}{a} \left(\frac{1}{b^3} - b \right) \left(\frac{1}{a} + a \right)^2 .$$

Taking into account that $x^4 + 1 - 2x^2 \cos \rho \leq (x^2 + 1)^2$ we obtain

$$B + 2A \geq \frac{(1 + a^2)^2 (1 + b^2)}{a^3 b} \left(\beta \frac{1 - b^2}{b} - \frac{1 - a^2}{a} \right) .$$

Let us consider the function $f_1(a, \rho, \beta) = \beta \frac{1 - b^2}{b} - \frac{1 - a^2}{a}$. From parts *b*) and *d*) in Lemma 4 we have that $b(a, \rho, \beta) \leq b(a, \pi, \beta) = \tanh(\beta \operatorname{arctanh}(a))$. Since the function $\frac{1 - b^2}{b}$ is decreasing in b we get

$$f_1(a, \rho, \beta) \geq \beta \frac{2}{\sinh(2\beta \operatorname{arctanh}(a))} - \frac{1 - a^2}{a} .$$

Next we compute

$$\frac{\partial f_1}{\partial \beta} = \frac{2 \cosh(2\beta \operatorname{arctanh}(a))}{\sinh(2\beta \operatorname{arctanh}(a))^2} (\tanh(2\beta \operatorname{arctanh}(a)) - 2\beta \operatorname{arctanh}(a)) .$$

Since $\tanh(x) - x \leq 0$ we deduce that $\frac{\partial f_1}{\partial \beta} \leq 0$ and so

$$f_1(a, \rho, \beta) \geq f_1(a, \rho, 1) = \frac{2}{\sinh(2 \operatorname{arctanh}(a))} - \frac{1 - a^2}{a} = 0 .$$

Thus $B + 2A \geq 0$ and then $\Delta \leq 0$. As $P(0, a, \rho, \beta) = A \geq 0$, this implies that $P(t, a, \rho, \beta) \geq 0$ and this concludes the proof of the claim.

Next we prove assertions on function *d*. In order to do this we need to establish some previous results. The first one consists of finding an upper bound for the point $\rho_0(\beta)$ defined in Sect. 3. Recall that $\rho_0(\beta)$ was the unique point in $]0, \pi[$ such that $h(1, \rho_0(\beta), \beta) = 0$. It is important to notice that from now on $b = b(a, \rho, \beta)$.

Lemma 5 $\rho_0(\beta) \leq \frac{\pi}{\beta + 1}$.

Proof: Taking into account CLAIM H.5, it suffices to prove that $H(\beta) = h(1, \frac{\pi}{\beta + 1}, \beta) \geq 0$. To prove this fact we will show that

$$-2 \cos \left(\frac{\pi}{2(\beta + 1)} \right) \sqrt{b^2 + \frac{1}{b^2} + 2 \cos \left(\frac{\pi}{\beta + 1} \right)} - 2 \cos \left(\frac{\pi}{\beta + 1} \right) \geq -2 ,$$

or equivalently

$$2 \sin\left(\frac{\pi}{2(\beta+1)}\right)^2 - \cos\left(\frac{\pi}{2(\beta+1)}\right) \sqrt{b^2 + \frac{1}{b^2} + 2 \cos\left(\frac{\pi}{\beta+1}\right)} \geq 0.$$

Thereby, consider the function

$$H_1(\beta) = 4 \sin\left(\frac{\pi}{2(\beta+1)}\right)^4 - \cos\left(\frac{\pi}{2(\beta+1)}\right)^2 \left(b^2 + \frac{1}{b^2} + 2 \cos\left(\frac{\pi}{\beta+1}\right)\right)$$

Using parts *b*) and *d*) of Lemma 4 we obtain that $\tan\left(\frac{\beta\pi}{4}\right) \leq b\left(1, \frac{\pi}{\beta+1}, \beta\right) \leq 1$. So, as the function $x^2 + \frac{1}{x^2}$ is decreasing in $]0, 1]$ we get after some computations

$$H_1(\beta) \geq H_2(\beta) = 4 \left(\tan\left(\frac{\pi}{2(\beta+1)}\right)^2 - \sin\left(\frac{\beta\pi}{2}\right)^{-2} \right).$$

By differentiating in the above expression we have

$$\frac{\partial H_2}{\partial \beta}(\beta) = 4\pi \left(\frac{\cos\left(\frac{\beta\pi}{2}\right)}{\sin\left(\frac{\beta\pi}{2}\right)^3} - \frac{\frac{1}{(\beta+1)^2} \sin\left(\frac{\pi}{2(\beta+1)}\right)}{\cos\left(\frac{\pi}{2(\beta+1)}\right)^3} \right) = \frac{4\pi}{\beta^2} \left(\frac{\beta^2 \cos\left(\frac{\beta\pi}{2}\right)}{\sin\left(\frac{\beta\pi}{2}\right)^3} - \frac{\frac{\beta^2}{(\beta+1)^2} \cos\left(\frac{\beta\pi}{2(\beta+1)}\right)}{\sin\left(\frac{\beta\pi}{2(\beta+1)}\right)^3} \right).$$

Now we define the function $H_3(x) = \frac{x^2 \cos\left(\frac{\pi}{2}x\right)}{\sin\left(\frac{\pi}{2}x\right)^3}$ for $x \in [0, 1]$. Let us see that H_2 is a decreasing function in x . Observe that

$$\frac{\partial H_3}{\partial x}(x) = \frac{x}{2} \csc\left(\frac{\pi}{2}x\right)^4 (2 \sin(\pi x) - \pi x(2 + \cos(\pi x))).$$

It is easy to see that $H_4(x) = 2 \sin(\pi x) - \pi x(2 + \cos(\pi x))$ is a concave function with $H_4(0) = 0$ and $H_4(1) = -\pi$. Therefore, we deduce that H_3 is decreasing. Hence, since $\frac{\beta}{\beta+1} \leq \beta$ we obtain

$$\frac{\partial H_2}{\partial \beta}(\beta) = \frac{4\pi}{\beta^2} \left(H_3(\beta) - H_3\left(\frac{\beta}{\beta+1}\right) \right) \leq 0.$$

Finally, taking into account $H_2(1) = 0$ we conclude that $H_1(\beta) \geq H_2(\beta) \geq 0$. □

Next, we will study in deep the function $a_3(a, \rho, \beta)$ presented in paragraph 2.3.

Lemma 6 *The function $a_3 : [0, 1] \times]0, \pi] \times [0, 1] \rightarrow \mathbb{R}$ fulfills the following properties:*

- a) $a_3(a, 0, \beta) = 0$,
- b) $a_3(a, \pi, \beta) = 0$,
- c) $a_3(1, \rho, \beta) = -\frac{1}{a_2} A_0$, where $0 \leq A_0 \leq 1$.

Proof: Recall that a_3 can be computed either by the expression (15) or expression (16). Observe that the denominator of (15) diverges to ∞ as $\rho \rightarrow \pi$ while the numerator goes to a constant. Therefore we obtain part *b*) of the lemma. On the other hand, when $a \rightarrow 1$ the expression (15) gives us $a_3(1, \rho, \beta) = -\frac{1}{a_2} A_0$, where

$$A_0 = \frac{\int_0^\rho \frac{\sqrt{2(\cos(t) - \cos(\rho))}}{b^2 + \frac{1}{b^2} + 2 \cos(t)} dt}{\int_0^\rho \frac{1}{\sqrt{2(\cos(t) - \cos(\rho))}} dt}$$

and $b = b(1, \rho, \beta)$. Clearly $A_0 \geq 0$. Furthermore, checking that the denominator in the above fraction is greater than the numerator we have that $A_0 \leq 1$ and so part *c*) is proved.

As before the denominator in the expression (16) diverges to ∞ as $\rho \rightarrow 0$ while the numerator goes to a constant. Thus we get statement *a*) of the lemma. \square

PROOF OF CLAIM D.1 . From (26) we have

$$d(a, \rho, \beta) = -\frac{1}{\lambda} \text{Im} \left(\int_{\alpha_1} g \Phi_3 \right) ,$$

where α_1 is the curve defined in paragraph 2.3. On the other hand, from (2) and (5) we have $(S_0^+)^*(g\Phi_3) = \frac{1}{g}\overline{\Phi_3}$. Hence, taking into account (25) and that $\gamma_1 = \alpha_1 - (S_0^+)_*(\alpha_1)$ we get

$$\int_{\gamma_1} g \Phi_3 = \int_{\alpha_1} g \Phi_3 - \int_{\alpha_1} \frac{1}{g} \overline{\Phi_3} = 2i \text{Im} \left(\int_{\alpha_1} g \Phi_3 \right) .$$

Therefore, we have

$$d(a, \rho, \beta) = \frac{i}{2\lambda} \int_{\gamma_1} g \Phi_3 = \frac{i}{2\lambda} \int_{\widehat{\gamma}_1} g \Phi_3 , \quad (34)$$

where $\widehat{\gamma}_1$ is a curve in $\overline{\mathcal{N}}$ homotopic to γ_1 that does not contain the point 1_+ . We can observe that when ρ approaches 0 the curve $\widehat{\gamma}_1$ is the boundary of a topological disk around the point 1_+ and thereby we obtain

$$d(a, 0, \beta) = \frac{i}{2\lambda} \lim_{\rho \rightarrow 0} \int_{\widehat{\gamma}_1} g \Phi_3 = \frac{-\pi}{\lambda} \text{Residue}(\lim_{\rho \rightarrow 0} g \Phi_3, 1_+) ,$$

Taking part *a*) of Lemma 6 into account, we have that, when ρ approaches 0, $\frac{dg}{g}$ is a holomorphic one-form in a neighborhood of 1_+ and furthermore $\lim_{\rho \rightarrow 0} g(1_+) = 1$. On the other hand, we have that $\lim_{\rho \rightarrow 0} \Phi_3$ has a simple pole at 1_+ and so

$$d(a, 0, \beta) = \frac{-\pi}{\lambda} \text{Residue}(\lim_{\rho \rightarrow 0} \Phi_3, 1_+) = \pi \frac{a(1+b^2)}{b(1+a^2)} > 0 .$$

PROOF OF CLAIM D.2 . Taking into account the assertion *b*) in Lemma 6, the Gauss map can be easily computed in the case $\rho = \pi$ and we obtain

$$g(z) = \frac{z + ib}{ibz + 1} \left(\frac{z - ia}{-iaz + 1} \right)^\beta ,$$

where b is given by part *b*) in Lemma 4 and we choose the branch of z^β satisfying $1^\beta = 1$. From here we infer that if $a \in [0, 1[$ then g has no poles in the curve α_1 . On the other hand, the expression of the one-form Φ_3 in the case $\rho = \pi$ is given by

$$\frac{1}{\lambda} \Phi_3(z) = -\frac{a}{b} \frac{(z + ib)(ibz + 1)}{(z - ia)(-iaz + 1)} \frac{dz}{z^2 + 1} .$$

Clearly, for $a \in [0, 1[$, this one-form has a simple pole at $z = \pm i$ that are the extremes of the curve α_1 . Let us compute the sign of $-\text{Im}(\text{Residue}(\frac{1}{\lambda} g \Phi_3, \pm i))$. Then, we get

$$\text{Residue} \left(\frac{1}{\lambda} g \Phi_3, i \right) = \frac{a(1+b)^2(1-a)^{\beta-1}}{2b(1+a)^{\beta+1}} e^{i\beta\frac{\pi}{2}} , \text{Residue} \left(\frac{1}{\lambda} g \Phi_3, -i \right) = -\frac{a(1-b)^2(1+a)^{\beta-1}}{2b(1-a)^{\beta+1}} e^{-i\beta\frac{\pi}{2}}$$

Thus we obtain $-\text{Im}(\text{Residue}(\frac{1}{\lambda}g\Phi_3, \pm i)) < 0$. Note that if $a = 1$, from statement b) of Lemma 4 we have that $b = 1$. Therefore, in the case $a = 1$ we have

$$\frac{1}{\lambda}g\Phi_3 = \frac{idz}{(-iz + 1)^\beta(z - i)^{2-\beta}}$$

and in this case we have a pole of order $2 - \beta$ at $z = i$ and as before we deduce that $d(1, \rho, \beta)$ diverges to $-\infty$ when $\rho \rightarrow \pi$.

PROOF OF CLAIM D.3 . As was indicated in Remark 1 our examples for parameters $(0, \rho, \beta)$ coincide with López-Martín examples for parameters $n = \frac{2}{\beta+1}$, $r = -\cos(\frac{\rho}{\beta+1})$ studied in [10] and [11]. Furthermore, our function d corresponds with function f studied in Lemma 3 of [11]. From the analysis of this function developed by López and the second author in that paper follows that there exists $\rho_1(\beta) \in]0, \pi[$ that verifies the conditions of CLAIM D.3.

PROOF OF CLAIM D.4 . Along this proof we assume $a = 1$ and $\rho \in]0, \rho_0(\beta)[$. Using (2), (5), (24) and that $\alpha_1 = -S_*(\delta) + \delta$ we obtain

$$d(a, \rho, \beta) = -\frac{1}{\lambda}\text{Im}\left(\int_1^{e^{i\frac{\rho}{2}}}\left(g + \frac{1}{g}\right)\Phi_3\right).$$

Hence we can write $d(a, \rho, \beta) = I_1 - I_2$, where

$$I_1 = \int_1^{e^{i\frac{\rho}{2}}}\left(|g| + \frac{1}{|g|}\right)\cos(\arg(g))\text{Im}\left(-\frac{1}{\lambda}\Phi_3\right) = \int_0^\rho\left(|g(t)| + \frac{1}{|g(t)|}\right)\cos(\theta(t))\text{Im}\left(-\frac{1}{\lambda}\Phi_3(e^{i\frac{t}{2}})\right),$$

$$I_2 = \int_1^{e^{i\frac{\rho}{2}}}\left(|g| - \frac{1}{|g|}\right)\sin(\arg(g))\text{Re}\left(\frac{1}{\lambda}\Phi_3\right) = \int_0^\rho\left(|g(t)| - \frac{1}{|g(t)|}\right)\sin(\theta(t))\text{Re}\left(\frac{1}{\lambda}\Phi_3(e^{i\frac{t}{2}})\right),$$

where $g(t) = g(e^{i\frac{t}{2}})$ and $\theta(t) = \arg(g(e^{i\frac{t}{2}})) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. First of all we get

$$\text{Re}\left(\frac{1}{\lambda}\Phi_3(e^{i\frac{t}{2}})\right) = \frac{2\cos(\frac{\rho}{2}) - c_2}{8\cos(\frac{t}{2})^2} > 0. \quad (35)$$

Furthermore, since $h(1, \rho, \beta) < 0$ for $\rho \in [0, \rho_0(\beta)[$ we deduce from (27) that $2\cos(\rho) - 2\cos(\frac{\rho}{2})c_2 > 2$ and therefore we obtain

$$\text{Im}\left(-\frac{1}{\lambda}\Phi_3(e^{i\frac{t}{2}})\right) = \frac{-2\cos(t) + 2\cos(\rho) - 2\cos(\frac{\rho}{2})c_2}{8\cos(\frac{t}{2})^2\sqrt{2(\cos(t) - \cos(\rho))}} > \frac{\sin(\frac{t}{2})^2}{2\cos(\frac{t}{2})^2\sqrt{2(\cos(t) - \cos(\rho))}} \geq 0. \quad (36)$$

Next, we will compute the expressions of $\theta(t)$ and $g(t)$. Regarding $\theta(t)$ we obtain

$$0 \leq \theta(t) = \arctan\left(\frac{1 - b^2}{1 + b^2}\tan\left(\frac{t}{2}\right)\right) \leq \frac{t}{2} \leq \frac{\pi}{2}. \quad (37)$$

On the other hand, $|g(t)|$ is given by

$$|g(t)| = \exp\left(\int_1^{e^{i\frac{t}{2}}}\text{Re}\left(\frac{dg}{g}\right)\right) = \exp(G(t, \rho, \beta)), \quad (38)$$

where

$$G(t, \rho, \beta) = -\frac{1}{2a_2}\int_0^t\left(\frac{\sqrt{2(\cos(s) - \cos(\rho))}}{b^2 + \frac{1}{b^2} + 2\cos(s)} - A_0\frac{1}{\sqrt{2(\cos(s) - \cos(\rho))}}\right)ds.$$

For this function we obtain the following facts:

Lemma 7 *The function G above described satisfies:*

a) $G(t, \rho, \beta) \geq 0$,

b) $G(t, \rho, \beta) \leq \frac{1-b^2}{\sqrt{b^4+1+2b^2 \cos(\rho)}} G(\rho, \beta)$, where $G(\rho, \beta) = \log \left(\sqrt{\frac{b^2+1+2b \sin(\frac{\rho}{2})}{b^2+1-2b \sin(\frac{\rho}{2})}} \right)$.

Proof: First of all we note that $G(0, \rho, \beta) = G(\rho, \rho, \beta) = 0$. Evidently, we have

$$\frac{\partial G}{\partial t}(t, \rho, \beta) = -\frac{1}{2a_2} \left(\frac{\sqrt{2(\cos(t) - \cos(\rho))}}{b^2 + \frac{1}{b^2} + 2 \cos(t)} - A_0 \frac{1}{\sqrt{2(\cos(t) - \cos(\rho))}} \right).$$

Thereby, if t_0 is a critical point of G as a function of t we have at this point

$$2(\cos(t_0) - \cos(\rho)) - A_0 \left(2 \cos(t_0) + b^2 + \frac{1}{b^2} \right) = 0. \quad (39)$$

Furthermore, the second derivative of G respect to t is

$$\begin{aligned} \frac{\partial^2 G}{\partial t^2}(t, \rho, \beta) &= -\frac{\sin(t)}{4\sqrt{2}a_2(b^2 + \frac{1}{b^2} + 2 \cos(t))^2(\cos(t) - \cos(\rho))^{\frac{3}{2}}} \left(-A_0(b^2 + \frac{1}{b^2} + 2 \cos(t))^2 \right. \\ &\quad \left. - 2(\cos(t) - \cos(\rho))(b^2 + \frac{1}{b^2} + 2 \cos(t)) + 8(\cos(t) - \cos(\rho))^2 \right) \end{aligned}$$

Therefore, at the point t_0 we get

$$\frac{\partial^2 G}{\partial t^2}(t_0, \rho, \beta) = -\frac{\sin(t_0)}{2\sqrt{2}a_2(\cos(t_0) - \cos(\rho))^{\frac{3}{2}}} A_0(A_0 - 1),$$

and since part *c)* of Lemma 6 guarantees that $0 \leq A_0 \leq 1$, we deduce that there exists at most one critical point of G as a function of t and this must be a maximum. From here follows statement *a)* of the lemma.

Now, we recall that $A_0 \geq 0$ and $\sqrt{2(\cos(s) - \cos(\rho))} \leq 2 \cos(\frac{s}{2})$. Then we have

$$G(t, \rho, \beta) \leq -\frac{1}{a_2} \int_0^t \frac{\cos(\frac{s}{2})}{b^2 + \frac{1}{b^2} + 2 \cos(s)} ds = \frac{1-b^2}{\sqrt{b^4+1+2b^2 \cos(\rho)}} \log \left(\sqrt{\frac{b^2+1+2b \sin(\frac{t}{2})}{b^2+1-2b \sin(\frac{t}{2})}} \right).$$

To conclude the proof of *b)* it suffices to note that $\log \left(\sqrt{\frac{b^2+1+2b \sin(\frac{t}{2})}{b^2+1-2b \sin(\frac{t}{2})}} \right)$ is increasing as a function of t . □

Now we will return to the study of *d)*. From (36) and (37) we get the following inequality for the integral I_1

$$I_1 > \int_0^\rho \frac{\sin(\frac{t}{2})^2}{\cos(\frac{t}{2})\sqrt{2(\cos(t) - \cos(\rho))}} dt = \frac{\pi}{2} \left(\sec\left(\frac{\rho}{2}\right) - 1 \right), \quad (40)$$

On the other hand, taking into account (35), assertions *a)* and *b)* of Lemma 7, and that $\exp(x) - \exp(-x)$ and $\sin(x)$ are increasing functions we obtain

$$\begin{aligned} I_2 &\leq \left(\exp\left(\frac{1-b^2}{\sqrt{b^4+1+2b^2 \cos(\rho)}} G(\rho, \beta)\right) - \exp\left(-\frac{1-b^2}{\sqrt{b^4+1+2b^2 \cos(\rho)}} G(\rho, \beta)\right) \right) \frac{2 \cos(\frac{\rho}{2}) - c_2}{8} \int_0^t \frac{\sin(\frac{s}{2})}{\cos(\frac{s}{2})^2} ds \\ &= \sinh \left(\frac{1-b^2}{\sqrt{b^4+1+2b^2 \cos(\rho)}} G(\rho, \beta) \right) \frac{(2 \cos(\frac{\rho}{2}) - c_2)}{2} \left(\sec\left(\frac{t}{2}\right) - 1 \right). \end{aligned}$$

As $\sec(x)$ is also increasing it follows

$$I_2 \leq \sinh\left(\frac{1-b^2}{\sqrt{b^4+1+2b^2\cos(\rho)}}G(\rho, \beta)\right) \frac{(2\cos(\frac{\rho}{2}) - c_2)}{2} \left(\sec(\frac{\rho}{2}) - 1\right). \quad (41)$$

As a consequence of (40) and (41), we have

$$d(1, \rho, \beta) = I_1 - I_2 > \frac{1}{2} \left(\sec(\frac{\rho}{2}) - 1\right) \left(\pi - (2\cos(\frac{\rho}{2}) - c_2) \sinh\left(\frac{1-b^2}{\sqrt{b^4+1+2b^2\cos(\rho)}}G(\rho, \beta)\right)\right).$$

We denote by $H(\rho, \beta) = (2\cos(\frac{\rho}{2}) - c_2) \sinh\left(\frac{1-b^2}{\sqrt{b^4+1+2b^2\cos(\rho)}}G(\rho, \beta)\right)$. Our next objective is to prove that $\pi - H(\rho, \beta)$ is non negative. We will distinguish several cases.

Suppose first that $\rho \in [0, 2\arcsin(\frac{\pi}{4})]$. Taking into account $\frac{1-b^2}{\sqrt{b^4+1+2b^2\cos(\rho)}} \leq 1$ and that $\sinh(x)$ is an increasing function we have

$$\pi - H(\rho, \beta) \geq \pi - \frac{2b\sin(\frac{\rho}{2})(2\cos(\frac{\rho}{2}) - c_2)}{\sqrt{b^4+1+2b^2\cos(\rho)}} = \pi - \frac{4b\sin(\frac{\rho}{2})\cos(\frac{\rho}{2})}{\sqrt{b^4+1+2b^2\cos(\rho)}} - 2\sin(\frac{\rho}{2}). \quad (42)$$

Consider now the function $k_1(\rho, \beta) = \frac{b}{\sqrt{b^4+1+2b^2\cos(\rho)}}$. By differentiating here we get

$$\frac{\partial k_1}{\partial \beta}(\rho, \beta) = \frac{1-b^4}{(b^4+1+2b^2\cos(\rho))^{\frac{3}{2}}} b_\beta.$$

From (29) we obtain that the preceding derivative is non negative. Therefore, taking part *a*) of Lemma 4 into account we get $k_1(\rho, \beta) \leq k_1(\rho, 1) = \frac{1}{2\cos(\frac{\rho}{2})}$. Thus, we can conclude

$$\pi - H(\rho, \beta) \geq \pi - 4\sin(\frac{\rho}{2}) \geq 0.$$

Using Lemma 5 we know that if $\beta \in [\beta_1, 1]$, where $\beta_1 = \frac{\pi}{2\arcsin(\frac{\pi}{4})} - 1$, we are in the preceding case. Thereby, we only have to study the cases where $\beta \in [0, \beta_1]$ and $\rho \in [2\arcsin(\frac{\pi}{4}), \pi]$. Observe that in the remainder cases $\cos(\rho) < 0$.

Now, we assume $\beta \in [0, 0.56]$ and $\rho \in [2\arcsin(\frac{\pi}{4}), \pi]$. Consider now the function $k_2(\rho, \beta) = \frac{1}{\sqrt{b^4+1+2b^2\cos(\rho)}}$. By differentiating here we get

$$\frac{\partial k_2}{\partial \beta}(\rho, \beta) = \frac{-2b(b^2 + \cos(\rho))}{(b^4+1+2b^2\cos(\rho))^{\frac{3}{2}}} b_\beta.$$

Recall that $b_\beta > 0$, there is a critical point when $b^2 = -\cos(\rho)$. In order to see the character of the critical point we compute the second derivative at a critical point and obtain

$$\frac{\partial^2 k_2}{\partial \beta^2}(\rho, \beta) = \frac{4\cos(\rho)}{\sin(\rho)^3} b_\beta < 0.$$

Hence, the critical point is a maximum and $k_2(\rho, \beta) \leq \frac{1}{\sin(\rho)}$. Then, from (42) we have

$$\pi - H(\rho, \beta) \geq \pi - 2b - 2\sin(\frac{\rho}{2}). \quad (43)$$

Using statement *d*) of Lemma 4 and Lemma 5 we have $b(1, \rho, \beta) \leq b(1, \frac{\pi}{\beta+1}, \beta)$. If we compute the derivative of this function respect to β and use (28) and (29) we obtain

$$\begin{aligned} \frac{\partial b}{\partial \beta}(1, \frac{\pi}{\beta+1}, \beta) &= -\frac{\pi}{(\beta+1)^2} b_\rho + b_\beta = \sqrt{b^4 + 1 + 2b^2 \cos(\frac{\pi}{\beta+1})} \left(\frac{\pi}{(\beta+1)^2} \int_0^b \frac{t^2 \sin(\frac{\pi}{\beta+1})}{(t^4 + 1 + 2t^2 \cos(\frac{\pi}{\beta+1}))^{\frac{3}{2}}} dt \right. \\ &\quad \left. + \int_0^1 \frac{(1-t^2)^2 + 2t^2 \cos(\frac{\pi}{2(\beta+1)})^2 (1 - \frac{\pi\beta}{(\beta+1)^2} \tan(\frac{\pi}{2(\beta+1)}))}{(t^4 + 1 + 2t^2 \cos(\frac{\pi}{\beta+1}))^{\frac{3}{2}}} dt \right) \end{aligned}$$

Let us prove $\frac{\partial b}{\partial \beta}(1, \frac{\pi}{\beta+1}, \beta) \geq 0$. To see this it suffices to prove that $k(\beta) = \frac{\pi\beta}{(\beta+1)^2} \tan(\frac{\pi}{2(\beta+1)}) \leq 1$. But this is very easy to see since k is an increasing function and $k(1) = \frac{\pi}{8} < 1$.

Summarizing we have that $b(1, \frac{\pi}{\beta+1}, \beta) \leq b(1, \frac{\pi}{1.56}, 0.56)$. Then substituting in (43) we get

$$\pi - H(\rho, \beta) \geq \pi - 2b(1, \frac{\pi}{1.56}, 0.56) - 2 > 0$$

Suppose now that $\beta \in [0.56, \beta_1]$ and $\rho \in [2 \arcsin(\frac{\pi}{4}), \pi]$. From this assumptions and Lemma 5, we can assert that $\rho \in [2 \arcsin(\frac{\pi}{4}), \frac{\pi}{1.56}]$. Let us consider the function $k_3(\rho, \beta) = 2 \cos(\frac{\rho}{2}) + c_2 = 2 \cos(\frac{\rho}{2}) + \frac{\sqrt{b^4 + 1 + 2b^2 \cos(\rho)}}{b}$. It is not hard to see that $\frac{\partial k_3}{\partial \rho}$ and $\frac{\partial k_3}{\partial \beta}$ are non negative. Since b_β and b_ρ where non negative also (see (29) and part *d*) in Lemma 4) we obtain that k_3 are a decreasing function on ρ and β . Therefore, $k_3(\rho, \beta) \leq k_3(2 \arcsin(\frac{\pi}{4}), 0.56) < 3.04$.

On the other hand, reasoning as before it is easy to see that $1 - b^2$ is a decreasing function on ρ and β . Then, taking into account the results obtained before for the function k_2 and that $\rho \in [2 \arcsin(\frac{\pi}{4}), \frac{\pi}{1.56}]$ we get

$$\frac{1 - b^2}{\sqrt{b^4 + 1 + 2b^2 \cos(\rho)}} \leq \frac{1 - b(1, 2 \arcsin(\frac{\pi}{4}), 0.56)^2}{\sin(\frac{\pi}{1.56})} < 0.79.$$

Finally, the same argument used with the preceding functions prove that $G(\rho, \beta)$ is an increasing function on ρ and β and so $G(\rho, \beta) \leq G(\frac{\pi}{1.56}, \beta_1) < 1.12$. Thus we have

$$\pi - H(\rho, \beta) \geq \pi - 3.04 \sinh(0.79 \cdot 1.12) = 0.08 > 0.$$

Lemma 8 *The maps h and t satisfy $\frac{h}{t} < \frac{1}{2}$.*

Proof: From (20), (22) and the definition of h given in (23), one has $\frac{h}{t} = \frac{2h}{\pi R}$, where $R = \frac{a^2}{1-a^4} (c_1 - c_2)$. First, we are going to check that $\frac{\partial}{\partial \rho} (\frac{h}{R}) > 0$ and $\frac{\partial}{\partial \beta} (\frac{h}{R}) > 0$ in $]0, 1[\times]0, \pi[\times]0, 1[$.

A direct computation gives:

$$\frac{\partial R}{\partial \rho} = -\frac{a^2}{1-a^4} \left[\sin(\rho) \left(\frac{1}{c_1} - \frac{1}{c_2} \right) + \frac{\partial c_2}{\partial b} \frac{\partial b}{\partial \rho} \right], \quad \frac{\partial R}{\partial \beta} = -\frac{a^2}{1-a^4} \frac{\partial c_2}{\partial b} \frac{\partial b}{\partial \beta}, \quad \frac{\partial c_2}{\partial b} = \frac{(b^4 - 1)}{b^3 c_2} > 0.$$

By *d*) in Lemma 4, we have $\frac{\partial b}{\partial \rho} \geq 0$, and so $\frac{\partial R}{\partial \rho} < 0$. The previous fact and CLAIM H.4 imply $\frac{\partial}{\partial \rho} (\frac{h}{R}) > 0$. On the other hand by differentiating in (27) we get

$$\frac{\partial h}{\partial \beta} = \int_0^1 \frac{c_1 t^2 \frac{\partial c_2}{\partial b} \frac{\partial b}{\partial \beta}}{(t^2 + a^2)(t^2 + \frac{1}{a^2}) \sqrt{t^4 + 1 - 2t^2 \cos \rho}} dt$$

Equation (29) gives that $\frac{\partial b}{\partial \beta} > 0$. Thus, $\frac{\partial R}{\partial \beta} < 0$ and $\frac{\partial h}{\partial \beta} > 0$. As a consequence $\frac{\partial}{\partial \beta} (\frac{h}{R}) > 0$.

Therefore, after an easy calculation, we have:

$$\frac{2h(a, \rho, \beta)}{\pi R(a, \rho, \beta)} < \frac{2h(a, \pi, 1)}{\pi R(a, \pi, 1)} = \frac{(-1 - a^2) \pi + 8a (\operatorname{arccot}(a) + \operatorname{arctan}(a))}{4a\pi}.$$

Furthermore

$$\frac{\partial}{\partial a} \left(\frac{2h(a, \pi, 1)}{\pi R(a, \pi, 1)} \right) = \frac{1 - a^2}{4a^2} > 0$$

which implies that

$$\frac{h(a, \rho, \beta)}{t(a, \rho, \beta)} < \frac{2h(a, \pi, 1)}{\pi R(a, \pi, 1)} < \frac{2h(1, \pi, 1)}{\pi R(1, \pi, 1)} = \frac{1}{2}.$$

□

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