Consensus and Flocking in Self-Alignment Dynamics
III. From kinetic description to hydrodynamics

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From particle to kinetic description

$$\frac{dv_i}{dt} = \alpha \sum_{j \neq i} a(x_i; \mathbf{x})(v_j - v_i) : \; a(x_i; \mathbf{x}) = \begin{cases} \dfrac{1}{N}\phi(|x_i - x_j|), & \text{Cucker-Smale} \\ \dfrac{\phi(|x_i - x_j|)}{\sum_k \phi(|x_i - x_k|)}, & \text{New model} \end{cases}$$

- Empirical distribution: $f^N(t, \mathbf{x}, \mathbf{v}) := \frac{1}{N}\sum_{j=1}^{N} \delta_{x_j(t)}(\mathbf{x}) \otimes \delta_{v_j(t)}(\mathbf{v})$

$$\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \alpha \int_{\mathbf{y}} \int_{\mathbf{w}} a(x_i; \mathbf{y})(\mathbf{w} - v_i) f^N(t, \mathbf{y}, \mathbf{w}) d\mathbf{y} d\mathbf{w}$$

- N is large enough to observe a limiting distribution:

$\lim f^N(t, \mathbf{x}, \mathbf{v}) = f(t, \mathbf{x}, \mathbf{v})$ satisfies a mean-field model,

$$\partial_t f(t, \mathbf{x}, \mathbf{v}) + \mathbf{v} \cdot \nabla_{\mathbf{x}} f(t, \mathbf{x}, \mathbf{v}) + \alpha \nabla_{\mathbf{v}} \cdot Q(f, f) = 0$$

- $Q(f, f)$ is a local quadratic interaction kernel:

Kinetic descriptions of flocking

$$\partial_t f(t,\mathbf{x},\mathbf{v}) + \mathbf{v}\cdot\nabla_{\mathbf{x}} f(t,\mathbf{x},\mathbf{v}) + \alpha \nabla_{\mathbf{v}}\cdot Q(f,f) = 0$$

$$Q(f,f) := \int_{\mathbb{R}^{2d}} a(\mathbf{x},\mathbf{y})(\mathbf{w}-\mathbf{v}) f(t,\mathbf{x},\mathbf{v}) f(t,\mathbf{y},\mathbf{w})\, d\mathbf{w}\, d\mathbf{y}$$

- Cucker-Smale model

$$a(\mathbf{x},\mathbf{y}) = \phi(|\mathbf{x}-\mathbf{y}|)$$

- The new model

$$a(\mathbf{x},\mathbf{y}) := \frac{\phi(|\mathbf{x}-\mathbf{y}|)}{\int_{\mathbb{R}^d} \phi(|\mathbf{x}-\mathbf{y}|)\rho(\mathbf{y})\, d\mathbf{y}}, \quad \rho(\mathbf{y}) := \int f(t,\mathbf{y},\mathbf{w})\, d\mathbf{w}$$

- Vicsek model [Degond & Motsch]:

$$Q(f,f) := (I - J_f \otimes J_f)\overline{J_f} f, \quad J_f := \int_{B_\rho}\int_{\mathbb{S}^d} \mathbf{w} f(t,\mathbf{x},\mathbf{v}) f(\mathbf{y},\mathbf{w})\, d\mathbf{w}\, d\mathbf{y}$$

Kinetic description of C-S flocking [S.-Y Ha and ET]

$$\begin{cases} \partial_t f(t,\mathbf{x},\mathbf{v}) + \mathbf{v}\cdot\nabla_{\mathbf{x}} f(t,\mathbf{x},\mathbf{v}) + \alpha\nabla_{\mathbf{v}}\cdot Q(f,f) = 0 \\ Q(f,f) := \displaystyle\int_{\mathbb{R}^{2d}} \phi(|\mathbf{x}-\mathbf{y}|)(\mathbf{w}-\mathbf{v})f(t,\mathbf{x},\mathbf{v})f(t,\mathbf{y},\mathbf{w})d\mathbf{w}d\mathbf{y} \end{cases}$$

- Global existence: $f_0(t,\mathbf{x},\mathbf{v}) \mapsto f(t,\mathbf{x},\mathbf{v})$
- Conservation of mass $\mathcal{M}_0(t) := \int f(t,\cdot)d\mathbf{v}d\mathbf{x}$
  and momentum $\mathcal{M}_1(t) := \int \mathbf{v}f(t,\cdot)d\mathbf{v}d\mathbf{x}$
- Energy fluctuation:

$$\vee(t) := \int_{\mathbb{R}^{2d}} |\mathbf{v}-\overline{\mathbf{u}}|^2 f(t,\mathbf{x},\mathbf{v})d\mathbf{v}d\mathbf{x}, \quad \text{average velocity: } \overline{\mathbf{u}} := \frac{\mathcal{M}_1}{\mathcal{M}_0}$$

- Flocking if $m(t) := \min \phi(|x_i(t) - x_j(t)|)$ decays sufficiently slow:

  if $\displaystyle\int^{\infty} \phi(s)ds = \infty$:

$$\frac{d}{dt}\vee(t) \leq -2\alpha m(t)\mathcal{M}_0\vee(t) \quad \mapsto \quad \vee(t) \to 0$$

From kinetic to hydrodynamic description of flocking

$$\begin{cases} \partial_t f(t,\mathbf{x},\mathbf{v}) + \mathbf{v}\cdot\nabla_{\mathbf{x}} f(t,\mathbf{x},\mathbf{v}) + \alpha\nabla_{\mathbf{v}}\cdot Q(f,f) = 0 \\ \\ Q(f,f) := \displaystyle\int_{\mathbb{R}^{2d}} \phi(\mathbf{x},\mathbf{y})(\mathbf{w}-\mathbf{v}) f(t,\mathbf{x},\mathbf{v}) f(t,\mathbf{y},\mathbf{w})\, d\mathbf{w}\, d\mathbf{y} \end{cases}$$

- closure of moments:

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ \mathbf{v} \\ \frac{1}{2}|\mathbf{v}|^2 \end{pmatrix} \partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \alpha\nabla_{\mathbf{v}}\cdot Q(f,f)\, d\mathbf{v} = 0$$

... is expressed in terms of:

mass $\rho(\mathbf{x},t) := \displaystyle\int_{\mathbb{R}^d} f(t,\mathbf{x},\mathbf{v})\, d\mathbf{v}$

momentum $\rho\mathbf{u}(\mathbf{x},t) := \displaystyle\int_{\mathbb{R}^d} \mathbf{v} f(t,\mathbf{x},\mathbf{v})\, dv$

energy $\rho E(\mathbf{x},t) := \displaystyle\frac{1}{2}\int_{\mathbb{R}^d} |\mathbf{v}|^2 f(t,\mathbf{x},\mathbf{v})\, d\mathbf{v}$

From kinetic to hydrodynamic description of flocking

$$\begin{aligned}
\text{mass :} \quad & \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) &&= 0 \\
\text{momentum :} \quad & \partial_t (\rho \mathbf{u}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u} \otimes \mathbf{u} + P) &&= \mathcal{R}, \quad \mathcal{R} = \mathcal{R}(\mathbf{u}) \\
\text{energy :} \quad & \partial_t (\rho E) + \nabla_{\mathbf{x}} \cdot (\rho E \mathbf{u} + P\mathbf{u} + q) &&= \mathcal{S}, \quad \mathcal{S} = \mathcal{S}(\mathbf{u}, \mathbf{u})
\end{aligned}$$

$$\begin{aligned}
\text{stress tensor } P = (p_{ij}) : \quad p_{ij} &= \int_{\mathbb{R}^d} (\mathbf{v}_i - \mathbf{u}_i)(\mathbf{v}_j - \mathbf{u}_j) f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v} \\
\text{heat flux vector } q = (q_i) : \quad q_i &= \int_{\mathbb{R}^d} (\mathbf{v}_i - \mathbf{u}_i) |\mathbf{v} - \mathbf{u}|^2 f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}
\end{aligned}$$

- with non-local source terms:

$$\mathcal{R}(\mathbf{u}) = -\alpha \int_{\mathbb{R}^d} a(\mathbf{x}, \mathbf{y}) \big(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\big) \rho(\mathbf{x}) \rho(\mathbf{y}) d\mathbf{y}$$

$$\mathcal{S}(\mathbf{u}, \mathbf{u}) = -\alpha \int_{\mathbb{R}^d} a(\mathbf{x}, \mathbf{y}) \Big(E(\mathbf{x}) + E(\mathbf{y}) - \mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{y})\Big) \rho(\mathbf{x}) \rho(\mathbf{y}) d\mathbf{y}$$

Flocking in the hydrodynamic description of C-S

- Cucker-Smale: Energy fluctuation:

$$\vee(t) := \int_{\mathbb{R}^{2d}} \left(\frac{1}{2}\big|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\big|^2 + e(\mathbf{x}) + e(\mathbf{y})\right)\rho(\mathbf{x})\rho(\mathbf{y})d\mathbf{y}d\mathbf{x}$$

Set $m(t) := \min_{\mathbf{x},\mathbf{y}\in\Omega(t)} \phi(|\mathbf{x}-\mathbf{y}|), \quad \Omega(t) := \{\mathbf{x}\in\mathbb{R}^d \mid f(t,\mathbf{x},\mathbf{v}) > 0\}$

$$\begin{aligned}\frac{d}{dt}\vee(t) &= -\alpha\mathcal{M}_0\int_{\mathbb{R}^d}\phi(|\mathbf{x}-\mathbf{y}|)\Big(\frac{1}{2}\big|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\big|^2+e(\mathbf{x})+e(\mathbf{y})\Big)\rho(\mathbf{x})\rho(\mathbf{y}) \\ &\leq -2\alpha\mathcal{M}_0 m(t)\vee(t)\end{aligned}$$

... unconditional flocking if $m(t) \sim \phi(t)$ decays sufficiently slow:

$$\int^{\infty}\phi(s)ds = \infty \quad \mapsto \quad |\mathbf{u}(\mathbf{x},t)-\mathbf{u}(\mathbf{y},t)|^2\rho(\mathbf{x},t)\rho(\mathbf{y},t) \to 0$$

- L^2 approach: the symmetry of $a(\mathbf{x},\mathbf{y}) = \phi(|\mathbf{x}-\mathbf{y}|)$ is essential

Hydrodynamic alignment through non-local means

- Mono-phase model $P \equiv 0$:

$$\rho_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0$$
$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} = \alpha \left(\int_{\mathbf{y}} a(\mathbf{x}, \mathbf{y}) \Big(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}) \Big) \rho(\mathbf{y}) d\mathbf{y} \right)$$

- Tendency to align w/ mean $\overline{\mathbf{u}}(\mathbf{x}) := \int_{\mathbf{y}} a(\mathbf{x}, \mathbf{y}) \mathbf{u}(\mathbf{y}) \, \rho(\mathbf{y}) d\mathbf{y}$

dictated by the influence function $a(\mathbf{x}, \mathbf{y}) = \begin{cases} \phi(|\mathbf{x} - \mathbf{y}|) \\ \frac{\phi(|\mathbf{x}-\mathbf{y}|)}{\int \phi(|\mathbf{x}-\mathbf{y}'|)\rho(\mathbf{y}')d\mathbf{y}'} \end{cases}$

- Localized: $a(\mathbf{x}, \mathbf{y}) = \varepsilon^{-d} \phi \left(\frac{|\mathbf{y} - \mathbf{x}|}{\varepsilon} \right) \quad \mapsto$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} = \varepsilon \left(\frac{1}{2} \rho \Delta \mathbf{u} + \nabla \rho \cdot \nabla \mathbf{u} \right)$$

- Singular influence: Fractional Laplacian (Caffarelli-Vasseur, Kiselev-Nazarov, ...)

$$a(x, y) = \phi(|x - y|), \qquad \phi(r) \sim r^{-d-2\theta}$$

- Non-local means: global vs. local $\phi \in L^\infty$ ...

Flocking of the new model – global interactions

THM. Set $\quad \mathrm{d}_{\mathbf{u}}(t) := \sup_{\mathrm{Supp}\,\rho(t)}\{|\mathbf{u}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{y})|.$

Let $\quad \eta(\mathbf{x},\mathbf{x}') := \int_{\mathrm{Supp}\,\rho(t)} \min\left\{a(\mathbf{x},\mathbf{y}), a(\mathbf{x}',\mathbf{y})\right\} d\mathbf{y}.$

Then if $\mathbf{u} \in C^1$ $\rightsquigarrow$ $\quad \frac{d}{dt}\mathrm{d}_{\mathbf{u}}(t) \le -\alpha \min_{\mathbf{x},\mathbf{x}'} \eta(\mathbf{x},\mathbf{x}')\, \mathrm{d}_{\mathbf{u}}(t).$

- Flocking for new model hydrodynamics w/global interactions:

$$\eta(\mathbf{x},\mathbf{x}') \ge \phi(\mathrm{d}_{\mathbf{x}}(t)) \;\mapsto\; \frac{d}{dt}\mathrm{d}_{\mathbf{u}}(t) \le -\alpha\phi(\mathrm{d}_{\mathbf{x}}(t))\mathrm{d}_{\mathbf{u}}(t)$$

- Unconditional flocking:

$$\text{if} \int^{\infty} \phi(s)ds = \infty \;\text{ then }\; \mathrm{d}_{\mathbf{u}}(t) \longrightarrow 0$$

- $\mathrm{d}_{\mathbf{u}}(t) \ll 1$ "expects" conditional regularity $\mapsto$ critical thresholds

Regularity of hydrodynamic alignment models

- The CS hydrodynamics:

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} = \alpha\left(\int_{\mathbf{y}} \phi(|\mathbf{x}-\mathbf{y}|)\big(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})\big)\rho(\mathbf{y})\,d\mathbf{y}\right)$$

- Conditional regularity in terms of $\mathrm{d}_{\mathbf{u}}(\mathbf{u}_0) := \sup_{x,y\in \mathrm{Supp}(\rho_0)} |\mathbf{u}_0(\mathbf{x}) - \mathbf{u}_0(\mathbf{y})|$:

- Critical Threshold - the <u>1D case</u> (Shochet, H. Liu, ET, D. Serre) :

$$u_t + uu_x = \alpha\left(\int_y \phi(|x-y|)\big(u(y)-u(x)\big)\,dy\right), \quad \rho\equiv 1$$

If $u_0'(x) > -\dfrac{\alpha + \sqrt{\alpha - 4\phi(0)\mathrm{d}_{\mathbf{u}}(u_0)}}{2} \mapsto$ global smooth solution

If $u_0'(x) < -\dfrac{\alpha + \sqrt{\alpha + 4\phi(0)\mathrm{d}_{\mathbf{u}}(u_0)}}{2} \mapsto$ finite time blowup

- $\alpha = 0$: "generic" breakdown for inviscid Burgers' eq. $u_0' < 0$
- $\alpha > 0$: CT – global solutions for "generic" initial configurations

Critical thresholds in Eulerian Dynamics

- Eulerian description: $\mathbf{u}(\mathbf{x},t) = \dfrac{d\mathbf{x}}{dt} = (u_1(\mathbf{x},t),\ldots,u_N(\mathbf{x},t))^\top$

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = \nabla_x\Phi : \quad \frac{\partial u_i}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial u_i}{\partial x_k} = \frac{\partial \Phi}{\partial x_i}, \quad i = 1,2,\ldots,N$$

⊙ velocity $\mathbf{u}(\mathbf{x},t)$ is governed by forcing $\mathbf{F} = \nabla_x\Phi : \ \Phi[\mathbf{u},\nabla_{\mathbf{x}}\mathbf{u},\ldots]$

Q.: whether smooth solutions develop singularity in a finite time?

Answer — possible scenarios:

No – global smooth solutions: $\mathbf{u}(\cdot,t)$ remains smooth for all time

Yes – finite time breakdown: shocks, singularities,.. $|\nabla_{\mathbf{x}}\mathbf{u}(\cdot,t_c)| \uparrow \infty$

- Critical threshold phenomena: regularity depends on initial configurations

One-dimensional Euler-Poisson equation

$$u_t + uu_x = -\kappa\phi_x, \quad x \in \mathbb{R}$$

• no pressure; zero background: $-\phi_{xx} = \rho, \quad \rho_t + (\rho u)_x = 0$
— smooth initial data: $\rho(x,0) = \rho_0(x) > 0, \quad u(x,0) = u_0(x)$

• Global smooth solution if

$$\boxed{u_0'(x) > -\sqrt{2\kappa\rho_0(x)}, \quad \forall x \in \mathbb{R}}$$

• Breakdown: if $\exists$ an x s.t. $u_0'(x) \leq -\sqrt{2\kappa\rho_0(x)}$
$\Rightarrow$ regularity breaks down at a finite $t = t_c$: $u(\cdot, t_c) \downarrow -\infty$

• Burgers equation $\kappa = 0$: 'generic' breakdown unless $u_0(x) \uparrow \forall x$

• Critical threshold ($\kappa > 0$):
Global solutions for large set of 'generic' initial configurations

Critical threshold in one-dimensional Euler-Poisson

- Mass equation: $\rho_t + (\rho u)_x = 0$ reads, $\boxed{d := u_x}$

$$(\partial_t + u\partial_x)\rho + u_x\rho = 0 \Longrightarrow \qquad \boxed{\rho' + d\rho = 0} \qquad (1)$$

- ∂_x(Balance equation: $u_t + uu_x = \kappa\phi_x$) reads

$$(\partial_t + u\partial_x)u_x + u_x^2 = \kappa\rho \Longrightarrow \qquad \boxed{d' + d^2 = \kappa\rho} \qquad (2)$$

⊙ Linear stability is of no help: $\lambda \begin{pmatrix} 0 & 0 \\ \kappa & 0 \end{pmatrix} = 0$

$$\frac{\rho \times (2) - d \times (1)}{\rho^2} = \kappa \quad \Longrightarrow \quad \frac{\rho d' - d\rho'}{\rho^2} = \left(\frac{d}{\rho}\right)' = \kappa$$

⊙ Decoupling: $\dfrac{d}{\rho} = \kappa t + d_0/\rho_0 \quad \Longrightarrow \quad d' + d^2 = \kappa \dfrac{d}{\kappa t + d_0/\rho_0}$

- Nonlinear resonance: $u_x = d = \dfrac{d_0 + \kappa\rho_0 t}{1 + d_0 t + \kappa\rho_0 \frac{t^2}{2}}, \qquad d_0 = u_0'$

- Geometry of characteristics: straight lines $(\kappa = 0) \rightarrow$ parabolas $(\kappa > 0)$

More on one-dimensional Euler-Poisson $u_t + uu_x = \Phi_x$

- Adding pressure: $\Phi[\rho, u] = -\kappa\phi + A\rho^{\gamma-1}, \ -\phi_{xx} = \rho$

Thm (w/Dongming Wei) Global smooth solution iff

$$u_0'(x) \gtrsim -\sqrt{2\kappa\rho_0(x)} + \sqrt{A}\frac{|\rho_0'(x)|}{\left(\sqrt{\rho_0}(x)\right)^{3-\gamma}}$$

Poisson and pressure compete: global regularity vs. breakdown

- Adding non-zero background: $-\phi_{xx} = \rho - c$: $|u_0'(x)| \leq \sqrt{\kappa\big(2\rho_0(x) - c\big)}$

⊙ Semi-classical limit NLSP:
$i\epsilon\psi_t^\epsilon = -\frac{\epsilon^2}{2}\Delta_x\psi^\epsilon - \kappa\left(\Delta_x^{-1}(|\psi^\epsilon|^2 - c)\right)\psi^\epsilon$

- WKB ansatz $\psi^\epsilon = A_0^\epsilon e^{iS^\epsilon/\epsilon}$: $\quad u := \nabla S^\epsilon, \ \rho := |A^\epsilon|^2$

$$\rho_t + \nabla\cdot(\rho u) = 0, \quad u_t + u\cdot\nabla u = \kappa\nabla\Delta_x^{-1}(\rho - c) + \frac{\epsilon^2}{2}\left[\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right]$$

- Classical limit with 1D sub-critical data:
$|S_0''(x)| \leq \sqrt{\kappa(2|A_0(x)|^2 - c)}$

1D threshold hydrodynamic alignment (C. Tan & ET)

- Self-alignment hydrodynamics with density:

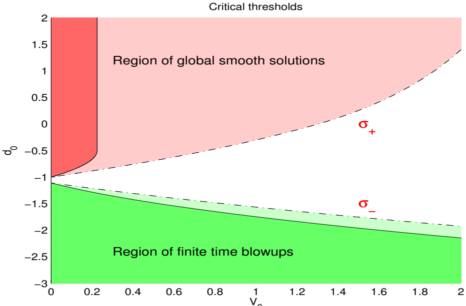
$$\rho_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0 \quad \text{subject to } \underline{\text{compactly supported}}\ \rho_0$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} = \alpha \left(\int_{\mathbf{y}} \phi(|\mathbf{x}-\mathbf{y}|)\big(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})\big)\rho(\mathbf{y})\, d\mathbf{y} \right)$$

$$u_0' := \inf_{x \in supp(\rho_0)} \partial_x u_0, \qquad \mathrm{d}_{\mathbf{u}}(u_0) := \sup_{x,y \in supp(\rho_0)} |u_0(x) - u_0(y)|$$

If $u_0'(x) > \sigma_+(\mathrm{d}_{\mathbf{u}}(u_0)) \mapsto$ global solution $(\rho, u) \in L^\infty(\mathbb{R}) \times C^1(\mathrm{Supp}(\rho))$

If $u_0'(x) < \sigma_-(\mathrm{d}_{\mathbf{u}}(u_0)) \mapsto$ finite time breakdown



Critical thresholds in multidimensional setup

- Eulerian dynamics $\mathbf{u}:\mathbb{R}^d \mapsto \mathbb{R}^d$: $\qquad \mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = \mathbf{F}$

- $\begin{cases} \text{Euler-Poisson}: & D\mathbf{F} = \alpha D^2\Delta^{-1}\rho \\ \text{Restricted Euler-Poisson}: & D\mathbf{F} = \dfrac{\alpha}{d}\rho I_{d\times d} \end{cases}$

⊙ 2D critical threshold in REP dynamics – in terms of $M := \dfrac{\partial u_i}{\partial x_j}$:

$\mathrm{div}(\mathbf{u}_0(x)) > \sigma_+\big(\rho_0(x), \eta_0(x)\big)$ $\quad \eta \mapsto$ spectral gap $:= \lambda_2(M) - \lambda_1(M)$

Critical surface: $\sigma_+(\rho,\eta) := sgn(\eta^2 - 2\alpha\rho)\sqrt{\eta^2 - 2\alpha\rho + 2\alpha\rho\ ln\left(\frac{2\alpha}{\eta^2}\right)}$

- Back to 2D self-alignment: $\mathbf{F} = \alpha\int\phi(|\mathbf{x}-\mathbf{y}|)(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}))\rho(\mathbf{y})d\mathbf{y}$

$\mapsto$ Global regularity for sub-critical data: $\mathrm{div}(\mathbf{u}_0(x)) > \sigma_+(\mathrm{d}_{\mathbf{u}}(\mathbf{u}_0), \omega_0)$

Opinion hydrodynamics

$$\frac{d}{dt}x_i(t) = \alpha \sum_j \frac{\phi_{ij}}{d_i}(\mathbf{x}) \cdot (x_j(t) - x_i(t)), \quad \sum_j a_{ij} = 1$$

- Aggregation equation:

$$\rho(t,\mathbf{x}) = \frac{1}{N}\sum \delta_{x_j(t)}(\mathbf{x}) \quad \rightsquigarrow \quad \rho_t + \nabla_{\mathbf{x}} \cdot (\mathbf{u}\rho) = 0$$

- Symmetric case: $\mathbf{u} = \nabla\Phi * \rho, \quad \Phi'(r) = r\phi(r)$

⊙ Bertozzi, Carrillo, Laurent,. . . . . . :
$\exists$ of regular solution: $\phi'(r)r \lesssim \phi(r)$

- Non-symmetric case: $\mathbf{u} = \dfrac{\int_{\mathbf{y}} \phi(\mathbf{x}-\mathbf{y})(\mathbf{y}-\mathbf{x})\rho(\mathbf{y})d\mathbf{y}}{\int_{\mathbf{y}} \phi(\mathbf{y}-\mathbf{x})\rho(\mathbf{y})d\mathbf{y}}$

⊙ Global ϕ - convergence towards (point) consensus; Local – open.

Non-local means, propagation of connectivity, ...

- Short term repulsion, long range attraction, ..., external forces:

mass : $\rho_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0$

momentum : $(\rho \mathbf{u})_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_{\mathbf{x}} p(\rho) = \mathcal{A}(\mathbf{u}) - \rho \nabla_{\mathbf{x}} U(\rho)$

⊙ Short range repulsion — $p(\rho) = \rho_* T \dfrac{\rho}{(\rho_* - \rho)_+}$

⊙ Long range attraction — $U(\rho) = \mu K(\mathbf{x}) * \rho(t, \mathbf{x})$

⊙ Velocity alignment $\mathcal{A}(\mathbf{u}) = \displaystyle\int_{\mathbb{R}^d} a(\mathbf{x}, \mathbf{y}) \big(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})\big) \rho(\mathbf{y}) \rho(\mathbf{x}) d\mathbf{y}$

- Ott, d'Orsogna, Bertozzi, Carrillo, L. Chayes, Laurent, Panferov...

- Heterophilious dynamics: propagation of connectivity

⊙ Counting active sets & formation of "islands"

- Stability (swarming of intelligent agents), critical thresholds, ...

KI-Net: Kinetic description of emerging challenges in multiscale problems of natural sciences

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The ultimate goal: development, analysis and computation of novel kinetic descriptions with particular focus on

- Quantum dynamics with applications to chemistry;
- Network dynamics with applications to social sciences;
- Kinetic models of biological processes.

THANK YOU